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MATHEMATICAL MODELLING OF A ROTATING NONLINEAR FLEXIBLE BEAM-LIKE WING

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Abstract. *The mathematical modelling of rotating nonlinear flexible beam-like wing with rectangular cross section is investigated here. The structure is mathematically modeled considering linear curvature and clamped-free boundary conditions. The flexible wing has an angle of attack which is considered constant during the numerical simulations. Nonlinearities resulting from the coupling between the angular velocity of the rotating axis and the transversal vibration of the beam are considered. A drag force and a lift force acting along the beam length are also included in the mathematical model. The drag force is modeled as a turbulent drag effect. The lift force is modeled as a potential derived force. These forces are velocity dependent nonlinear excitations acting on the wing.*

1 INTRODUCTION

Investigations about the dynamic behavior and control of nonlinear flexible beam-like structures represent an area of research of continuing interest to researchers and scientists around the world due to the wide range of application of such structures in different areas, such as aerospace and oceanic engineering. The objectives of these studies are in general the design of more lightweight and faster structures for oceanic, aerospace and robotics applications. Some of these applications are: lightweight robotic manipulators, solar panels or antennas in satellites, helicopter blades and the ISS. In fact, in an increasingly competitive world, new demands are constantly created on stability, dynamics and control techniques.

Low inherent damping, small natural frequencies, and extreme light weights are among the common characteristics of these systems which make them vulnerable to any external or internal disturbances (such as slewing maneuvers with great velocities, impacts, fluid interaction, etc.). Robotic manipulators with such characteristics are easy to carry out, need smaller actuators and can reach objectives in a greater workspace since they are thinner and longer than the rigid ones usually used for the same task.

In dealing with these kind of rotating structures, the interaction between the angular displacement, θ, also called slewing angle, and the flexible structure deflection variable, $v(x,t)$, can be very important in some cases, as in high angular speed maneuvers [1-3]. The inclusion of the drag and lift effects, in this case, incorporates, although in a simple manner, the interaction between the structure and the surrounding fluid as, for example, the air or the water. This interaction plays an important role as the fluid dissipates the motion kinetic energy and executes work on the system that can significantly alter the control performance and efficiency.

2 THE GEOMETRIC MODEL

The geometric model of the dynamic system investigated in this work is presented in Figure 1.

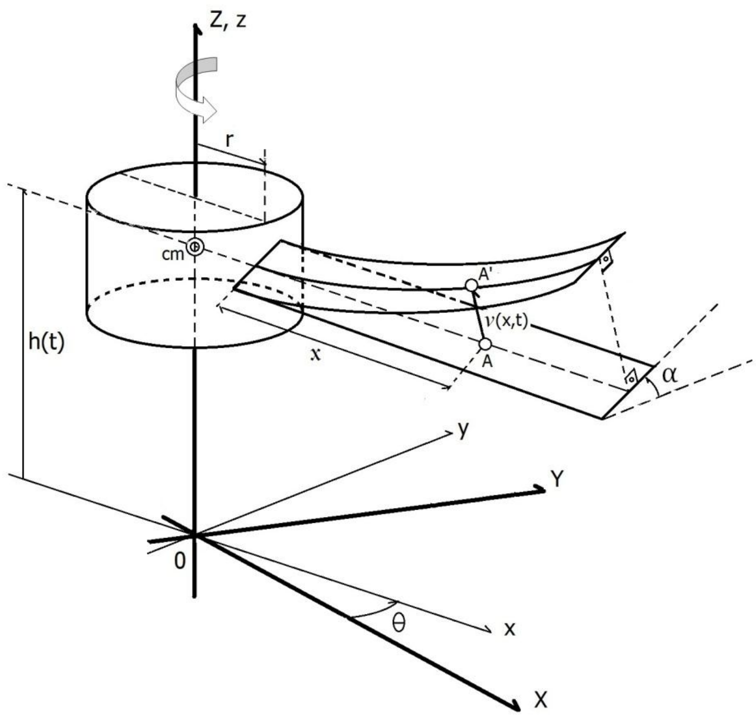

Figure 1. The rotating flexible beam system interacting with a fluid.

This system comprises a rigid hub and a flexible beam-like structure in rotation about Z axis and interacting with a fluid. In this system, the fluid effect on the motion is represented by the drag force, $F_{drag}(x, t)$ and lift L(x,t). These effects are not represented in the figure.

The drag and lift forces, as considered in this work, are functions of the velocities v(x, t) and θ(t). The lift is also a function of the velocity *h*(t). The model for the lift uses strip theory, and no tridimensional effects are included.

In this figure, the inertial axis is represented by XYZ, the moving axis (attached to the rotating axis and moving with it) is represented by xyz.

3 THE MATHEMATICAL MODEL

The governing equations of motion are obtained through the energy method [4-8]. To apply this method one needs to know the kinetic, potential and strain energies stored in the hub and in the flexible structure during the time evolution.

The total kinetic energy, T, of the system is given by:

$$T = \frac{1}{2} m_{hub} \dot{h}^2 + \frac{1}{2} I_{hub} \dot{\theta}^2 + \frac{1}{2} \int_0^{\ell} \rho A \Big[\dot{\theta}^2 (\mathrm{r}+\mathrm{x})^2 - 2\dot{\theta}(\mathrm{r}+\mathrm{x}) \dot{v} \sin\alpha + \dot{v}^2 + $$
$$+ v^2 \sin^2\alpha \dot{\theta}^2 + 2\dot{v}\cos\alpha \dot{h} + \dot{h}^2 \Big] dx \tag{1}$$

In Equation (1), θ represents the angular displacement of the hub axis, *h* represents the displacement of the hub, *r* represents the radius of the hub, α represents the angle of attack of the beam, *v*(x, t) represents the transversal displacement of the beam, ρ represents the density of the material that composes the beam, *A* represents the beam cross section area and *l* represents the non deflected length of the beam.

In this model, linear curvature is assumed in the modeling of the flexible structure [9-11]. The total potential energy, *V*, of the hub and of the rotating beam is given by:

$$V = m_{hub} g h + \frac{1}{2} \int_0^{\ell} \rho A g (h + v\cos\alpha) dx + \frac{1}{2} \int_0^{\ell} EI v''^2 dx \tag{2}$$

where *E* represents the Young's modulus of the material that composes the beam, and *I* represents the moment of inertia of the cross-section area of the beam.

The lagrangian, *L*, therefore, is given by:

$$\mathrm{L} = \mathrm{T} - \mathrm{V} \tag{3}$$

or, substituting Eq.(1) and Eq.(2) into Eq.(3):

$$L = \frac{1}{2} m_{hub} \dot{h}^2 + \frac{1}{2} I_{hub} \dot{\theta}^2 - m_{hub} g h + \frac{1}{2} \int_0^{\ell} \rho A \Big[\dot{\theta}^2 (\mathrm{r}+\mathrm{x})^2 - 2\dot{\theta}(\mathrm{r}+\mathrm{x}) \dot{v} \sin\alpha + \dot{v}^2 + $$
$$v^2 \sin^2\alpha \dot{\theta}^2 + 2\dot{v}\cos\alpha \dot{h} + \dot{h}^2 - g(h + v\cos\alpha) - \frac{EI}{\rho A} v''^2 \Big] dx \tag{4}$$

The drag effect is inserted in Lagrange's equations through the Rayleigh dissipation function, R, given by:

$$D_{section} = \frac{1}{2}\, kU^2 \tag{5}$$

In Eq. (5), the proportionality constant k is given by:

$$k = \rho A C_D \tag{6}$$

with ρ representing the density of the fluid, A representing the area of the orthographic projection of the beam and C_D representing the nondimensional drag coefficient.

In fact, the correct value of the constant k depends on the form, dimensions and attitude of the body. One can expect to derive an analytical closed form for the flow over the structure only in the case of creeping flow, what is certainly not our case. So, one adopts the representation given by Eq. (6), of the constant k, in order to have an estimative for the drag force, based on the fact that the Reynolds number (Re) of the flow, induced by the movement of the slewing structure, will present Reynolds number values from $0 < Re < 10^3$, so that one considers here the dissipative forces being proportional to the velocity.

The drag coefficient, C_D, is calculated, approximately, through the empirical relation [12]:

$$C_D = 1.1 + 0.02[h/w + w/h]$$

where h, is the height and w is the width of the beam-like structure, for the linear velocity case. It should be noted that the expression is valid in the range $1/30 < h/w < 30$.

In the same manner one can also include the force component normal to the flow, called lift. The lift force is introduced, also, as a generalized force.

Let a cross section of the beam be at a distance x from the hub. The lift generated by this beam section is given by

$$L_{section} = \frac{1}{2}\rho U^2 c C_L(\beta) \tag{7}$$

where c is the beam cross sectional width (mean chord), U is the modulus of the fluid velocity as seen by the beam cross section and C_L is the lift coefficient for the beam cross section. U^2 is given by:

$$U^2 = [\dot{\theta}\,(r + x) - \dot{v}\,\sin\alpha]^2 + [\dot{v}\,\cos\alpha + \dot{h}]^2 \tag{8}$$

and the angle β is the angle of attack seen by the beam cross section. It is given by

$$\beta(x,t) = \alpha - \tan^{-1}\left(\frac{\dot{v}\cos\alpha + \dot{h}}{\dot{\theta}\,(r+x) - \dot{v}\sin\alpha}\right) \tag{9}$$

which varies with the cross section position x along the beam position and time. One assumes the horizontal velocity much larger than the vertical velocity ($\left|\dot{\theta}\,(r + x) - \dot{v}\,\sin\alpha\right| \gg |\dot{v}\,\cos\alpha + \dot{h}|$). Then, the angle of attack is $\beta = \alpha$ and the sectional lift is in the vertical direction and the sectional drag is in the horizontal direction. This assumption simplifies the model.

The virtual work generated by the cross sectional lift force and drag force is

$$\delta W_{section} = \{\vec{L}_{section} + \vec{D}_{section}\}.\{[(r+x)\delta\theta - \delta v\ \sin\alpha]\vec{j} + [\delta v\ \cos\alpha + \delta h]\vec{k}\} \tag{10}$$

With the assumption in the above paragraph, the sectional lift force is in the *z* direction, while the drag force is in *y* direction so that the virtual force due to the sectional lift and drag forces is given by:

$$\delta W_{section} = \frac{1}{2}\rho\, U^2 c\, C_L(\alpha)\,[\delta v\ \cos\alpha + \delta h] + \frac{1}{2}\, k\, U^2[(r+x)\delta\theta - \delta v\ \sin\alpha] \tag{11}$$

In order to obtain the total virtual work due to the lift and drag force along the beam, one needs to integrate the above expression along the beam. One obtains:

$$\delta W = \frac{1}{2}\rho c \int_0^L \{[\dot{\theta}(r+x) - \dot{v}\sin\alpha]^2 + [\dot{v}\cos\alpha + \dot{h}]^2\}\, C_L(\alpha)[\delta v\cos\alpha + \delta h]dx + \frac{1}{2}\, k\int_0^L \{[\dot{\theta}(r+x) - \dot{v}\sin\alpha]^2 + [\dot{v}\cos\alpha + \dot{h}]^2\}\,[(r+x)\delta\theta - \delta v\sin\alpha]dx. \tag{12}$$

Notice that $C_L(x,t)$ is written as a function of *x* and *t* since $\beta = \beta(x,t)$. It is easy to identify the generalized force associated with the virtual displacement δh. With respect to the virtual displacement δv one obtains a generalized force per unit of length. The virtual work due to the lift force is written as:

$$\delta W = F_h\delta h + \int_0^L f_v \delta v\, dx + F_D\delta\theta \tag{13}$$

where F_h ,f_v and F_D are given by:

$$F_h = \frac{1}{2}\rho c \int_0^L \{[\dot{\theta}(r+x) - \dot{v}\sin\alpha]^2 + [\dot{v}\cos\alpha + \dot{h}]^2\}\, C_L(\alpha)dx \tag{14}$$

$$f_v = \frac{1}{2}\rho c\{[\dot{\theta}(r+x) - \dot{v}\sin\alpha]^2 + [\dot{v}\cos\alpha + \dot{h}]^2\}\,[C_L(\alpha)\cos\alpha - \frac{1}{2}k\sin\alpha] \tag{15}$$

$$F_D = \frac{1}{2}k\int_0^L \{[\dot{\theta}(r+x) - \dot{v}\sin\alpha]^2 + [\dot{v}\cos\alpha + \dot{h}]^2\}\,[(r+x)]dx \tag{16}$$

The extended Hamilton Principle is given by:

$$\int_{t_1}^{t_2}\left[\delta W_e + \delta L\right]dt = 0 \tag{17}$$

where W_e is the work done by external loads on the bodies, and t_1, t_2 the initial and final times.

Substituting (4) and (13) into (16) results the governing equations of motion for the variables h(t), θ (t) and *v*(x, t) respectively given by:

$$(m_{hub} + \rho AL)\ddot{h} + \rho A\cos\alpha\int_0^L \ddot{v}dx + \rho AgL - m_{hub}g = F_h \tag{18}$$

$$\left(I_{hub} + \rho A\int_0^L (r+x)^2 dx + \rho A\sin^2\alpha\int_0^L v^2 dx \right)\ddot{\theta} - \rho A\sin\alpha\int_0^L (r+x)\ddot{v}dx + 2\rho A\sin^2\alpha\dot{\theta}\int_0^L v\dot{v}dx = F_D \tag{19}$$

$$\ddot{v} + (r+x)\sin\alpha\ddot{\theta} + \cos\alpha\ddot{h} - v\sin^2\alpha\dot{\theta}^2 + \rho Ag\cos\alpha + \frac{EI}{\rho A}v^{iv} = f_v \tag{20}$$

The boundary conditions for the beam are given by:

$$v(0,t) = 0 \tag{21}$$

$$v'(0,t) = 0 \tag{22}$$

$$v''(L,t) = 0 \tag{23}$$

$$v'''(L,t) = 0 \tag{24}$$

4 CONCLUSIONS

A mathematical model consisting of a flexible beam like wing under the influence of aerodynamic drag and lift forces was derived through the energy method yielding a deterministic system of coupled nonlinear equations for the description of its dynamics.

This system of equations, although carrying simplifying assumptions, contains the fundamental physical processes of the flexible beam dynamics, considering its interaction with the surrounding fluid, where drag and lift forces are modeled as potential derived forces. Is our purpose to develop a fluid structure interaction model that contains the fundamental physical characteristics of the problem, but, at the same time, can be solved using simple numerical techniques

Our next step is to implement a State Dependent Riccati Equation (SDRE) nonlinear control method in order to eliminate the vibrations on the structure while controlling its altitude position and angular velocity.

One expects that such models can help to develop efficient, physical based, control strategies for rotary wing and fixed wing vibration, as also for the underwater robotic platforms.

REFERENCES

[1] Telli, S., Kopmaz, O., “On The Mathematical Modelling Of Beams Rotating About a Fixed Axis”, Mathematical and Computational Applications, Vol. 9, No. 3, pp. 333-347, 2004.

[2] Low, K. H., “Vibration Analysis of a Tip-Loaded Beam Attached to a Rotating Joint”, Computers and Structures, vol. 52, no. 5, pp 955-968, 1994.

[3] Putter, S., Manor, H., “Natural Frequencies of Radial Rotating Beams”, Journal of Sound and Vibrations”, 56(2), pp 175-185, 1978.

[4] Meirovitch, L., “Analytical Methods in Vibrations”, Macmillan Publishing Co., Inc., New York, 1967.

[5] Greenwood, D. T., “Classical Dynamics”, Dover Publications, Inc., New York, 1977.

[6] Fox, C., “An Introduction to the Calculus of Variations”, Dover Publications, Inc., New York, 1987.

[7] Thomson, W. T., “Theory of Vibration with Applications”, Prentice Hall, New Jersey, 1988.

[8] Craig, Jr., R. R., “Structural Dynamics - An Introduction to Computer Methods", John Wiley and Sons, 1981.

[9] Fenili A., “Mathematical Modeling and Analysis of the Ideal and Nonideal Behavior of Slewing Flexible Structures”, Ph.D Thesis, University of Campinas (UNICAMP), Faculty of Mechanical Engineering. Brazil, 2000. In Portuguese.

[10] Popov, E. P., "Introdução à Mecânica dos Sólidos", Editora Edgar Blücher Ltda, 1978.

[11] Sah, J. J., Mayne, R. W., "Modeling of a Slewing Motor-Beam System", Proceedings of the International Computers in Engineering Conference, Boston, pp 481-486, 1990.

[12] Holmes, J., “Wind loading of Structures”, Spon Press, New York, 2003.