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DYNAMIC STABILITY, POST-CRITICAL BEHAVIOR AND RECOVERY OF SYSTEMS IN ENGINEERING

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Abstract. *Aim of the study is to present a brief overview concerning the Dynamic Stability problem, its basic definitions and principles, important phenomena, research motivation and application in engineering. Relation with relevant systems being prone to stability loss encountered in other branches like in physics, other natural sciences and engineering is pointed out as well.*

The theoretical background applicable in many disciplines is presented. In the paper the most frequently used methods of Dynamic Stability analysis are reminded in relation with individual dynamic systems widely discussed in various engineering branches. In particular Lyapunov function and exponent procedures, Routh-Hurwitz, Lienard and other theorems together with demonstrations are outlined. Possibilities of analytical and numerical procedures are mentioned together with possible feedback coming from experimental research and testing.

Systems widely encountered in engineering are presented in a form of mathematical models. The analysis of their Dynamic Stability and post-critical behavior are gone through. Stability limits, bifurcation points, quasi-periodic response processes and chaotic regimes are discussed. Limit Cycle existence and stability are examined together with their separating role as attractor and repulser.

Two levels of stability loss (possible recovery, final inevitable collapse) as they can be observed at softening systems are pointed out. Time limited excitation and respective transition effects (seismic excitation) are also discussed together with evaluation of possible system reliability improvement. Dynamic Stability investigation of double degree of freedom aero-elastic systems in linear using several approaches is briefly highlighted. Further systems modeling problems arising in engineering of transport means are outlined as well. A few hints for applications are given. Some open problems and possible future research strategy are outlined.

1 INTRODUCTION

Phenomena of stability have been attracting researchers and engineers nearly for three centuries. The famous Swiss mathematician L.P. Euler opened this domain in 1744 discovering a critical axial central load which brings a simply supported beam into collapse (widely known Euler load), see [1]. A great number of renowned scientists followed the founder of this discipline until now including the most respected world mathematicians, physicians and engineers.

Although stability can be considered as very compact and consistent discipline having very rigorous common background, many branches separated themselves during the time focusing to special attributes of potential engineering applications or to possible abstract theoretical investigations. Finally we can speak now about various subject fields which have been developing rather independently one from each other. Each of them is stressing some other attributes and details of the basic theory, prefers some other tools, leads towards different numerical procedures and their variants. Despite of that a strong link is still ruling in common basic principles and many investigation methods are popular not only on a research but also on a practical application level.

Although problems of stability came almost exclusively from Civil Engineering for more than first hundred years, other branches recognized very soon that various phenomena of stability are emerging also in their area. Mechanical Engineering and Dynamics of stiff bodies motivated appearing of a discipline dealing with a stability of motion. Rigorous theoretical grounds have been laid by A.M. Lyapunov - Russia (1857 - 1918), respecting studies of J.S. Lagrange and W.R. Hamilton, see Figure 1. Subsequently a number of popular problems of Dynamic Stability have been studied as a consequence of needs in engineering. During the twentieth century and mainly its second half nearly all branches of engineering, physics and other natural sciences ascertained that problems of stability are decisive very often from the viewpoint of their systems reliability, technological processes feasibility and social acceptability. It can be stated that in the contemporary time various forms of stability problems are treated almost everywhere taking advantage of very similar background. Indeed, although they are factually far away, all of them are based on principles related either with energy or entropy stationary points or their impacts in relevant differential or algebraic formulations, matrices, fractions, etc.

Let us mention some examples of stability problems decisive for the subject existence or functionality concerning systems outside of Mechanics of Solids or Fluids: Plasma physics: stability of hot plasma (Tokamak, technological plasma devices), see e.g. [2], [3], [4]. Stability of chain reaction (nuclear reactors, testing facilities), see e.g. [5], [6]. Particle accelerators, lasers and other types of resonators, see e.g. [7]. Electro-magnetic field stability in particular domains and interacting with other fields, problems of magnetic fluids, see e.g. [8]. Optics: stability in fibre glass optics, see e.g. [9]. Mechanics of fracture. Chemistry: stability of reactors with respect to various attributes (temperature, concentrations, phase transforming, etc.), [10] and other papers by the same author. Biology: stability of various types of cultivation, stability of physiological processes, see e.g. [11], [12], [13]. Social disciplines: stability of predator-prey equilibrium, see e.g. [14], [15], [16], [17]; stability of economical processes, stability in prognostication, see e.g. [18], [19]. Regulation, stability of control-loop (intervention delay, strategies of information keeping/canceling), see e.g. [20], [21], [22], [23], control of road and other types of traffic, see e.g. [24]. See also mathematical studies discussing problem of bifurcation in control systems with single or multiple delay, [25], [26]. The above citations are accidental only. Hundreds references more can be found in relevant journals and proceedings. A lot of monographs, text-books and reports oriented to individual areas above appeared during several decades worldwide.

Take a note that a very unpleasant phenomenon represents the numerical stability loss. Although it concerns rather a mathematical problem, probably everybody encountered this effect during numerical experiments. There are several significantly different types of the numerical

Figure 1: Leonhard Paul Euler: 1707 Basel - 1783 Sankt Petersburg; Joseph-Louis Lagrange: 1736 Turin 1813 Paris; Carl Friedrich Gauss: 1777 Braunschweig 1855, Göttingen; William Rowan Hamilton: 1805 Dublin 1865 Dublin; Jules Henri Poincaré: 1854 Nancy 1912 Paris; Aleksandr Mikhailovich Lyapunov: 1857 Yaroslavl 1918 Odessa; Stephen Prokopovich Timoshenko: 1878 Shpotivka, Ukraine - 1972 Wuppertal.

stability loss depending on numerical process which are carried out. Despite this category is basically not very far from the physical and other types of stability loss phenomena, this area after all keeps rather independent. They develop separately due to specific problem formulations and methods of investigation, see e.g. [6]. The special position take cases where stability itself and numerical stability intermingle. They need to be treated individually respecting particular sources of instability.

2 DYNAMIC STABILITY IN MECHANICS

2.1 Domain of Investigation

Theoretical and experimental research of stability problems in mechanics as well as immediate application in relevant engineering domains is a process which lasts basically since the above mentioned discovery of the Euler load in the year 1744 as remembered in the previous section. It is characteristic from the viewpoint of Rational as well as Computational Mechanics that almost everybody among the famous scientists involved in mechanics contributed significantly to its development. Most of theorems, principles, equations, functionals and functions bear their names. See a few of them on pictures in the Figure 1.

In the contemporary time the stability of mechanical systems (or may be interacting with other physical fields) also represents several areas which are not fully separated having either way the common theoretical basis, but still their development follows different attributes and employs other mathematical formulations and solution procedures. Although they interact very often and a sharp limit between them can be hardly traced, following fields can be outlined:

(x) Dynamic Stability of multi-body systems. This field is predominantly oriented to problems of motion stability, reaching certain stability limits and investigation of post-critical processes. We have many monographs at our disposal, e.g. [27], [28], [29]. See also monograph by

Poincaré [30] and other partial original resources by Lyapunov and others. Stability of Limit Cycles (LC) together with perturbation of trajectory and regimes of chaotic motion is discussed. Many problems of aero-elastic stability treated at the level of one or several degrees of freedom can be included into this field. Linear and non-linear approaches are used being dependent on a particular problem. However, many problems of aero-elastic stability cannot be included in this paragraph and should be considered as integral parts of remaining categories.
(xx) Problems of buckling of large deformable systems, like frameworks, plates, shells, etc. This field is a follow-up of classical studies since the beginning of the whole discipline. Formulations are predominantly static, however tasks of Dynamic Stability of weak deformable systems are also subject of investigation. Very comprehensive book informs about the contemporary knowledge, see [31]. Let us remember Monographs and numerous papers by Koiter, e.g. [32], [33], etc., see also [34]. Stability and possible collapse of laminated and reinforced structures can be considered as a part of this field. Stable or collapsing state of a structure due to crack propagation or creep influence could fall to this field representing a special problem of the stability loss. The fracture mechanics as a very close discipline can formulate the crash for the most part as a problem of stability loss and post-critical behavior of a system with successively propagating cracks until collapse arises. FEM in many variants belongs now to main instruments used for analysis itself, see well known monograph [35] and numerous papers dealing with special problems, for example [36]. However many semi-analytic methods in non-linear mechanics have been developed and successfully used in the past. A lot of literature is available see e.g. [37], [38], [39], [40]. New approaches emerge abandoning traditional formulations being based on non-conventional principles (neural networks, information gaps, convex models, etc.), e.g. [41]. They look to be effective on boundary between deterministic and stochastic domain when very limited information about system imperfection are available. Interesting synthesis of both fields (x) and (xx) presents the monograph [42].
(xxx) Stability of flow. Many problems of fluid streaming and interacting with stiff obstacles make subject of the area of Fluid Induced Vibration. This assignment implicates problems of stability in micro- and macro-scale, emerging of turbulence and its stable/unstable regimes, appearance of explicit vortices, e.g. [43], etc. Well known books and papers have been published by Païdoussis, see [44], [45]. Hundreds problems of streaming stability related with mechanical and civil engineering or architecture are investigated all over, see e.g. high level study concerning the Cosserat fluid motion [46]. Large Eddy Simulation is frequently used method of modern investigation using the most powerful computer technology, see e.g. monograph [47]. Many large scale atmospheric and sea stream phenomena belong to this category of stability research. They are for instance typhoon formation, movement and extinction, vortex shading behind islands in sea current, stability of sea streaming regarding climatologic processes, Coriolis force produced air streaming, e.g. [48], etc. Stability of non-linear waves, in particular of solitons, e.g. [7] and their interaction (modeling by stiff body mechanics), which is a phenomenon very stable in a prismatic channel (Korteweg de Vries) with a possible sudden vanishing. Many special problems are emerging in supersonic flow where a shock wave stability loss could have far reaching implications.

Problems of the Dynamic Stability in mechanics are very diverse. Except formulation of basic principles and development of rigorous but rather abstract mathematical background, e.g. [49], [50], [51], [52], they include mostly research of stability limits, bifurcation points, critical and post-critical system behavior, ways of collapsing on local or global level, reliability and possible recovery of systems, LCs. A rigorous mathematical background can be found in quite recent monographs [53], [54]. A large majority of cases need to work in a stable state to avoid any danger of collapse and to keep the basic reliability. However take a note that some systems on the contrary need to run in post-critical state for their desired functionality (various resonators, vibrators, experimental devices, regulators, musical instruments, etc.).

Critical state and their accompanying effects can be reached by various ways depending on basic physical and mathematical formulation. From this viewpoint several classes should be indicated, which differ mainly in the energy supply form and interaction with surroundings either of solid state or flowing nature. It implicates very often also a range of principles and procedures suitable for development of particular stability criteria and expected phenomena typical for the problem investigated.

The critical state can be reached usually due to self-exciting processes or external excitation of any type (interaction of interacting systems with relative motion, auto-parametric systems, etc.) whatever problem of solid or fluid mechanics is handled including their interaction. Obviously a lot of sub-classes exist within both of them. Nevertheless important shades of meaning represent the type of excitation time history. In particular deterministic and stochastic cases should be investigated. Therefore deterministic and stochastic approaches are to be taken into account, see many monographs, e.g. [55], [56], [57], [58] and papers, e.g. [59], [60], [61], [62] and many others. Anyway combination of both especially for non-linear systems leads often to specific problems which should be carefully analyzed with respect to rate of both excitation types. Such cases being solved using methods of the stochastic mechanics bring us to problems of Stochastic Stability. It applies to problems of turbulent flow induced vibration in various regimes.

2.2 Lyapunov Second Method

Let us touch on some principal and popular methods encountered and widely used in mechanics of stiff and deformable bodies concerning the domain (x) as denoted in the previous sub-section. Therefore we focus to area of Dynamic Stability of stiff and deformable systems including selected problems of flow induced vibrations of systems with one or a few degrees of freedom. Although there doesn't exist any sharp delimitation, some attitudes are typical for the area (x) after all, as they look to be more effective than the others.

Doubtlessly in the Dynamic Stability of systems following the above domain (x), the Lyapunov Second Method (LSM) is widely known being based on the Lyapunov Function (LF). In addition to the LSM itself there exist a lot of derived or associated methods which don't look very often to be related or produced by LSM any more. In a basic form the LSM is basically not related with physical principles, but treats the stability of a system of ordinary differential equations and tries to answer the question of whether the solution remains arbitrarily close to an equilibrium solution after being disturbed. For this purpose a couple of functions is carried out: function $V(\mathbf{x})$ - (Lyapunov Function) and its total derivative with $2n$ scalar variables:

$$V(\mathbf{x}), \quad \dot{V}(\mathbf{x}) = \boldsymbol{\nabla} V(\mathbf{x}) \cdot \dot{\mathbf{x}}(t), \quad \mathbf{x} \in \mathbb{R}^{2n} . \tag{1}$$

Inspecting properties of this couple (dynamic system with n degrees of freedom) a decision concerning stability can be made. Our explanation will occupy with analytic purposes of LSM only. However take a note that originally it has been intended for a general area concerning problems of synthesis (proofs of basic principles, synthesis of theorems, etc.).

The rigorous definition and proof of Lyapunov theorems is extensive and can be found in a number of monographs, for instance [20], [63], [21], [28], [64]. For a certain generalization, see [65]. Let us remember very briefly the second Lyapunov theorem. We consider the unforced dynamic system with n degrees of freedom:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}), \quad \mathbf{f}(\mathbf{x}) \in \mathbb{R}^{2n} , \tag{2}$$

which has an equilibrium point $\mathbf{x}(0) = \mathbf{0}$. The system is stable for every $\mathbf{x}_0 \in \mathfrak{R}$ if the system never goes outside the region $\mathfrak{R}$ and asymptotically stable if ($\lim_{t\to\infty} \|\mathbf{x}(t)\| \to 0$ - trajectory tends to origin). The system is globally stable if it is both asymptotically stable and the region $\mathfrak{R}$ is the entire state space. If a LF $V(\mathbf{x})$ exists and satisfies following conditions: (i) $V(\mathbf{x})$

is continuously differentiable in all components of the vector $\mathbf{x}$, (ii) $V(\mathbf{x})$ is positive definite, (iii) $\dot{V}(\mathbf{x})$ is negative definite; then the origin is asymptotically stable. If the $\dot{V}(\mathbf{x})$ is negative semi-definite, the origin is globally stable.

Analogously LSM can be developed also for problems of Stochastic Dynamics. With reference to widely known monographs, see for instance [55], [56], [57], one can outline a problem formulation of stability in the stochastic domain as follows:

We have a stochastic differential system (SDF):

$$\dot{\mathbf{x}}(t) \;=\; \mathbf{f}(\mathbf{x}) \;+\; \mathbf{G}(\mathbf{x}) \cdot \boldsymbol{\varphi}(t), \quad \mathbf{f}(\mathbf{x}) \in \mathbb{R}^{2n},\; \mathbf{G}(\mathbf{x}) \in \mathbb{R}^{2n,m},\; \boldsymbol{\varphi}(t) \in \mathbb{R}^{m}\;, \tag{3}$$

where $\boldsymbol{\varphi}(t)$ is a vector of m independent Gaussian white noises. The Fokker-Planck (FP) stochastic differential can be composed:

$$\mathbf{D}(P(\mathbf{x},t)) \;=\; -\sum_{i=1}^{m} \frac{\partial \varkappa_i(\mathbf{x}) P(\mathbf{x},t)}{\partial x_i} \;+\; \sum_{ij=1}^{m} \frac{\partial^2 \varkappa_{ij}(\mathbf{x}) P(\mathbf{x},t)}{\partial x_i \partial x_j}\;, \tag{4}$$

where $P(\mathbf{x},t)$ is a cross-probability density of the response coordinates $\mathbf{x}$, while $\varkappa_i, \varkappa_{ij}$ are drift and diffusion coefficients respectively. Stochastic Stability in probability can be now approximately defined on the basis of probability density of the system Eq. (3) response components. Therefore instead of $V(\mathbf{x}), \dot{V}(\mathbf{x})$ the pair $P(\mathbf{x},t), \mathbf{D}(P(\mathbf{x},t))$ should be treated.

In principle the right side of Eq. (1) or Eq. (4) can be interpreted as a time "increase" of a certain deterministic or random generalized energy flux which the system is either taking up (positive) or loosing (negative). In the latter case the system is roughly speaking asymptotically stable in probability.

It is necessary to realize that the LF is mostly not unique. The above conditions for the same system can be satisfied by a number of pairs. In such a case each pair may yield different quantitative information about the stability characteristics of the system. Primarily the principal problem is to construct the pair $V(\mathbf{x}), \dot{V}(\mathbf{x})$ in such a way that it possesses certain properties listed above. Nevertheless regarding non-linear problems, each one represents a new target as the pair must be newly constructed at least for every class of systems. Moreover, the adequate choice of the $V(\mathbf{x})$ depends significantly upon experience and shrewdness of authors. It should be emphasized that the pair $V(\mathbf{x}), \dot{V}(\mathbf{x})$ or $P(\mathbf{x},t), \mathbf{D}(P(\mathbf{x},t))$ represents reaching the satisfactory conditions and the difficulty arises when the necessary conditions cannot be exhibited. The difference between those can be so large, that satisfactory conditions lead to useless or physically paradoxical criteria.

Consequently, while application of the LSM for synthesis doesn't need any further supplements, it appears consistent that relating with analytical purposes a number of methods emerged as tools for construction of the LF. In principle they are based on a purely analytic type constructions or using various energy or entropy related analogies. During the time several comprehensive overview papers have been published, see e.g. [66], [67], [68]. There are not any sharp limits separating their groups, but still they enable to be roughly classified as follows:

(i) Chetayev methods (integral methods), see [27]. The LF is obtained, either directly or indirectly, from the first integral of the system Eq. (2). These methods can be used for both deterministic as well as for stochastic problems, even thought the latter type of application is significantly more complicated, see e.g. [60], [69].

(ii) Krasovskii type methods (quadratic forms), see [63]. The construction is based upon quadratic forms derived directly from the system Eq. (2). In original form they are rather intended for systems with weak non-linearity. However some of these methods can be expanded using additional forms of the even degree making homogeneous functions in Euler meaning. These methods are transparent and enable relatively good insight into the whole mechanism of

the LSM during a particular analysis. Similarly like (i) also some of Krasovskii methods are suitable for deterministic and stochastic problems.
(iii) Zubov type methods (partial differential equations), see [70]. The Zubov methods are based on the fact that the LF can be obtained as a solution of a non-homogeneous partial differential equation (PDE) of the first order following Eq. (1) with the left side which is assigned by a user with respect to system Eq. (2). The type of PDE being of the first order allows solution procedure using method of characteristics as a rule. Nevertheless the character of these methods predestines those to deterministic problems only. Although in principle they could be developed for FP operator as well, the effective solution of a large PDE of the second order is practically impossible even if not excluded.
(iv) Lur'e - Postnikov type methods (canonical forms). These methods are rather specific and based upon certain canonical equations. They cannot be mostly generalized for larger class of systems than that for which they have been developed. On the other hand if the proper canonical equations can be found, they provide very powerful and effective tool for practical use. No applications in area of stochastic problems are known.
(v) Miscellaneous mostly case oriented methods. Many rather heuristic methods and special problem oriented procedures. A lot of references can be found predominantly in aeronautic and control engineering.

Obviously the above division and overview is not exhausting, there are additional ways to define the constructive methods. A typical alternate, which is very popular might be based upon the first selection of a trial function. Indeed, some methods start by selecting $V(\mathbf{x})$, others by selecting $\boldsymbol{\nabla} V(\mathbf{x})$ or by $\dot{V}(\mathbf{x})$. Anyway, it can be observed that there are essentially two basic approaches to construct a LF:
(i) A suggestion of a function being a candidate for a LF is done. If it doesn't satisfy the necessary conditions, it is abandoned.
(ii) A trial function is formed and to satisfy the desired properties either certain conditions are imposed on the system equations, or a certain components of the candidate function are adjusted to make the function the LF.

In practice various combination of above procedures combined also with numerical support are used. Concerning numerical way of LF construction, first papers appeared in early sixtieth, e.g. [71]. However a massive intervention of numerical approaches came with powerful computer technology not sooner than in late eighties, e.g. [72]. During this period, however, a wide employment of Lyapunov exponent oriented procedures has been launched as a quick and powerful tool of deterministic as well as Stochastic Stability investigation. On the other hand, as in application of every numerical method, internal structure of the problem remains completely hidden, although a quick and relatively precise quantitative answer is obtained. Both strategies are living now next to each other complementing mutually.

It should be remembered that the background of numerical procedures changed principally during decades. While in sixtieth they have been mostly oriented to partial steps rather accelerating constituent elements of analytical techniques, the contemporary methods are conceptually different being based on direct numerical methods, see e.g. [73], [59], [61], [74], in particular various modifications of FEM are frequently addressed. Concerning Lyapunov exponent, it became very often an integral part of simulation packages watching on line a procedure of numerical integration and evaluating the relevant exponent or its real part. The integration interval can be selected arbitrarily, however its suitable length is significantly dependent on type of analysis we are doing (deterministic stationary/non-stationary, quasi-periodic, chaotic, stochastic, etc.). There are rich references in literature, see e.g. monographs [75], [76], numerous papers, e.g. [77], [78], [79], [80] or web-site presentations and algorithms, e.g. [81]. On the other hand semi-analytic procedures appeared working with partial expressions, which are based on Liouville formula (Bendixon formula is a very special case) having an exponential form very close to expression serving for the Lyapunov exponent testing, see e.g. [82]. This approach seems to be promising for investigation of LC stability, see the 3rd section.

2.3 Routh-Hurwitz method and Silvester determinants

A large area of linear systems stability should be treated separately. The reason is that besides the original linear (or linearized) systems serving as direct model of a physical or engineering system, the linear systems also arise as result of the first order perturbation procedure investigating nature and neighborhood of singular points of non-linear systems Eq. (2). Let us recall the Eq. (2) and suppose that it holds in a singular point:

$$\mathrm{d}\mathbf{f}(\mathbf{x}_0)/\mathrm{d}t \;=\; \mathbf{0} \quad\Longrightarrow\quad \mathbf{f}(\mathbf{x}_0) \;=\; \mathbf{0}\,. \tag{5}$$

The trajectories $\mathbf{x}$ in the neighborhood of the point $\mathbf{x}_0$ with respect to perturbation $\boldsymbol{\xi}$ can be written on level of the first degree of approximation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{x}+\boldsymbol{\xi}\right)|_{\mathbf{x}=\mathbf{x}_0} = \mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_0} + \mathbf{B}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_0}\cdot\boldsymbol{\xi}, \quad \mathbf{B}(\mathbf{x}) = \left[\frac{\partial\mathbf{f}(\mathbf{x})}{\partial x_1},...,\frac{\partial\mathbf{f}(\mathbf{x})}{\partial x_{2n}}\right] \in \mathbb{R}^{2n,2n}\,, \tag{6}$$

which results regarding Eq. (5) in an equation for perturbations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi} \;=\; \mathbf{B}(\mathbf{x}_0)\cdot\boldsymbol{\xi}\,, \tag{7}$$

where $\mathbf{B}(\mathbf{x}_0)$ is the Jacobi matrix in the point $\mathbf{x}_0$. A care should be taken if either $\mathbf{B}(\mathbf{x}_0)$ has some multiple eigen-values or the neighborhood of $\mathbf{x}_0$ cannot be satisfactorily characterized by means of a linear approximation. The latter case can be illustrated on a simple example. It can be shown that the system:

$$\dot{x_1} = -x_2 + \alpha\cdot x_1^3, \quad \dot{y} = x_1 + \alpha\cdot x_2^3, \quad \alpha = \text{const.}\,, \tag{8}$$

is stable or unstable in the point $(0,0)$ if $\alpha < 0$ or $\alpha > 0$ respectively and labile for $\alpha = 0$, i.e. if linear approximation in $(0,0)$ and its neighborhood is adopted. Consequently, adopting the linear approximation, any influence of the parametr α is avoided, although in the original system is decisive. For details, see monograph [54].

Therefore treating either the problem Eq. (2) in the neighborhood of $\mathbf{x}_0$ (it means that Eq. (7) is in use) or the original linear problem, we can write following normal differential system:

$$\dot{\mathbf{x}}(t) \;=\; \mathbf{A}\cdot\mathbf{x} \;+\; \mathbf{F}(t), \quad \mathbf{A}\in\mathbb{R}^{2n,2n},\; \mathbf{F}(t)\in\mathbb{R}^{2n}\,, \tag{9}$$

where $\mathbf{A}$ is a square regular matrix. The LF for particular cases or some classes could be constructed. It would bring us in principle from an analysis of a differential system stability analysis to investigation of eigen-value properties of the matrix $\mathbf{A}$. A couple theorems have been deduced and exactly proved in order to show an equivalency of both strategies concerning regular linear systems. So the problem concerns the rules requesting negative real part of all eigen-values, provided the system should retain stability. Stability could be sometimes influenced also by external influences and so the term $\mathbf{F}(t)$ is incorporated into Eq. (9).

To examine the matrix eigen-values means to inspect a characteristic polynomial:

$$a_0\lambda^{2n} + a_1\lambda^{2n-1} + a_2\lambda^{2n-2} + ...a_{2n-2}\lambda^2 + a_{2n-1}\lambda + a_{2n} \;=\; 0\,. \tag{10}$$

Routh-Hurwitz (RH) determinants composed of coefficients $a_0 - a_{2n}$ can be used for testing the eigen-value properties in particular the sign of their real parts. Analogously also Silvestr determinants can be used, which is a method quite close to previous one. Substantiation and exact mathematical background are explained in well known monographs, for instance: [29],

[83], [84] and others. With respect to notation in Eq. (10), $2n$ RH determinants and conditions can be outlined as follows:

$$a_1 > 0, \begin{vmatrix} a_1, & a_3 \\ a_0, & a_2 \end{vmatrix} > 0, \begin{vmatrix} a_1, & a_3, & a_5 \\ a_0, & a_2, & a_4 \\ 0, & a_1, & a_3, \end{vmatrix} > 0, \ldots, \begin{vmatrix} a_1, & a_3, & a_5, & \ldots, & 0 \\ a_0, & a_2, & a_4, & \ldots, & 0 \\ 0, & a_1, & a_3, & \ldots, & 0 \\ 0, & a_0, & a_2, & \ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots, & .. \\ \ldots & \ldots & \ldots & \ldots, & a_{2n} \end{vmatrix} > 0\,. \tag{11}$$

A very important fact is, that the RH method produces a closed set of $2n$ stability conditions which are necessary and satisfactory in the same time. Therefore should one of them is not fulfilled, the system Eq. (9) is not stable.

It is obvious however, that algebraic structure of conditions being provided by RH or Silvestr determinants using Eq. (11) is very complicated as a rule. For this reason it is useful to combine RH method with other theorems in order to facilitate the final set of conditions which are to be investigated. One of the most effective seems to be the Descartes rule specifying position of the polynomial Eq. (10) root real parts on the basis of coefficients $a_0 - a_{2n}$ sign changes. Descartes rule doesn't provide a complete set of conditions (at least one is always missing) and hence it cannot be used independently. Nevertheless certain combination with the RH method is possible. Moreover combining RH method, Descartes rule and Vieta's formulae, one can obtain applicable tool for detailed investigation of linear system Eq. (9) Dynamic Stability. Unfortunately a practical usage of this tool in original analytic form is still limited due their complexity. Systems with Double Degree of Freedom (DDOF) or polynomial Eq. (10) of the fourth degree can be analyzed, as we will see later. Nevertheless analysis of more complicated systems by means of this tool is not yet realistic. However the system of conditions can be simplified once again using the Lienard theorem, see e.g. [83]. Its purpose lies in a possibility to reduce the number of RH conditions in the whole set for the sake of Descartes conditions providing simpler criteria referring isolated coefficients $a_0 - a_{2n}$ only. Polynomials of the 6th and 8th degree have been analyzed in such a way, see [85]. Anyway it should be stressed a general strong point of this procedure, which enables to interrupt analytic way in certain points and to continue numerically. This combination proved its effectiveness also in cases of systems with higher degrees of freedom than those mentioned above.

2.4 Case related Dynamic Stability loss testing

There are many engineering systems where case related criteria and procedures are the most effective being worthy to pay them attention. Their principles are diverse, nevertheless all of them are related with "enormous or infinite" increase of the system response at least in some coordinates. Enumerating some of them we can experience their relation with procedures based on LF method or RH determinants. Nevertheless they are simple and transparent and therefore applicable by very wide society of researchers. As for mathematical formulation some typical variants can be mentioned:

(i) The denominator of a fraction expressing a solution (or its certain part) vanishes. Typical case represents the primal Euler stability problem, where the criterium follows from the vanishing determinant in the relevant fraction, when integration constants are solved. Many other cases can be referred. Searching for stability limits of simple non-linear systems, usually fraction denominator being a determinant of algebraic system for response parameters is tested, i.e. unstable branches of Duffing or Van der Pol oscillators and their generalized versions including

some Multi-Degree of Freedom (MDOF) systems are got through. Some problems of critical speed of inertial load moving on a rail-road resting on massless elastic support etc., are analyzed using these principles. The strength of this attitude consists in a high transparency and quite often in simplicity of results. On the other hand only very simple problems can be treated in such a way.
(ii) The absolute value of the response is infinitely rising due to spatial resonance and insufficient internal damping. It can be encountered in periodic systems excited by a chain of identical forces with identical distance moving with a suitable velocity. This effect is observed very often in railway engineering, see for instance [86], in pipeline systems conveying fluid or in stochastic resonance driven systems.
(iii) Integral of the inverse transform looses its existence. Many problems of wave propagation in 2D and 3D continuum due to moving load. Transition through the critical velocity of the load movement can be effectively tested tracking the respective integrals existence, see e.g. [87]. A special procedure limited to cases, where a solution method is based on integral transform in spatial coordinates.

3 LIMIT CYCLE STABILITY

3.1 Perturbation of Limit Cycle trajectory

Limit Cycles (LC) are encountered very often when studying multi-degree of freedom non-linear systems (MDOF). Their features should be investigated carefully, especially the basic existence, their emergence and extinction, or shape and type identification. These factors are closely related to the stable or unstable character of the LC in question. The aspect of stability or non-stability impresses on LC the character of an attractor or a repulser. A few special effects can accompany LC. It is primarily the stability rate which should be observed carefully. Being at low level it can result in quasi-periodic processes moving around the LC and leading to a certain toroidal formation around the LC, see e.g. [88], [82]. In this case an easy transition into the chaotic state is possible as demonstrated in papers, e.g. [89], as well as in monographs, e.g. [90]. On the other hand the response of many non-linear systems cannot include LC for principal reasons. Indeed, regarding the same dynamic system, LCs can disappear due to an internal damping increase or to changes in excitation strategy.

Let us consider an autonomous Hamiltonian dynamic system with n degrees of freedom. Assuming that the equivalent normal system of $2n$ differential equations exists, Eq. (2) can be rewritten in the form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \;, \qquad \mathbf{f}(\mathbf{x}) \;=\; [f_1(\mathbf{x}), ..., f_{2n}(\mathbf{x})]^T \;, \\ \mathbf{x} &= [x_1, ..., x_{2n}]^T \;, \quad x_j \;=\; x_j(t) \;; \quad j \in (1, 2n) \;, \end{aligned} \tag{12}$$

X - $2n$ space arithmetized by coordinates x_j,
$x_{2i-1}(t)$ - displacements; $i \in (1, n)$, $\quad x_{2i}(t)$ - velocities,
$f_j(x_1, ..., x_{2n})$ - smooth functions of state variables $\mathbf{x}$; $\quad f_{2i-1}(\mathbf{x}) \;=\; x_{2i}$.

Only single and smooth LCs in the system (1) are supposed being described by a radius vector in the space X:

$$L_0 : \quad x_j = \varphi_j(s) \;; \quad j \in (1, 2n) \;, \tag{13}$$

where s is a coordinate along the LC.

To investigate stability of the LC, a toroidal domain around the loop Eq. (13) with an arbitrarily narrow cross-section is constructed. This geometry is arithmetized by the vector $\mathbf{q}(s) = [\varphi_1'(s), ..., \varphi_{2n}'(s)] \;\in\; \mathbb{R}^{2n}$ tangential to the LC together with a system of $(2n-1)$ vectors $\mathbf{n}_j(s) \;\in\; \mathbb{R}^{2n}$ which are orthogonal to the tangential vector $\mathbf{q}(s)$ and orthogonal mutually

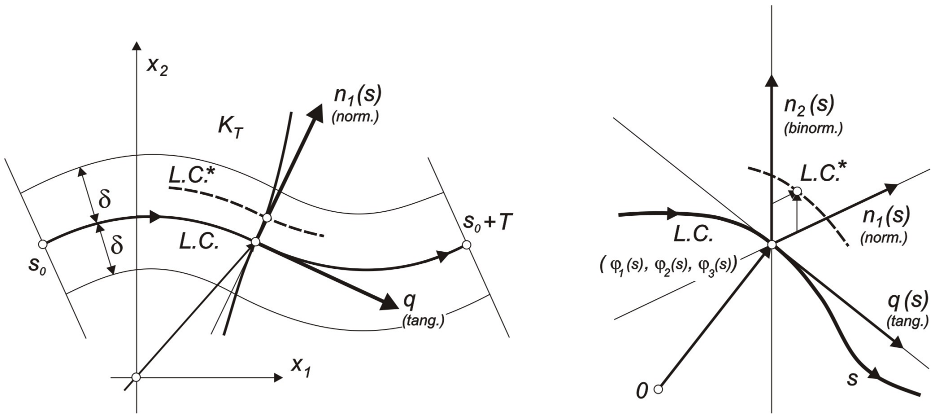


Figure 2: Limit Cycle trajectory in 2D and 3D and local coordinates in the relevant hyper-cube K.

as well, see Figure 2. Coordinates on vectors $\mathbf{n}_j$ are given by parameters n_j summarized into the vector $\mathbf{n}$. Vectors $\mathbf{q}$ and $\mathbf{n}_j$ can be gathered into a square matrix:

$$\mathbf{Q} \ = \ [\mathbf{q}, \mathbf{n}_1, ..., \mathbf{n}_{2n-1}] \ \in \ \mathbb{R}^{2n,2n} \ . \tag{14}$$

Elements of the matrix $\mathbf{Q}$ can be evaluated using widely known rules of differential geometry. The LC being mildly deflected from its trajectory L_0 can be described as follows:

$$L_0^* : \ x_j^* = \varphi_j^*(s, \mathbf{n}) = \varphi_j(s) + \sum_{k=1}^{2n-1} n_k \kappa_{jk}(s) \ ; \ j \in (1, 2n) \ ; \ \mathbf{n} = [n_1, ..., n_{2n-1}] \ , \tag{15}$$

where $\kappa_{jk} = \kappa_{jk}(s)$ are components of the vectors $\mathbf{n}_k$ and $-\delta < n_k < \delta$ values of respective coordinates on the vectors $\mathbf{n}_k$.

Respecting the system Eq. (12) the following differential system involving the perturbed LC state can be carried out:

$$\frac{\mathrm{d}x_j^*}{\mathrm{d}t} \ = \ \frac{\partial \varphi_j^*}{\partial s} \cdot \frac{\mathrm{d}s}{\mathrm{d}t} + \sum_{k=1}^{2n-1} \frac{\partial \varphi_j^*}{\partial n_k} \cdot \frac{\mathrm{d}n_k}{\mathrm{d}t} = P_j^* \ , \tag{16}$$

where $P_j^* \ = \ P_j^*(\varphi_1^*(s, \mathbf{n}), ..., \varphi_{2n}^*(s, \mathbf{n})) \ ; \ j \ \in \ (1, 2n)$. Eqs (16) represents the governing system of the LC perturbed trajectory in the X domain, where $|L_0^* - L_0| < \delta$ for every $s \in (0, T)$.

Substituting to Eqs (16) for φ_j^* following Eq. (15), an easy rearrangement gives:

$$\left(\varphi_j' + \sum_{k=1}^{2n-1} n_k \cdot \kappa_{jk}' \right) \cdot \frac{\mathrm{d}s}{\mathrm{d}t} \ + \ \sum_{k=1}^{2n-1} \kappa_{jk} \cdot \frac{\mathrm{d}n_k}{\mathrm{d}t} \ = \ P_j^* \ . \tag{17}$$

A painful derivation brings us to differential system:

$$\frac{\mathrm{d}n_j}{\mathrm{d}s} \ = \ D_j(s, \mathbf{n}) \cdot D_s(s, \mathbf{n})^{-1} \ = \ r_j(s, \mathbf{n}) \ ; \qquad j \in (1, 2n-1) \ , \tag{18}$$

where $D_s(s, \mathbf{n}), \ \ D_j(s, \mathbf{n})$ are determinants of the system matrix subsistent to individual unknowns in the system Eq. (17) with respect to right-hand side $\mathbf{P}^*$.

The system (18) describes behavior of the LC in the $\delta-$neighbourhood of the basic trajectory L_0 in the form of small deviation in orthogonal basis $\mathbf{n}_j$ with regard to stationary point $\mathbf{n} = \mathbf{0}$.

Therefore, the decision concerning LC stability will be made on the basis of the system (18). The system is strongly non-linear. Hence the neighborhood of the singular point $\mathbf{n} = \mathbf{0}$ can be inspected by means of linear approximation characterizing small deflections from $L_0(s)$ or point $\mathbf{n} = \mathbf{0}$. The relevant linear differential system can be solved in a form:

$$\mathbf{N}(s) \;=\; \mathbf{N}_0 \cdot \exp\left(\int_0^s \mathbf{R}(\theta)\,\mathrm{d}\theta\right), \quad \mathbf{N}(s) \in \mathbb{R}^{2n-1,2n-1}\,, \tag{19}$$

where $\mathbf{R}$ is the Jacobi matrix associated with the system (18). Every of $2n-1$ columns of the matrix $\mathbf{N}(s)$ represents a sensitivity of normals $\mathbf{n}_j$ around L_0 for relevant set of initial conditions.

Decreasing or increasing values of $\mathbf{N}(s)$ components indicates stable or unstable LC. The decision can be adopted verifying the state for the full length of the LC, i.e. $s = T$. In principal the LC is stable if the real part of all eigen-values of $\mathbf{N}(s)$ for every s is negative. In this particular case, the detailed evaluation can be tuned adjusting the matrix of initial values $\mathbf{N}_0$ separating the influence of individual imperfection components.

A certain information can be obtained evaluating the determinant $\det[\mathbf{N}(s)] = D_N(s)$. Employing the Liouville theorem the following equation can be deduced:

$$\frac{\mathrm{d}}{\mathrm{d}s}D_N \;=\; \frac{1}{2n-1}T_R(s)\cdot D_N \;\Longrightarrow\; D_N(s) \;=\; \frac{1}{2n-1}D_{N,0}\cdot\exp\left(\int_0^s T_R(\theta)\,\mathrm{d}\theta\right)\,, \tag{20}$$

where $T_R(\theta)$ is the trace of the Jacobi matrix $\mathbf{R}(s)$. Hence the LC is stable if it holds:

$$Re\{\int_0^s T_R(\theta)\} < 0\,. \tag{21}$$

for $s = T$. For $n = 1$, e.g. for a Single Degree of Freedom (SDOF) system, both variants (19) and (21) are identical. However for $n > 1$ the criterium (21) should be used with care. The negative trace Re$\{T_R(\theta)\}$ doesn't necessarily guarantee that the real part of at least one eigen-value is positive. Consequently, the condition (21) provides only general information. This condition should be considered as necessary but not satisfactory. Another way can represent the case when eigen-values and eigen-vectors of the matrix $\mathbf{R}$ are available. Then the normal space can be rotated adequately and all coordinates $j \in (1, 2n-1)$ assessed separately.

Some analogy with the Lyapunov exponent testing can be observed taking into account the exponential character of formulae Eq. (19) and (21), see e.g. [77], [72], [91], [52] and particularly [80]. Some numerical approaches very close to those discussed here by an analytical way are available in literature, e.g. [92]. A common influence of both looks to be very promising and can be fruitful for development of some hybrid procedures.

3.2 Demonstration of a system with single degree of freedom

The governing system in compliance with Eq. (12) gains the form:

$$\dot{x}_1 \;=\; x_2\,, \quad \dot{x}_2 \;=\; P_2(x_1, x_2)\,. \tag{22}$$

For a possible Limit Cycle, see Eq. (13), and consequently for L_0 trajectory and its perturbed modification in δ neighbourhood L_0^* can be written:

$$\begin{array}{rclcrcl} x_1 & = & \varphi_1(s) & \longrightarrow & \varphi_1^{*'}(s,n_1) & = & \varphi_2^*(s,n_1)\,, \\ x_2 & = & \varphi_2(s) & & \varphi_2^{*'}(s,n_1) & = & P_2^*(s,n_1)\,. \end{array} \tag{23}$$

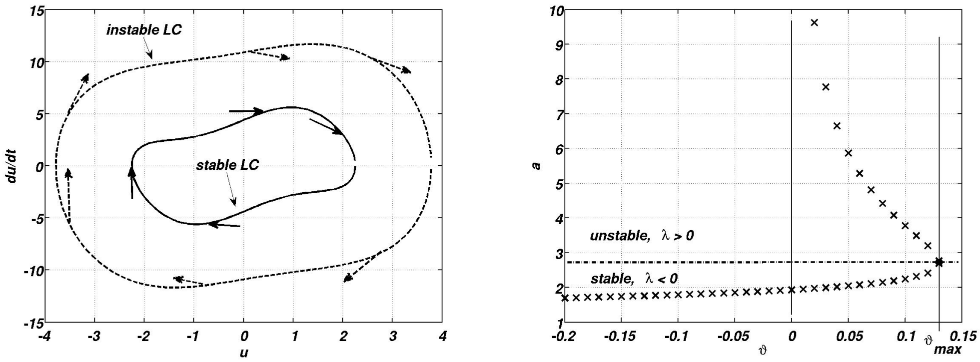


Figure 3: Attractive and repulsive LCs; left: stable and unstable Limit Cycles meet at a certain point creating the separatrix manifold; right: stability diagram $a = f(\vartheta)$ for ϑ varying in the interval $\vartheta \in \langle -0.2, \vartheta_{max} \rangle$

Geometry of a narrow strip bordering the Limit Cycle in plane (x_1, x_2) is obvious from Figure 2. Using Eq. (15), then φ_1^*, φ_2^* containing linear approximation of a small perturbation can be defined:

$$\mathbf{Q} \;=\; [\mathbf{q}, \mathbf{n}_1] \;=\; \begin{bmatrix} \varphi_1', & \varphi_2' \\ \varphi_2', & -\varphi_1' \end{bmatrix} \;\text{(a)}\;, \quad \longrightarrow \quad \begin{array}{rcl} \varphi_1^*(s, n_1) & = & \varphi_1(s) + n_1\varphi_2'(s) \\ \varphi_2^*(s, n_1) & = & \varphi_2(s) - n_1\varphi_1'(s) \end{array} \;\text{(b)}\;. \qquad (24)$$

Following the procedure outlined in the previous sub-section one can obtain the expression characterizing behavior of the trajectory linear approximation in the normal direction with respect to $L_0(s)$:

$$\frac{\partial \tilde{n}_1(s|0, n_{1,0})}{\partial n_{1,0}}\Big|_{n_{1,0}=0} \;=\; \frac{\varphi_1'^2(0) + \varphi_2'^2(0)}{\varphi_1'^2(s) + \varphi_2'^2(s)} \exp\left(\int_0^s P_2^{x2}(\theta)\mathrm{d}\theta\right). \qquad (25)$$

If the LC is stable, then taking one full period $s \in (0, T)$ the fraction in front of exponential equals one. The stability itself is indicated by a negative real part of the integral, as it is requested by formula Eq. (19) or by criterium Eq. (21). Structure of formula (25) corresponds with the general form (19) or (20) and remembers the Bendixon's formula.

Let us now demonstrate this approach on the SDOF system described by the Van der Pol - Duffing type equation. This equation appears very often in various applications, for instance in aero-elasticity, auto-parametric studies and in other areas, see e.g. [93]:

$$\ddot{u} + \alpha u + \beta u^3 = (\eta - \nu u^2 + \vartheta u^4)\dot{u}\,. \qquad (26)$$

The right-hand side of Eq. (26) includes the fourth degree of the response and encompass possibly both stable and unstable LC as they are observed experimentally in various branches. The existence of LCs is predetermined by a certain ratio of parameters η, ν, ϑ. With respect to Eqs (22), Eq. (26) can be rewritten as follows:

$$\dot{x}_1 \;=\; x_2\;, \quad \dot{x}_2 \;=\; -\alpha x_1 - \beta x_1^3 + (\eta - \nu x_1^2 + \vartheta x_1^4)x_2 = P_2(x_1, x_2). \qquad (27)$$

Referring to formula Eq. (25), the expected LC is stable if the real part of relevant integral along the whole period is negative. Therefore the following inequality should be resolved:

$$\lambda = \int_0^T (\eta - \nu x_1^2 + \vartheta x_1^4)\mathrm{d}t < 0\;. \qquad (28)$$

The integral can be evaluated in a closed form. Then for the limit separating stable and unstable part in a space of parameters η, ν, ϑ following can be written (a_0 is an amplitude of x_1):

$$\eta - \frac{1}{2}\nu a_0^2 + \frac{3}{8}\vartheta a_0^4 = 0 \ . \tag{29}$$

Some results are depicted in Figure 3. The stability diagram and LCs are typical for an aeroelastic system with parameters $\eta > 0, \nu > 0$ as usually encountered in bluff body aerodynamics, when a post-critical state is investigated. Follow the right picture in Figure 3: provided that $\vartheta < 0$, only one stable LC exists. If $0 < \vartheta < \vartheta_{max}$, then two simple LCs exist, having the shape demonstrated in left picture of Figure 3. The inner LC is stable and acts as an attractor in the area closed by the outer LC. This means that every system positions inside the inner LC approaches this stable post-critical state. The outer LC is unstable having a repulsing character. If it is crossed over, the response is boosted ad infinitum. The ϑ_{max} provides a twofold semi-stable LC and acts as a strong energy barrier. When this barrier is exceeded by way of further energy supply or adequate change of above parameters, the LCs doesn't exist any more and the final stability loss occurs.

4 AUTO-PARAMETRIC SYSTEMS

4.1 General properties and definitions

Many structures and systems widely used in engineering and other branches have a character of auto-parametric systems. With respect to special properties they make up a separate group within the non-linear systems. Investigation of auto-parametric systems has proceeded intensively during last three decades. Nevertheless the first studies devoted to these effects appeared during years 1968-1985, see for instance [94], [95], [96]. A great number of papers and monographs published Tondl together with co-authors. Let us mention some of them, [97], [98], [99]. Contribution of other authors was also remarkable, see e.g. [100], [101], etc. Certain inspiration, special procedures and some results can be found among others also in papers by the author of this study, e.g. [102], [103], [93], etc. Detailed analysis of particular auto-parametric systems is included into well known monographs, for example [64], [104], [105] and others.

Auto-parametric are MDOF systems, which consist of several (at least of two) sub-systems. Dynamic behavior of individual sub-systems is independent until the system response finds in sub-critical regime. In such a case the solution of relevant differential system is called semi-trivial. It means that solution of one part is non-trivial (primary sub-system), while that of the second remains trivial (secondary sub-system). Under these circumstances the sub-systems or parts don't influence each other.

Overstepping a certain limit of the semi-trivial solution stability, or passing through a particular bifurcation point in space of system or excitation parameters, the semi-trivial solution can lose its Dynamic Stability. The response becomes non-trivial in general on the whole system. This state activates non-linear links of the primary and secondary sub-systems. The system response takes a regime of the auto-parametric resonance or the post-critical state.

The primary sub-system is subdued to excitation. The excitation structure is not arbitrary and must be in accordance with structure of the system itself and respect its auto-parametric character. The primary as well as the secondary sub-systems can be either linear or non-linear. Nevertheless links between them are always non-linear. Therefore interaction of both can be expressed in a form of the following symbolic differential system:

$$\begin{aligned} \mathbf{L}_1(\mathbf{x}, t) \quad &+ \ \mathbf{K}_{12}(\mathbf{x}, \mathbf{y}) \ = \ 0 \ , \\ \mathbf{K}_{21}(\mathbf{x}, \mathbf{y}) \ &+ \ \mathbf{L}_2(\mathbf{y}) \quad\ \ = \ 0 \ , \end{aligned} \tag{30}$$

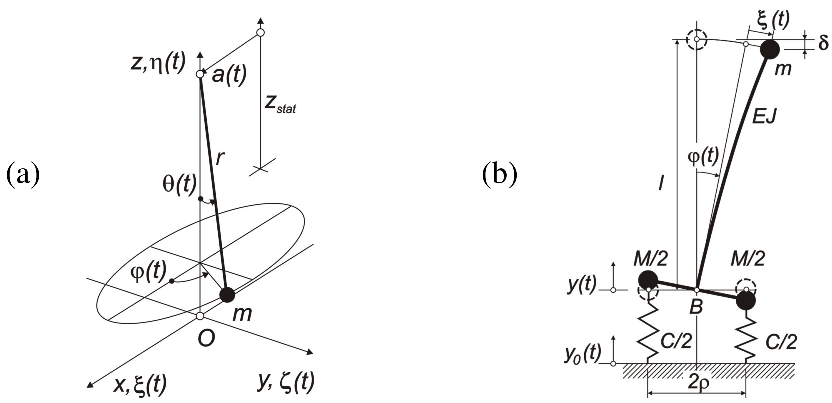


Figure 4: Simple auto-parametric systems with two sub-systems; (a) spherical pendulum, (b) inverse pendulum.

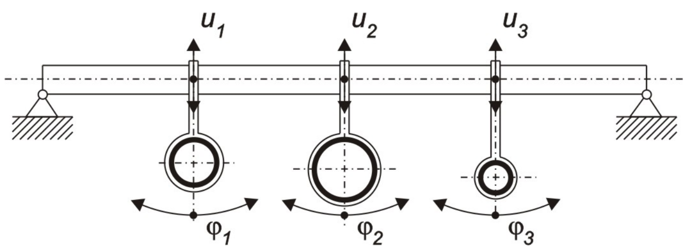


Figure 5: Simple beam with several secondary sub-subsystems.

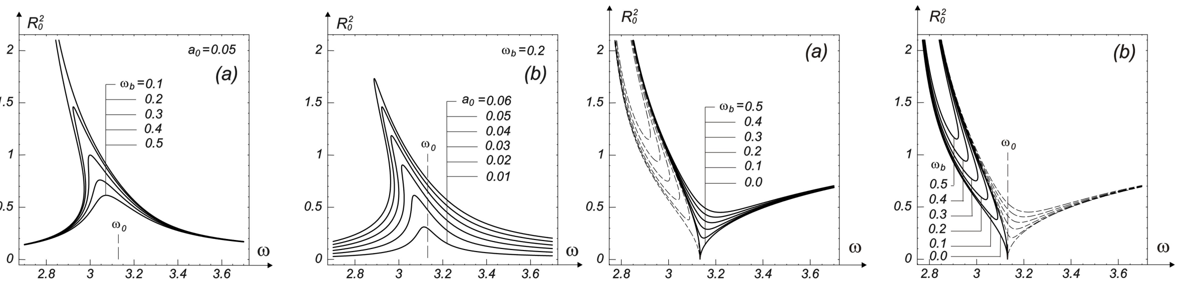
where $\mathbf{L}_1(\mathbf{x}, t)$, $\mathbf{L}_2(\mathbf{y}, t)$ are linear or non-linear differential operators acting on vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{K}_{12}(\mathbf{x}, \mathbf{y}), \mathbf{K}_{21}(\mathbf{x}, \mathbf{y})$ are linking non-linear operators. Let us suppose that $\mathbf{L}_1(\mathbf{x}, t)$ represents the primary and $\mathbf{L}_2(\mathbf{y}, t)$ the secondary sub-system. Both systems in Figure 4 correspond with structure of Eqs (30).

In order to be entitled to call the system(30) auto-parametric, there should exist a semi-trivial solution $\mathbf{x}_{st}, \mathbf{y}_{st}$ under certain conditions. So it should hold:

$$\begin{gathered} \mathbf{L}_1(\mathbf{x}, t) = 0,\ \lim_{t\to\infty} \mathbf{x} = \mathbf{x}_{st},\ \mathbf{y}_{st} = 0\ , \\ \mathbf{K}_{12}(\mathbf{x}_{st}, 0) = \mathbf{K}_{21}(\mathbf{x}_{st}, 0) = 0, \quad \mathbf{L}_2(0) = 0\ . \end{gathered} \tag{31}$$

Sweeping up the excitation frequency from zero, provided amplitudes of excitation are rising, or changing system parameters, we can pass through any bifurcation point which can implicate stability loss of the semi-trivial solution. At this moment amplitudes of $\mathbf{x}$ increase dramatically and non-linear terms $\mathbf{K}_{12}(\mathbf{x}, \mathbf{y}), \mathbf{K}_{21}(\mathbf{x}, \mathbf{y})$ in Eqs (30) start to influence the response of the whole system. The vector $\mathbf{y}$ becomes also non-trivial which brings the system into the post-critical state or into auto-parametric resonance. Basic definitions and results on level of the Rational Mechanics can be found in monographs e.g. [27], for extension into the domain of the stochastic dynamics, see other monographs for instance [55], [56], or papers [60], [80] and others.

It is obvious that auto-parametric system can consist of more than two sub-systems. In such a case the number of operator equations in the system Eqs (30) will be higher in accordance with the number of sub-systems. The number of state vectors $(\mathbf{x}, \mathbf{y}, ...)$ increases correspondingly. A simple beam with suspended pipelines, see Figure 5, can serve as an example. Another case is an aircraft wing with suspended engines.



(a) (b) (c) (d)

Figure 6: Resonance curves and stability limits of semi-trivial solution; pictures (a) and (b) are resonance curves as amplitudes of the semi-trivial solution (non-zero part) - (a) fixed excitation amplitude, variable damping, (b) fixed damping, variable excitation amplitude; pictures (c) and (d) are semi-trivial solution stability limits for various damping values - (c) ζ limit (plane yz - transverse), (d) ξ limit (plane xz -longitudinal).

4.2 Spherical pendulum and associated systems

Let us consider a spherical pendulum with two degrees of freedom moving on a spherical surface with a center in the pendulum suspension point, see Figure 4a. Provided the suspension point moves in a horizontal plane, the pendulum can serve as a vibration absorber in the relevant plane, as it is very frequently used in civil or mechanical engineering, see monographs [106], [29], [107] and papers [108], [109], [110], etc. During several decades a number of methods to project such devices using one or more pendulum, see e.g. [111]. If the function of a pendulum is not limited into one direction only, its behavior is by far more complicated as it would follow from cited resources, which are limited to linear formulations and planar function as a rule. Post-critical effects emerging in one or more resonance domains can be very dangerous, because pendulum, when losing stability of its semi-trivial response, loses also its original purpose and it can influence dynamics of the structure negatively.

We suppose that the horizontal excitation in the suspension point is kinematic and characterized by function $a = a(t)$. The mathematical model follows from equilibrium of mechanical energy. The system is Hamiltonian, see e.g. [29], [112], so kinetic and potential energy can be written in a form:

$$\begin{aligned} T &= \frac{m}{2}[r^2(\dot\theta^2 + \dot\varphi^2 \sin^2\theta) + 2r\dot a(\dot\theta\cos\theta\cos\varphi - \dot\varphi\sin\theta\sin\varphi) + \dot a^2]\ , & (a) \\ V &= mgr(1-\cos\theta)\ . & (b) \end{aligned} \tag{32}$$

m, r - pendulum mass or suspension length respectively,
$a = a(t)$ - horizontal kinematic excitation in the suspension point.

Two Langrangian equations follow from expressions Eqs (32). The viscous damping is included using quadratic Rayleigh function.

The Lagrangian system should be approximately transformed into Cartesian coordinate system x, y. For details, see e.g. [103], [93] and Figure 4a. Some longer derivation leads to following approach:

$$\begin{aligned} \ddot\xi + \frac{1}{2r^2}\xi(\xi^2+\zeta^2)^{\bullet\bullet} + 2\omega_b\dot\xi + \frac{g}{r}(\xi + \frac{1}{2r^2}\xi(\xi^2+\zeta^2)) &= -\ddot a\ , & (a) \\ \ddot\zeta + \frac{1}{2r^2}\zeta(\xi^2+\zeta^2)^{\bullet\bullet} + 2\omega_b\dot\zeta + \frac{g}{r}(\zeta + \frac{1}{2r^2}\zeta(\xi^2+\zeta^2)) &= 0\ , & (b) \end{aligned} \tag{33}$$

which is an approximation on the level $O(\varepsilon^6)$; $\varepsilon^2 = (\xi^2+\zeta^2)/r^2$.

Semi-trivial solution of Eqs (33) can be searched using the harmonic balance method (mono-harmonic is used, for comprehensive mathematical background see [54]). A few results depicted

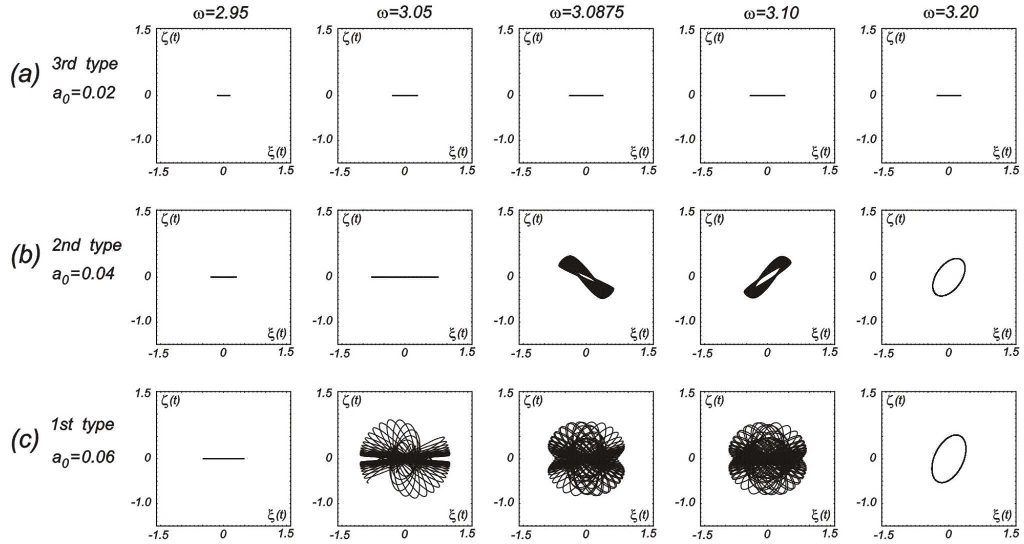


Figure 7: Three types of the resonance domain for the damping factor $\omega_b = 0.3$; response time history depicted in the ground plane (xy); row (a) - 3rd type, row (b) - 2nd type, row (c) - 1st type.

in Figure 6a,b show basic character of the semi-trivial solution having shape of resonance curves of a non-linear SDOF system. In the second step the stability of the semi-trivial solution is assessed. Stability limits make possible to determine parts of resonance curves where the existence of stable steady state solution is guaranteed. Hence we write solution of the system Eqs (33) in the form of a linear approximation combining the semi-trivial solution ξ_0 and a small deviation u, v in both coordinates (the first order perturbation procedure), see e.g. [27], [113], [114] and many more resources dealing with Dynamic Stability.

$$
\begin{aligned}
\xi &= \xi_0 + u \quad , \quad u(t) = u_c \cos\omega t + u_s \sin\omega t \; , && (a) \\
\zeta &= 0 + v \quad , \quad v(t) = v_c \cos\omega t + v_s \sin\omega t \; , && (b) \\
& R_A^2 = u_c^2 + u_s^2 + v_c^2 + v_s^2, \quad S_A^2 = u_s \cdot v_c - u_c \cdot v_s \; .
\end{aligned} \tag{34}
$$

The approximation (34) should be put into the system Eqs (33) and the operation of the harmonic balance is applied once again, see e.g. [51], [54].

Let us refer some aspects typical for post-critical behavior of the system in the resonance domain. Three types of resonance domains have been identified with respect to relation of resonance curves and stability limits, see Figure 6. Figure 7 demonstrates the time history of the pendulum response projected to the plane xy: row (a) - 3rd type, row (b) -2nd type, row (c) - 1st type. The system response in the 1st type (3rd row) is dramatic. It reveals chaotic, quasi-periodic type and regimes of the Limit Cycle (LC), see [115], [116]. Should the resonance curve oversteps both stability limits, see Figure 6c,d, five different regimes (a-e) are passed through, see Figure 8a. The second type of the resonance domain complies with the situation when only the limit ζ is crossed. Two different regimes can be encountered. The 3rd type of the resonance domain comes up when neither of both stability limits is crossed. No special regime occurs. The semi-trivial solution is always stable a continuously links with sub- and super- resonance intervals of the excitation frequency no violating stability limits.

Let us take a note that in the post-critical state the chaotic response regimes variability can be very large. A careful analysis is necessary, see e.g. papers [89], [117], or monographs [118], [119]. If numerical methods are used, their selection or modifications should be closely considered and during the computational procedure checked all the time using various instruments,

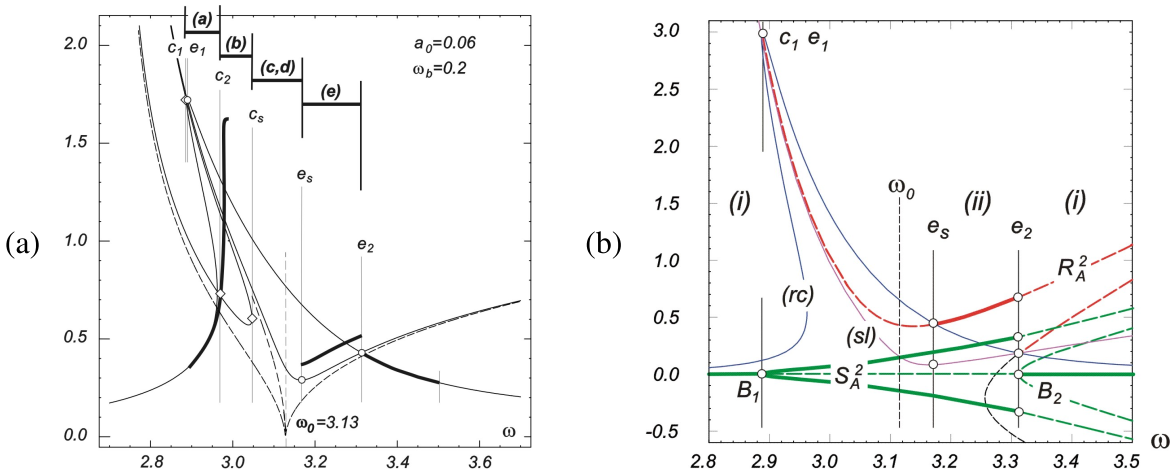


Figure 8: Left picture (a): Resonance domain of the 1st type $\omega\epsilon(c_1, e_2)$ for $a_0 = 0.06, \omega_b = 0.2$; sub-resonance $\omega\epsilon(0, c_1)$ and super-resonance interval $\omega\epsilon(e_2, \infty)$ with stable semi-trivial solution; Right picture (b): bifurcation diagram of R_A^2, S_A^2 - generalized amplitude and phase shift of the post-critical response; stable part - solid curve, unstable part - dashed curve; $(rc), (sl)$ - resonance curve and ζ stability limit; B_1, B_2 - bifurcation points of S_A^2; $B_1 \equiv c_1, e_1$, $B_2 \equiv e_2$; e_s - minimum of the ζ stability limit and curves (rc) and R_A^2 intersection point; (ii) - limit cycles, see Figure 7 $\omega = 3.20$ rows (b,c).

for instance Lyapunov exponent, cyclic tests, etc., see e.g. [77], [76] and many other references. To application of the Lyapunov exponent are devoted among others [72], [78], [79], [120], [81].

The bifurcation diagram depicted in the Figure 8b outlines some main attributes of the system from the viewpoint of the Dynamic Stability and transitions between response regimes in the post-critical domain. These attributes can be gathered evaluating the generalized amplitude R_A^2 and phase shift of both response components S_A^2, see Eq. (34), passing through the resonance domain. The green curves S_A^2 exhibit double bifurcation of the pitchfork type. Stable or unstable part is plotted by solid or dashed lines respectively. The stable trivial solution $(S_A^2 = 0)$ within the frequency interval $\omega\epsilon(0, B_1)$ indicates the response in the vertical plane. In the interval $\omega\epsilon(B_1, B_2)$ there exist two stable non-trivial and one unstable trivial solution. Starting from point $\omega = B_2$ one stable trivial and four unstable non-trivial solution S_A^2 exist. It is in accordance with R_A^2. The only difference reveals in the interval $\omega\epsilon(B_1, e_s)$, where R_A^2 changes very quickly. It means that in this interval no LC exists. The diagram of S_A^2 can be interpreted as follows: intervals ω where the trivial S_A^2 is stable, represent intervals of the stable semi-trivial (or planar in the vertical plane) solution, while within the interval $B_1 - B_2$ the post-critical regime is ruling.

Results of this analysis indicate an important recommendation for a practical design of dynamic vibration absorbers. The device should be projected in such a way that any crossings of resonance curves and stability limits are avoided (3rd type in Figure 7).

4.3 Heavy ball moving in a spherical dish

Spherical pendulum discussed in the previous subsection is applied very often as a dynamic vibration absorber at slender structures excited by dynamic effects of wind or may be also by a seismic attack.

Although it is very effective and reliable, it cannot be applied due to vertical dimensions, when an absorber should be installed as a supplementary equipment. Also horizontal constructions, like foot bridges, cannot accept any absorber of the pendulum type. These shortcomings can be avoided using the absorber of ball type. The basic principle comes out of a rolling movement of a metallic ball of a radius r inside of a metallic rubber coated dish of a radius $R > r$. This system is closed in an airtight case. Such a device is practically maintenance free. Its

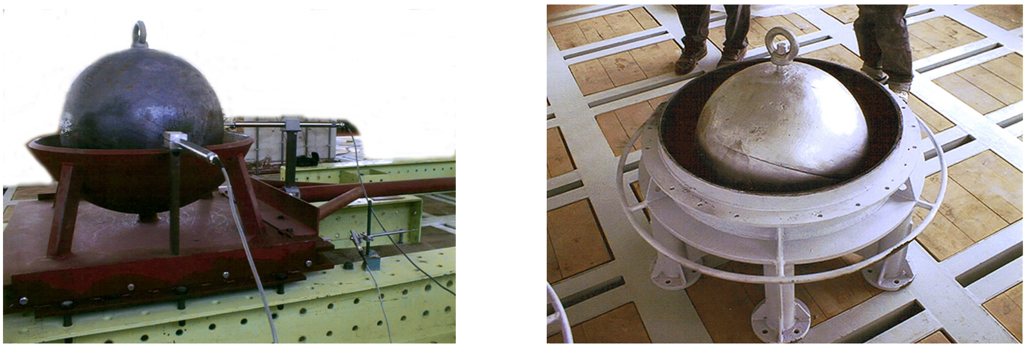


Figure 9: Ball absorber in the dynamic testing laboratory, before installation on a TV tower.

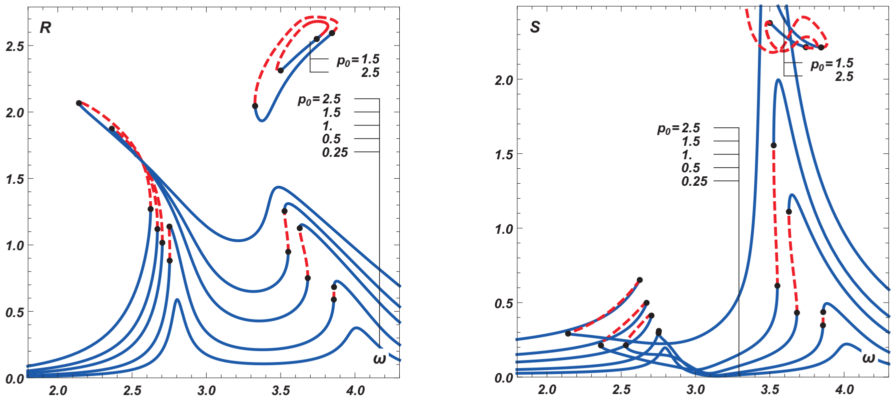


Figure 10: Non-linear resonance curves describing the stationary response of the system for excitation amplitudes $p_0 = 0.25, 0.5, 1, 1.5, 2.5$. Stable branches are shown as solid blue curves, unstable parts are indicated as the red dashed curves. Amplitudes $R = \sqrt{\alpha^2 + \beta^2}$ are shown in the left part of the figure, amplitudes $S = \sqrt{\gamma^2 + \delta^2}$ are on the right.

vertical dimension is relatively very small and can be used also in such cases where a pendulum absorber is inapplicable. First papers dealing with the theory and practical aspects of ball absorbers have been published during the last decade of the twentieth century, see [121] and [122].

Dynamics of this device can be hardly described in a linear state although for the first view its behavior is similar to the pendulum type absorber. A number of problems related with movement stability, bifurcations, auto-parametric resonances and at least but not last with dish and ball surface imperfections should be analyzed. The system should be considered as a non-holonomic strongly non-linear system. The system of governing equation follows from the Hamiltonian functional either using classical Lagrangian equations completed by non-holonomic constraints of contact ball - dish via Lagrangian multipliers or using the Gibbs-Appel procedure to obtain the governing differential system, for details see [29]. Some papers dealing with the basic mathematical background appeared recently, see e.g. [123], [124], [125] dealing with 2D as well as with the full 3D versions.

Two photos demonstrating the real shape of the device during tests in the dynamic laboratory are presented, see Figure 9. A certain outline of resonance curves are depicted in Figure 10. Red dashed or blue solid curves represent unstable or stable parts of resonance curves. Notice in the left picture a separated "island" of resonance curves. This effect can emerge on certain non-linear systems only. It represent a weakly stable process. This "island" can disappear easily when changing damping parameters.

4.4 Inverse pendulum related systems

Many studies devoted to dynamics of slender structures (towers, masts, chimneys, bridges, etc.) relating with earthquake attack have been published. They are dealing predominantly with an influence of horizontal excitation components. On the other hand a strong vertical excitation component especially in the earthquake epicentrum area can be decisive. It reveals that the origin of problems consists in auto-parametric resonance effects.

In sub-critical linear regime the vertical and horizontal response components are independent. If no horizontal excitation is taken into account, no horizontal response component is observed. The semi-trivial solution gives a full image of the structure behavior. If an amplitude of a vertical harmonic excitation in a structure foundation exceeds a certain limit, vertical response looses its Dynamic Stability and a dominant horizontal response component through the non-linear interaction of both component is generated. Therefore the system has a character of the auto-parametric system.

Hamiltonian theoretical model with three DOF is considered, see Figure 4b. Composing the Hamiltonian functional, following Lagrange equations, see [103], [126] can be formulated:

$$\begin{aligned} \ddot{\zeta} - \Upsilon_0(\dot{\psi}\varphi)^{\bullet} + \omega_0^2(\zeta - \zeta_0 + \eta_c(\dot{\zeta} - \dot{\zeta}_0)) &= 0, \qquad (a) \\ \ddot{\varphi} - \Upsilon_1(\dot{\zeta}\varphi)^{\bullet} + \Upsilon_1\ddot{\psi} + \Upsilon_1\dot{\zeta}\dot{\psi} - \Upsilon_1\omega_2^2\psi + \omega_1^2(\varphi + \eta_c\dot{\varphi}) &= 0, \qquad (b) \\ \ddot{\psi} - (\dot{\zeta}\varphi)^{\bullet} + \omega_3^2(\psi - \varphi + \eta_e(\dot{\psi} - \dot{\varphi})) &= 0, \qquad (c) \end{aligned} \tag{35}$$

where following notations and approximations have been used:

$$\begin{gathered} \zeta_0 = y_0/l,\ \zeta = y/l,\ \varphi,\ \psi = \varphi + \xi/l, \qquad \sin\varphi \approx \varphi\ , \quad \cos\varphi \approx 1\ , \\ \Upsilon_0 = \frac{m}{M+m},\ \Upsilon_1 = \frac{m \cdot l^2}{M \cdot \varrho^2},\ \omega_0^2 = \frac{C}{M+m},\ \omega_1^2 = \frac{C}{M},\ \omega_2^2 = \frac{g}{l},\ \omega_3^2 = \frac{6EJ}{m \cdot l^3}\ . \end{gathered} \tag{36}$$

The harmonic balance method (ω and $\omega/2$ sub-harmonics are considered) provides the semi-trivial solution:

$$\zeta_s = a_c \cdot \cos\omega t + a_s \cdot \sin\omega t\ , \quad \psi = 0\ , \quad \varphi = 0\ , \tag{37}$$

where symbols have following meaning:

$$a_c = -\frac{a_0\omega_0^2}{\delta}\omega^3\eta_c\ , \quad a_s = \frac{a_0\omega_0^2}{\delta}(\omega_0^2 - \omega^2 + \omega_0^2\omega^2\eta_c^2)\ , \quad \delta = (\omega^2 - \omega_0^2)^2 + \omega_0^4\omega^2\eta_c^2\ , \tag{38}$$

which can be seen in form of the amplitude $R_0^2 = a_c^2 + a_s^2$ in the Figure 11. Employing the linear level of the perturbation method, a system of two linear differential equations for perturbations of horizontal components p, s of $\varphi(t), \psi(t)$ can be carry out:

$$\begin{array}{llllll} \ddot{p} & +(\Upsilon_1\omega_3^2\eta_e + \omega_1^2\eta_c)\dot{p} & -\Upsilon_1(\omega_3^2\eta_e - \dot{\zeta}_s)\dot{s} & +(\Upsilon_1\omega_3^2 + \omega_1^2)p & -\Upsilon_1(\omega_2^2 + \omega_3^2)s & = 0,\ (a) \\ \ddot{s} & -(\omega_3^2\eta_e + \dot{\zeta}_s)\dot{p} & +\omega_3^2\eta_e\dot{s} & -(\omega_3^2 + \ddot{\zeta}_s)p & +\omega_3^2 s & = 0.\ (b) \end{array} \tag{39}$$

Three coefficients include harmonic components due to $\ddot{\zeta}_s, \dot{\zeta}_s$ terms being given by Eq. (37). The system Eq. (39) is of the Mathieu type and its solution stability should be verified, see for instance [89], [51], [60], [80].

Results of two typical examples have been plotted in the Figure 11. Resonance curves for a few excitation amplitudes a_0 are plotted in black while the stability limits following out from Eqs (39) are shown in red color. Points s_1, s_2 and s_3, s_4 in the left picture represent lower and upper limits of intervals where the semi-trivial solution becomes instable and post-critical

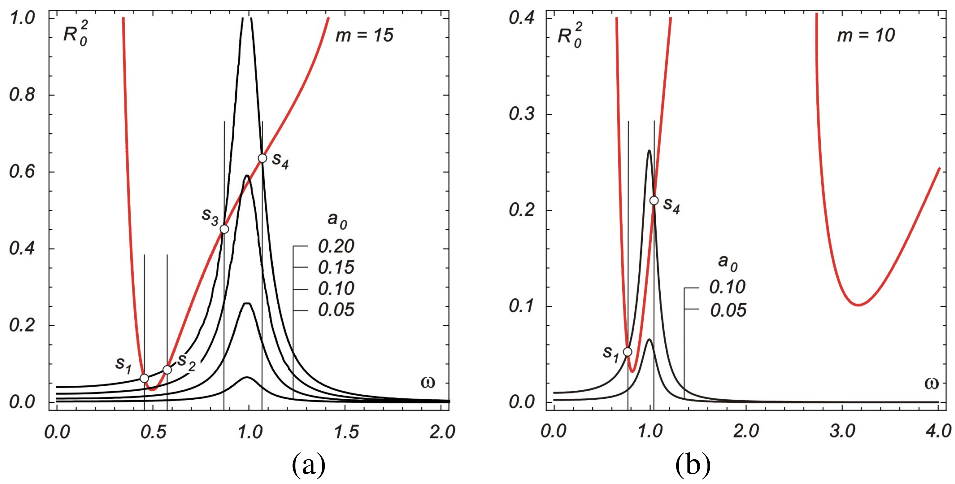


Figure 11: Relation of resonance curves (black) and stability limits (red); picture (a): $M = 3990, m = 15, \eta_c = 0.2, \eta_e = 0.2, l = 20, g = 9.81, EJ = 10000, C = 4000 > gml/r^2 = 2943$; $a_0 = 0.2 \Rightarrow s_1 = 0.4548, s_2 = 0.5718, s_3 = 0.8671, s_4 = 1.0680$; picture (b): $m = 10; C = 4000 > gml/r^2 = 1962$; $a_0 = 0.1 \Rightarrow s_1 = 0.7652, s_4 = 1.0396$.

response should be investigated. It is obvious that the most sensitive interval is in the domain of the basic eigen-value ω_0 as we can see in the left picture - interval $\omega \in (s_3, s_4)$ and also in the right picture - interval $\omega \in (s_1, s_4)$. However the stability limit indicates also another instability interval $\omega \in (s_1, s_2)$ in lower frequency range. This fact is related with a hypothetical "eigen frequency" in the component φ and represents a sub-harmonic resonance. The right picture demonstrates two separate parts of the particular stability limit.

This preliminary analysis shows that within resonance domains the semi-trivial response can remain in force although post-critical response type is rather typical. Three basic categories of the post-critical response can be noticed:

(i) *Response amplitudes are either weakly variable or constant.*
In such a case the system response can be studied by means of the harmonic balance method. It eliminates an oscillatory response character investigating (approximately) amplitudes only. To make a decision between single and multi harmonic approximation is a delicate matter. Anyway several verifications by means of simulation show that limitation on ω and $\omega/2$ sub-harmonics is satisfactory. Some more interesting experiences can be found in [115]. Should the internal resonance or its proximity is analyzed, valuable results have been obtained by authors of [127],

Going through the harmonic balance procedure a differential system for amplitudes: $\mathbf{X}(t) = [R_c(t), R_s(t), P_c(t), P_s(t), S_c(t), S_s(t)]^T$ can be obtained:

$$\mathbf{H}(\mathbf{X})\,\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{K}(\mathbf{X})\,; \qquad \mathbf{H}(\mathbf{X}) \in \mathbb{R}^{6,6},\;\; \mathbf{K}(\mathbf{X}) \in \mathbb{R}^{6}\,. \tag{40}$$

The detailed structure of the matrix $\mathbf{H}(\mathbf{X})$ and the vector $\mathbf{K}(\mathbf{X})$ is complicated. For detailed form, see [103].

The system Eq. (40) for amplitudes $\mathbf{X}(t)$ is meaningful if they are functions of a "slow time", in other words if their changes within one period $T = 2\pi/\omega$ are small or vanishing. In such a case two response regimes corresponding to harmonic vertical excitation can be encountered:

(i1) Weakly non-stationary response. The system Eq. (40) cannot be solved analytically and numerical procedure must be applied. As a feedback the variability rate of the $\mathbf{X}(t)$ components is to be checked. If some periods are comparable (or even shorter) with T, results should be rejected. In general, shapes of amplitudes $\mathbf{X}(t)$ are deterministic and periodic with possibly small perturbations.

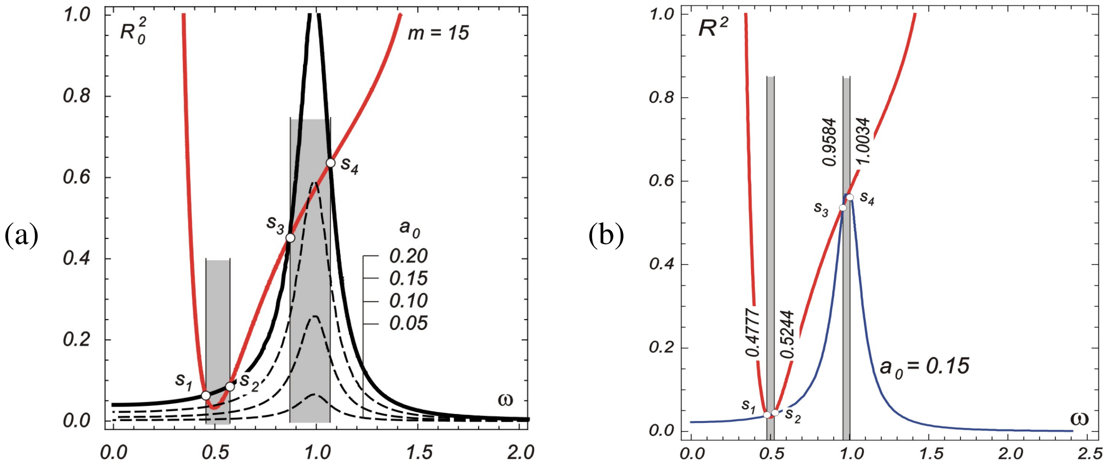


Figure 12: Instability intervals; picture (a): large excitation amplitude ($a_0 = 0.20$); picture (b) medium excitation amplitude ($a_0 = 0.15$).

(i2) Stationary response (special case of the previous one) The time derivative of the amplitudes $\mathbf{X}(t)$ vanishes and therefore the remaining non-linear algebraic system should be solved:

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} \equiv 0 \Rightarrow \quad \mathbf{K}(\mathbf{X}) = 0 \, ; \qquad \mathbf{K}, \; \mathbf{X} \text{ - constant fields .} \tag{41}$$

Both above cases can cover Limit Cycles which are typical for certain types of the post-critical behavior. System Eq. (40) can provide stable as well as unstable Limit Cycles, see e.g. [128].

(ii) *Response amplitudes are strongly variable.*
Provided that the response is of any other type than those in section 3.1., numerical integration of the original system (1.4) is necessary. The following most important regimes of the response can be observed:

(ii1) Transition processes: The systems are asymptotically approaching the stationary state due to sudden energy supply/loss, etc. Response shape is rather deterministic. For some aspects of transition processes and their asymptotic properties, see e.g. [129], [130], [131].

(ii2) Quasi periodic regime - energy periodic transflux between DOFs: Beating effects arose due to two or more DOFs interaction. A structure of one period can be complicated. However, the time history repeats and the length of individual periods fluctuates within very limited interval, e.g. [132].

(ii3) Chaotic regime: Its existence can be identified by means of Lyapunov exponent testing. For various technics based on the Lyapunov exponent see e.g. [77], [75], [119], [78], [81].

(iii) *Response amplitudes blow up.*
This case can be treated as the special case of the previous paragraph (ii). Due to unstable nature of the system under study, excitation which crosses certain limit leads to the collapse of the structure. However, it appears, that the structure can withstand a certain time limited non-stationary response.

Let us demonstrate the typical system behavior under excitation with a large and medium excitation amplitudes. Basic features of the response have been outlined in Figure 12. For detailed discussion including bifurcation and post-critical response analysis, see [126].

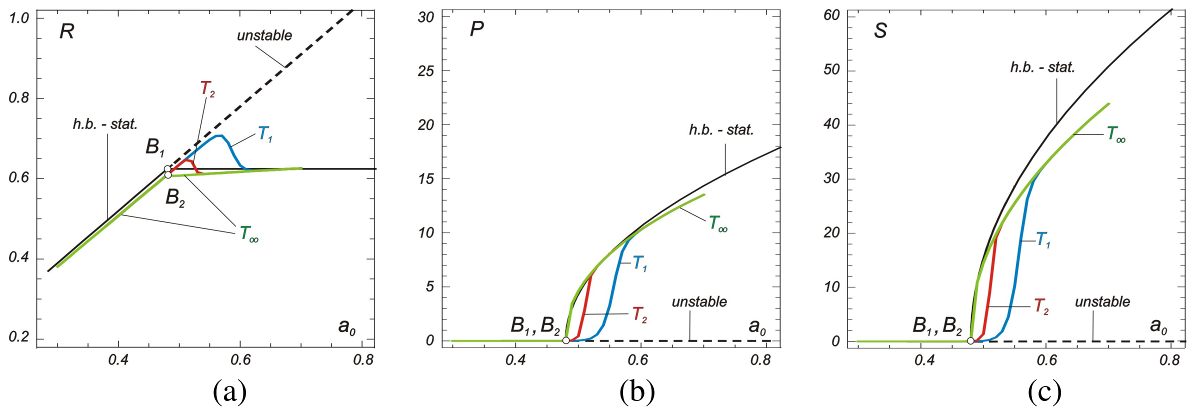


Figure 13: Process of the stability loss and of the post-critical response; picture (a) bifurcation diagram of the vertical response component ζ - amplitude R; pictures (b), (c): bifurcation diagrams of horizontal (rotation) components φ, ψ - amplitudes P, S.

The excitation with the amplitude $a_0 = 0.20$ can be considered as large providing the response amplitude plotted by a bold solid black curve, see Figure 12a. The system is unstable within intervals $\omega \in (s_1, s_2)$ and $\omega \in (s_3, s_4)$, see shadowed areas. It follows from detailed analysis that the system response depicted in the left part of the Figure finds out of critical limit. Consequently, this system should be rejected for reliability reasons.

The same system has been investigated for the excitation amplitude $a_0 = 0.15$. Frequency response curve of the semi-trivial solution intersects the red stability limit once again, see Figure 12b. Two instability intervals arise. However, their width became significantly smaller than in the previous case. Detailed evaluation shows, that the system recovery is probable. Therefore this case can be concerned acceptable, although the stability limit of the semi-trivial solution has been broken. The post-critical state enables to be investigated using differential or algebraic system Eqs (41) or (40) respectively. However some detailed properties need to be investigated using simulation. Observing time history of the response components, see Figure 12a, the transition effects should be analysed.

The detailed investigation of the transition process from the semi-trivial to post-critical response should be done for time limited excitation period. The reason is that the dangerous post-critical state needs a certain time since the excitation beginning to fully evolve. Provided that the excitation finishes within this time interval, the system recovery could be possible. Evolution of the response components ζ- vertical and φ, ψ- horizontal or their amplitudes R, P, S for frequency $\omega = 1.1700$ (roughly the middle of "instability interval") with increasing excitation amplitude a_0 is demonstrated in Figures 13a-c. Starting from bifurcation point B_1, two branches can be found. Dashed black curves represent unstable part corresponding to semi-trivial solution, while solid black curves (nearly horizontal for R and rising curvilinear for P, S) demonstrate stable post-critical solution.

Evaluating the system response for the same parameters and homogeneous initial conditions by means of several methods, nearly the same results are provided, when a long time elapsed $T \to \infty$ (green curves) and a steady state is concerned. If the period of excitation is limited, an effect of transition from the semi-trivial until post-critical stable state proceeds, see red and blue curves in Figures 13b,c. For short excitation period T_1 (blue) the excitation amplitude a_0 can significantly exceed the bifurcation point B_2 and the response has still the semi-trivial character. The response is approaching the steady state (black dashed curves). The medium time T_2 of excitation leads to an intermediate response character (red curves). It can be ascertained that for time limited excitations, e.g. earthquake attack, higher excitation amplitudes can be admissible than those obtained for the steady state vibrations.

From a practical point of view it should be remembered, that the post-critical regime only in

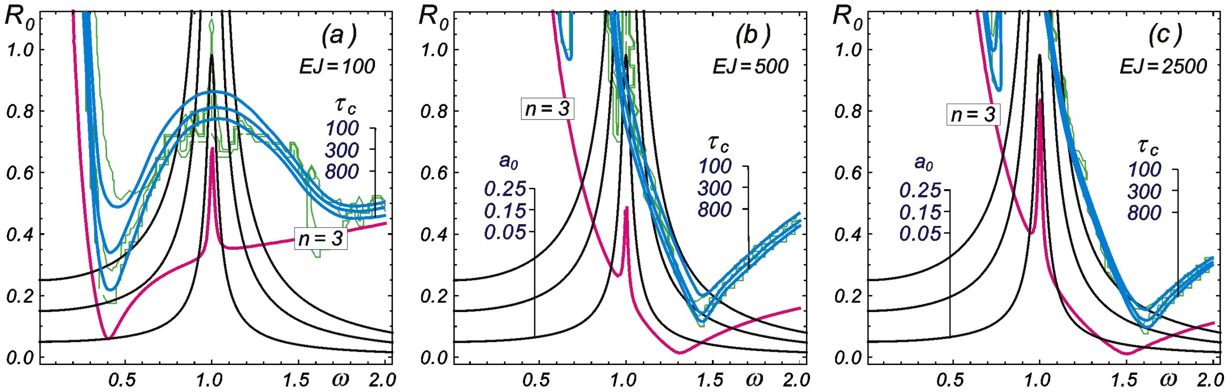


Figure 14: Outer stability limit (limit of irreversibility) of the system - blue curves for the increasing console bending stiffness; other parameters $\eta_e = 0,05$, $\varrho = 0,2$.

a very narrow area adjoined semi-trivial response stability limit is admissible before the ultimate critical limit or the limit of irreversibility is reached. Beyond that the final collapse of the system is inevitable. On the other hand some reserve can be regarded in a limited time of the seismic attack.

As it has been mentioned above, the post-critical regime can be of two types. Both of them are governed by the full differential system Eq. (35). The first type means a response process running within a certain limits around the semi-trivial solution. When the excitation is stopped, the system is able to recover and to return to a standstill. Overstepping the limit of irreversibility (or the outer stability limit) the second regime emerges leading to inevitable collapse of the system. The response becomes non-periodic rising exponentially beyond all limits. The mathematical model is not able any more to give a true picture of such terminal states. Its applicability finishes shortly after the limit of the irreversibility, despite it is able to determine its shape.

However to trace this limit the analytical investigation of the system Eq. (35) does not probably provide any understandable results. Therefore simulation processes should be undertaken in order to outline this limit. Numerical solution of the system Eq. (35) in full version has been multiply performed as long as the numerical process fails due to numerical stability loss. This collapse occurs in a certain time from the beginning of the integration, because the cumulative errors lose an ability to eliminate themselves. So that the moment when this state occurs indicate that the limit of irreversibility has been reached.

Some results have been plotted in Figure 14 for three console bending stiffness. The stability limit of the semi-trivial solution has been plotted (red curve). Blue curves represent limits of irreversibility, see pictures (a)-(c). Special problems emerged for low bending stiffness when the eigen-frequency ω_0 oversteps the first bending eigen-frequency of the console.

4.5 Means of transport

Dynamic Stability is a crucial aspect of transport means. Within this area the auto-parametric systems and their stability hold a decisive position. Although basic principles are identical and particular studies can be evolved from ideas outlined in the first parts of this study, individual categories of vehicles are very different from the viewpoint of sensitivity against the stability loss. Most of them consist of two or more interacting sub-systems with different motion velocity. This fact stresses some attributes of every category. However it leads every time to auto-parametric system stability problems and stability due to auto-oscillating effects.

Let us scan very briefly through the main categories of them. There should be referred auto-parametric stability of ships against the capsizing of several types, see first papers devoted to this problem [133], [134]. Later more sophisticated models have been introduced [135] and at last appeared studies focussed to non-linear coupling of pitch and roll modes [95], [136],

[137], [97]. See also monographs, e.g. [105] and many more. High speed ships and submarines represent a special category of auto-parametric and general systems stability.

Dynamic Stability related mostly with auto-parametric systems is decisive in automotive engineering. It concerns tire deformability and contact with road, stability of many other details and stability of a vehicle as a whole moving on a roadway. These interacting systems (vehicle - road) are modeled as complicated non-holonomic systems with non-linear links. Sources of their Dynamic Stability loss can be both auto-parametric resonance as well as auto-oscillating processes. For detailed information see monographs, e.g. [138], [139], [140]. Papers appear in journals of general orientation and problem oriented magazines, for instance [141], [142], [143].

Obviously stability of aircrafts and space vehicles would be a subject of several very special keynote lectures. Concerning literature, see some of numerous general monographs [144], [145], [146], [147], [148], [149], or computational mechanics oriented books like [150]. Hundreds of special papers which appear permanently in general but mostly in aeronautics oriented journals, e.g. [151], [152], [153], etc.

Regarding railway cars and systems, rich references can be evaluated as well. They are dealing with detailed problems, e.g. stability of contact rail-wheels as well as with stability of the train as a whole moving along the deformable track. Let us refer for example [141], [154] or a survey paper [155]. Special category in the railway transport stands the magnetic levitation system. A lot of special papers have been published dealing with stability of this system. Some review of the knowledge has been presented in [156].

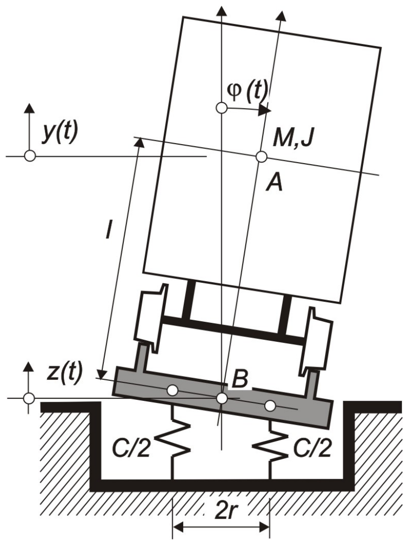


Figure 15: Cross-section of a railway car and a track.

In order to illustrate a bit the area of transport means we outline at least some aspects of high speed railway car stability on an elastic track, see Figure 15. The car moving along the railway track is subdued to effects of its deterministic and stochastic unevenness and deformability. On the contact of wheels and rail very complex processes are ruling. It implies strong vertical and horizontal excitation of the car and track. Of course, also other processes related with geometry, transverse-movement of the wheel set, dry friction in the contact, etc. All of them are necessary to be taken into account. Nevertheless one of the most important phenomena influencing a safe riding is the transverse stability. To express the core of this phenomenon, we simplify the model as much as possible.

The constant run speed of the car is considered. The perfect joining of the bogie and car case is supposed. The car is modeled as a stiff plate of mass M and eccentric inertia moment J with reference to point B. The plate has two degrees of freedom $y(t), \varphi(t)$. The point A is the center of gravity. Horizontal component of the point B movement is excluded. The track is understood as a beam with uniformly distributed mass and constant inertia moment along the axis. The track subsoil is of the Winkler type with the linear stiffness and internal viscosity. This subsoil model corresponds with that introduced into railway dynamics by Timoshenko and many other authors, although contemporary view is circumspect and rather model with continuously distributed mass is recommended. In the contact wheel - track kinematic excitation arise due to track unevenness, being characterized by processes $y(t), z(t)$. Remaining characteristics are obvious from the chart in Figure 15.

Interaction of two sub-systems arises: (i) car moving along the track axis (ii) standing track, which moves due its deformability in vertical direction only. The semi-trivial solution (or the stable state) can be expected until the moment, when the first horizontal response components emerge. As the post-critical response can rise due to auto-parametric resonance effects, then overcoming a certain critical limit, irreversible capsizing or derailment of the car can occur.

Taking into account the above simplifications, one can write out expressions for kinetic and potential energies of the sub-system (car) moving along the track axis. Completing the viscose damping by means of the Rayleigh function, following Lagrangian differential system can be carried out:

$$\begin{aligned} M\ddot{y} + b_y\dot{y} - Ml\cdot(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2) \;&=\; F - Mg \;, \\ J\ddot{\varphi} + b_\varphi\varphi - Ml\cdot(\ddot{y}\sin\varphi + \dot{y}\dot{\varphi}) - Ml\cdot\sin\varphi \;&=\; Q \;, \end{aligned} \tag{42}$$

where F, Q is the force and the moment acting in the contact wheel - track.

Car movement, Eq. (42), is described in coordinates x, t moving by velocity c_s with respect to standing coordinates x_s, t_s. Moving and standing coordinates are related by well known transformation: $(x_s = x + c_s\cdot t,\ t_s = t) \longleftrightarrow (x = x_s - c_s\cdot t_s,\ t = t_s)$.

The track is considered as Euler beam on a mass-less Winkler subsoil, horizontal displacement is neglected, torsional deformability is respected, see above. With reference to classical literature following differential system can be written:

$$\begin{aligned} EIz'''' + C_z\cdot z + \mu\cdot(c_s^2\cdot z'' + 2c_s\cdot\dot{z}_s' + \ddot{z}) + b_z(c_s\cdot z' + \dot{z}) \;&=\; -F\cdot\delta(0)\;, && (a) \\ GK\psi'' + C_z r^2\cdot\psi - \mu K(c_s^2\psi'' + 2c_s\cdot\dot{\psi}' + \ddot{\psi}) + b_\psi(c_s\cdot\cdot\psi' + \dot{\psi}) \;&=\; -Q\cdot\delta(0)\;, && (b) \\ y \;-\; z\cdot\delta(0) \;&=\; \alpha(c_s t)\;, && (c) \\ \varphi \;-\; \psi\cdot\delta(0) \;&=\; 0\;, && (d) \end{aligned} \tag{43}$$

where $z_s = z_s(x_s, t_s), \psi_s = \psi_s(x_s, t_s)$. Eqs (43c,d) express kinematic links coming from unevenness of the track structure, which are originally described by process $\alpha(x_s)$. Unevenness are introduced by means of Eq. (43c) being supposed identical at both rails.

Hence the governing system consists of two Eqs (42) and of four Eqs (43). It includes six unknown response components $y(t), \varphi(t), z(x,t), \psi(x,t), F(t), K(t)$ which are referred to moving coordinates.

Getting through system Eqs (42), (43), we learn that the primary system includes three response components $y(t), z(x,t), F(t)$. Together with trivial components $\varphi(t), \psi(x,t), K(t)$ it makes the semi-trivial solution. This status lasts until the stability limit is reached. Then a complicated process of transition into the post-critical response takes place. The post-critical state is worthy to be investigated until the irreversibility limit is reached and the final collapse of the system starts.

Let us take a note that the system investigated can lose the Dynamic Stability also due to non-symmetric character of operators Eqs (43a,b) containing non-conservative terms. Such types of critical speeds are related with propagation velocities of bending waves in the track resting on an elastic subsoil. This associated problem has been investigated separately for a long time very intensively using various mathematical models. It is very important to be aware that both groups of critical speeds can intermingle. Consequently, non-conservative character of the problem can lead to critical speeds which are lower than those following from auto-parametric properties of the system and vice versa. In the former case the decision will be made within the domain of semi-trivial solutions avoiding any horizontal movement of the car. The response will take place in the vertical plane only. In the latter case the mechanism of stability loss of any semi-trivial solution would be decisive. Post-critical solution will include large horizontal components.

5 SELF-EXCITED SYSTEMS

5.1 Moving systems with interaction

Self excited systems are in principle systems being able to absorb energy from ambient environment or due to interaction with other sub-systems. The absorbed energy is higher than energy loss with respect to internal damping and other reasons. The limit separating auto-parametric and self-excited systems is not sharp and many of them can appertain to both categories. A typical separating aspect of the self-excited system stands that sub-critical state doesn't mean necessarily existence of semi-trivial state. The whole system can exhibit non-trivial response in all components, but being strictly limited in a certain area. Overstepping a certain critical limit, response structure changes considerably representing either unacceptable state from the viewpoint of the system safety and reliability or on the contrary a necessary condition for its basic applicability.

There are several principles of self-excited vibration classified with respect to set-up of mathematical models. A couple of basic categories not sharply separated can be mentioned:

(i) SDOF as well as more complicated systems where the damping parameter (or parameters) are response velocity dependent being maximal in a standstill. Every deviation from the zero point leads to decrease of the effective damping level and can result in self-excitation. For instance positive: string musical instruments (see [157] and many references in this book), negative: some effects in conveyor belts, [158], [159], moving string [160] or in disc brakes [161].

(ii) Self-exciting effects within DDOF and more complicated systems due special structure of stiffness and damping terms. In case of linear systems non-conservative and gyroscopic terms can lead to overstepping of certain stability limits and post-critical self-excited response emerges. As examples can serve a majority of aero-elastic systems where well known effects can occur: flutter, divergence, galloping, etc. Nevertheless many aero-elastic effects cannot be modeled in the linear domain and non-linear approach is necessary to be adopted. Among many other papers see e.g. [162], [62], [163], [164], [165] and obviously monographs [107], [166], [167], [168], [169], see also the next sub-section. Many effects encountered in transport means can be included into this category: shimmy and many related effects, e.g. [170], [171], etc. Systems with following forces like various reactive propulsion systems, e.g. [172], [45]. A large area which belongs to this category represent problems of machine tools self-excitation. Successful suppression of those is a crucial condition for a high quality machining, see e.g. [173].

(iii) Vortex shedding. Very large and complex problems emerging in any scale. These phenomena occur due to stability loss in local area of dynamic pressure field in fluid when interacting with stiff or deformable bodies. A large number of various references exist as monographs, e.g. [43], papers, various laboratory reports, etc. Vortex shedding can be observed in micro-, macro- and giga-scale. For instance vortex shedding has been observed in streaming sea behind a circle-like shaped island, see [48]. Many variants of vortices are investigated with respect to particular conditions ruling in the stream (laminar, turbulent).

5.2 Linear non-symmetric systems

As a demonstration of a widely studied problems on scientific as well as engineering level, main features of an aero-elastic DDOF system is briefly presented. It concerns a model of a prismatic slender beam in a wind cross flow. From the viewpoint of the above classification, it concerns the type (ii). Engineering structures like bridge decks, high rise buildings, towers, chimneys and others can be prone to reach a stability limit due to certain wind speed. Beyond that the structure loses its Dynamic Stability with a high probability of a collapse. Because it is strictly forbidden to draw near this limit, the post-critical states don't need to be examined and the detect the limit itself and structure behavior on it looks to be satisfactory (similarly like aircraft wings). Therefore a linear approach is fitting in such a case.

The vibration of a prismatic sections in the cross flow is often result of interaction of the response and the wind forces varying in time. This interaction causes the non-conservative and gyroscopic character of the vibration. The wind engineering community describes several types of aero-elastic oscillation using technical language as: flutter of the bridge deck, rotational galloping, galloping of cables or vortex induced vibration, see e.g. papers [174], [175], [176], [177], [178] or monographs e.g. [107], [166], [167], [168], [169] and many others.

Two parallel ways of linear aero-elastic problem, each of them bringing some advantages, can be formulated. This duality of time and frequency domain formulation of the self-excited wind forces was and still is investigated, see e.g. [179]. In the first approach, applying direct and indirect aero-elastic derivatives, see e.g. [180] and [181], "combined time/frequency" system has to be solved by means of iteration to determine directly critical stream velocities and relevant frequency. In the second approach, in time domain formulation of self-excited forces on a bridge deck the indicial functions are adopted, see e.g. [175], [182], [183], and [184]. The majority of these models have a linear character (explicit or hidden) being based on various types of convolution or other superposition related principles.

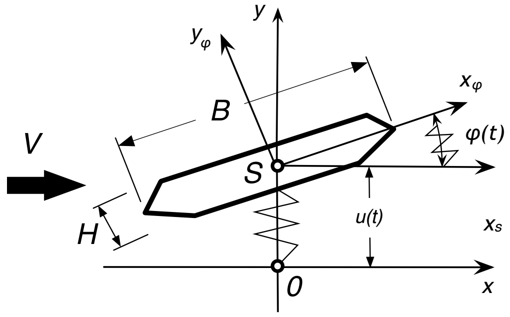


Figure 16: Schematic DDOF model of a bridge girder under wind loading; bridge girder is axially symmetric or almost symmetric with possible response in heave u (vertical direction) and pitch φ (rotation around S point).

Authors of [178] tried to formulate a general linear model being based on principles of the Rational Dynamics integrating all above phenomena which later follow as special cases from an analysis of the general model. Effects of gyroscopic and non-conservative forces of aero-elastic origin are qualitatively studied using the linearized DDOF model as in Figure 16. In this way vibrations of a beam are considered as vibration of a beam in heave and pitch mode. Skipping all details, see [178], following linear governing system can be formulated:

$$\begin{aligned} \ddot{u} + b_m \cdot \dot{u} - hq \cdot \dot{\varphi} + \omega_u^2 \cdot u - p \cdot \varphi \ &= \ 0 \ , \\ \ddot{\varphi} + q \cdot \dot{u} + b_I \cdot \dot{\varphi} + gp \cdot u + \omega_\varphi^2 \cdot \varphi \ &= \ 0 \ , \end{aligned} \tag{44}$$

where u, φ are motion components and remaining symbols represent system parameters of obvious meaning. The characteristic equation can be carried out from the system Eq. (44):

$$\begin{aligned} D = \lambda^4 + \lambda^3(b_m + b_I) + \lambda^2(\omega_u^2 + \omega_\varphi^2 + b_m b_I + hq^2) + \\ + \lambda(\omega_u^2 b_I + \omega_\varphi^2 b_m + (1 + gh)pq) + \omega_u^2\omega_\varphi^2 + gp^2 \ = \ 0 \ . \end{aligned} \tag{45}$$

The resulting characteristic equation represents a polynomial with roots $\lambda_1, \lambda_2, \lambda_3$ and λ_4. The trivial solution of system Eq. (44) is stable only if the real part of all four roots is negative. Using Routh-Hurwitz determinants and Descartes theorem see section 2.3, the set of conditions reads:

$$\begin{aligned} &\alpha_1 = b_m + b_I > 0 \ , && \text{(a)} \\ &\alpha_2 = \omega_u^2 + \omega_\varphi^2 + b_m b_I + hq^2 > 0 \ , && \text{(b)} \\ &\alpha_3 = \omega_u^2 b_I + \omega_\varphi^2 b_m + (1 + gh)pq > 0 \ , && \text{(c)} \\ &\alpha_4 = \omega_u^2\omega_\varphi^2 + gp^2 > 0 \ , && \text{(d)} \\ &\alpha_5 = \alpha_1\alpha_2\alpha_3 - \alpha_3^2 - \alpha_1^2\alpha_4 > 0 \ . && \text{(e)} \end{aligned} \tag{46}$$

Stability conditions Eqs (46) are schematically outlined in Figure 17, in the plane $\omega_u^2 \times \omega_\varphi^2$. Looking through the picture one can see the influence of individual parameters on the stability of the basic system while creating limits identifying the change in stability character. Conditions $\alpha_i \geq 0$, treated usually separately in the literature, have now gained general meaning. Stability conditions may intersect mutually and thus create separated instability domains in which individual generalized forces need not be necessarily positive due to non-conservative and gyroscopic influences. Therefore traditionally discussed types of Dynamic Stability loss appear here as special cases of one general mechanism treated above. So that going through one can detect all types of instability dealing with linear model.

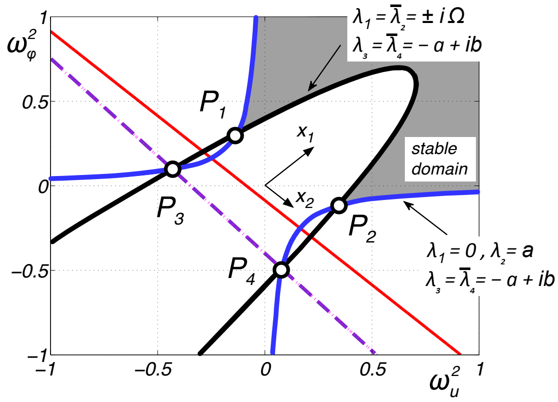


Figure 17: The stability conditions in the plane $\omega_u^2 \times \omega_\varphi^2$ with the intersection points $P_1 - P_4$ of particular interest; α_2–red line, α_3–magenta dashed line, α_4–blue hyperbola, α_5–black parabola; parameter a in the figure is always positive indicating decaying transition process no influencing the steady state response.

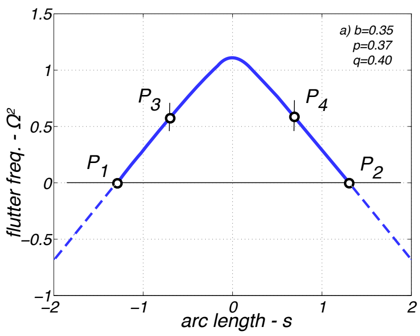
For instance crossing over the parabolic limit the solution of Eq. (45) yields two pairs of complex conjugate roots:

$$\lambda_{1,2} = \pm \mathrm{i}\Omega\,, \qquad \lambda_{3,4} = \frac{1}{2}\left(-(b_m + b_I) \pm 2\mathrm{i}\sqrt{\frac{\omega_u^2\omega_\varphi^2 + gp^2}{\Omega^2}}\right)\,. \tag{47}$$

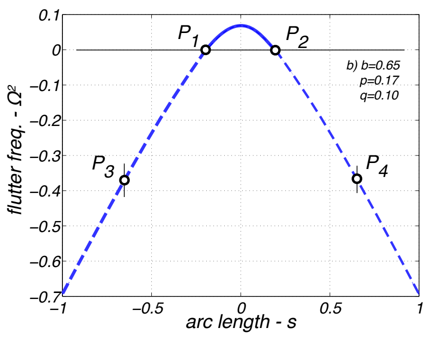
We can see that the Hopf bifurcation occurs and the flutter of the frequency Ω emerges:

$$\Omega^2 = \frac{\omega_u^2 b_I + \omega_\varphi^2 b_m + (1 + gh)qp}{b_m + b_I}\,. \tag{48}$$

Real part of the roots $\lambda_{3,4}$ on the parabola are negative. Therefore the amplitude of corresponding solutions with $t \to \infty$ vanishes and only non-damped response being given by the Hopf bifurcation remains non-trivial. Let us note that moving along the parabola, the flutter frequency and phase shift of the response component changes as it is depicted in Figures 18 and 19. The maximum of the frequency Ω occurs when apex of the parabola is reached. Due to parameter variability the basic configuration of the stability diagram can change considerably

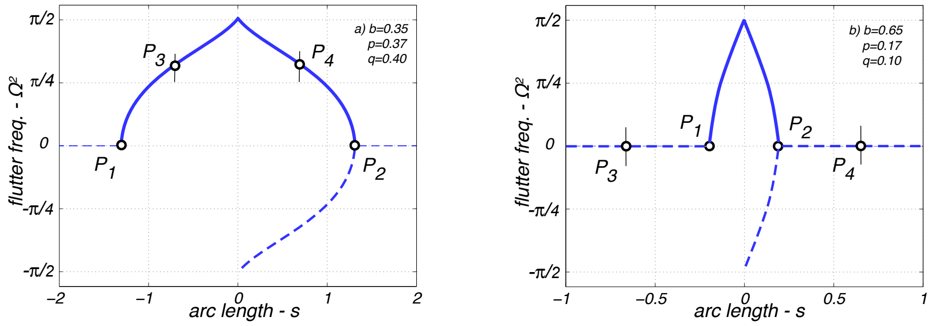


(a)



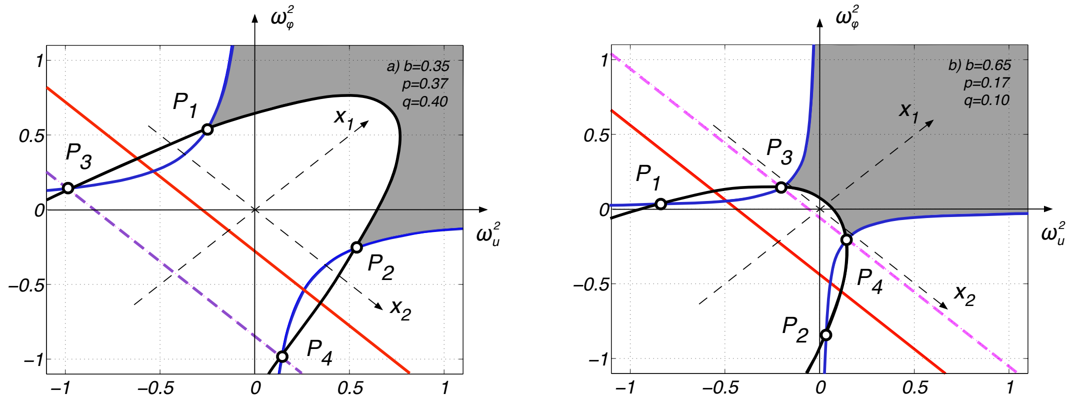
(b)

Figure 18: The flutter frequency Ω^2 as a function of the arc length s on the parabolic condition; the origin ($s = 0$) is on the vertex of the parabola; (a) sub-critical damping b; (b) super-critical damping b.



(a) (b)

Figure 19: The phase shift θ as a function of the arc length s on the parabolic condition; the origin ($s = 0$) is on the vertex of the parabola; (a) sub-critical damping b; (b) super-critical damping b.



(a) (b)

Figure 20: Relation of individual stability limits and points $P_1 - P_4$ position; (a) sub-critical damping b; (b) super-critical damping b.

and sub-critical and super-critical damping in the flutter meaning can be observed, see Figure 20.

Similarly the hyperbolic limit characterizes the divergence. Combinations of individual system parameters can provide a large range of general and special types of the stability loss with significant variability of the response parameters on the stability limit. Also intersection points of the parabola and hyperbola $(P_1 - P_4)$ are worthy to be discussed separately providing valuable information concerning interaction of flutter and divergence under special conditions.

6 CONCLUSIONS

The aim of the Dynamic Stability of engineering systems, its place in the Rational Mechanics and in engineering area is outlined. A short overview of selected problems, methods and results are presented in the study. Relation with other branches outside of civil and mechanical engineering are remembered, in particular with physics, chemistry, control and traffic engineering and other branches. Famous persons who founded, discovered the main principles and proved the main theorems are remembered with respect. The most frequently applied methods are mentioned together with some possibilities of their application in research and engineering application. Lyapunov second method is particularly stressed and some possibilities of the Lyapunov function construction are enumerated on the level of deterministic as well as stochastic approaches. Other popular methods are reminded as well. Domains of Limit Cycle stability including their possible attractive or repulsing character are also outlined together with a simple example of application in engineering. Similarly auto-parametric systems as one the most important area within the non-linear dynamic systems are introduced. Their problems with semi-trivial solution stability and post-critical response types are discussed along with quasi-periodic and other cyclic-stationary response types. Special attention is paid to large number of resonance domain regimes. Also self-excited systems are outlined together with frequently encountered applications in engineering.

Dynamic Stability is a domain attracting researchers and top engineers many decades. Many interesting results have been achieved, which shifted the level of contemporary knowledge in the basic research not only in mechanics, but also in other branches which accepted many of methods and procedures discovered in mechanics. Moreover many important problems in practical engineering have been solved supporting safety and reliability of structures.

Nevertheless many open problems still exist. Although analytic approaches are still powerful, limits of their practical applicability are even more obvious. These limitations should be solved in cooperation with numerical procedures. This step, however, opened a new category of problems related primarily with the basic new concept of solution strategy. Always some question exists if completely new methods independent on existing analytical modeling should be invented or to combine analytical approaches with powerful computational technology. The decision is not easy because strength and weakness of both should be understood in connection with particular problems. Therefore both categories will always co-exist complementing each other.

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