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APPLICATIONS OF STATISTICAL DYNAMICS METHODS IN "LIGHTWEIGHT" STRUCTURES DESIGN

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Abstract. *One of the aircraft design problems is to establish simple and convenient techniques of finding rational primary structures. These tools are especially important at the design stage, while there are not yet exact data about the stiffness of the structure and environment loads but you should choose the best option of the structure layout among many possible ones. The random response of the spacecraft dynamic model has been studied using the methods of statistical dynamics. The regularities obtained enable to define criterion for selecting the more acceptable variant of the structure exposed to dynamic environment. The simple and convenient technique bases on estimates not widely used in the practice of design and allows to developer to identify quickly the most reasonable option among several alternatives. Moreover the full and expensive calculation of loads and stresses in structural elements is not required.*

1 INTRODUCTION

For the initial preliminary design of the aircraft and spacecraft structure, one of the most important cases of loading condition is the flight in turbulent air [1] which consists of two streams: a steady horizontal wind and occasional gusts. The reaction of the aircraft on the action of a steady wind is a special case of structural response to the discrete impulse. The levels of shear forces, bending moments, angles of attack and transverse accelerations during the flight in turbulent air depend on many factors [1]: the state of the atmosphere, the trajectory parameters, inertial and dynamic characteristics of the aircraft, parameters and the type of control system, aerodynamic properties of aircraft and etc. To estimate the structural response to the exposure of turbulence, the dynamic system transfer functions should be obtained in response to the input signal, which is a unitary sine gust. There are two approaches to solving the problem of determining the bearing capacity of structures during the passage of the area of maximum dynamic pressure. The first one is based on the actual wind profiles for the launching site and uses the numerical integration of the complete system of differential equations of motion. The second one is to use the normalized profiles of the wind flows, i.e. using the normalization of the required bearing capacity of the structure. When calculating the structural response to external influence normalized, bending moments and shear forces are represented as the sum of static components, which are caused by the action of steady wind, and of dynamic components, which are caused by the deviation of the wind speed from the mean.

To predict the values of the external influence functions by earlier stage of an aircraft development is a complex task involved with many issues of mechanics of motion, aerodynamics, dynamics, structure, and even meteorology. The design of an aircraft and its dynamic characteristics has a substantial influence upon the values of the loads [2]. Therefore, the levels of dynamic loads can be adjusted by changing the structural lay-out or load path within certain limits, i.e. in principle, a reduction of the structure weight can be received by reducing the bearing capacity of its parts. As the structural layout of any aircraft depends on its bearing capacity in itself, a more efficient structure design may be found by successive approximations. It is important to choose the most efficient layout by earlier stage of the development in order to avoid as far as possible design improvements in the later stages of development, and thus reduce development time and overhead costs of the project as a whole. The difficulty about the problem is to solve it based on preliminary models and loads, which repeatedly can be corrected later. Fortunately, there are factors that can be taken into account early in the development of design when assessing the options under consideration. One of these is discussed in the article.

2 THE ELASTIC MODEL RESPONSE FOR STOCHASTIC EFFECTS

2.1 Stochastic properties of turbulent atmosphere

For flight in the atmosphere, one of the main external factors is the turbulent air, i.e. atmospheric turbulence. As it is a random and ergodic process, the calculation of its stochastic characteristics can be computed using temporal averaging instead of ensemble averaging, if the realization is quite large of time. The average values are calculated using formulas of the theory of random processes [3].

The constant component of the wind vector includes long-haul ordered flows of air and it can be taken into account in the equations of motion. The variable component is a random function of the coordinates of the selected point of the atmosphere and time. Because of the random projections of wind are quite a little in comparison with the speed of movement velocity, the field of turbulent gusts can be reduced to a frozen, changeless in space, model. Due to this assumption, the spatial frequency can be used instead of the temporal angular frequency when describing frequencies of atmospheric turbulence. The advantage of this approach is that the velocity field of wind gusts can be determined with spatial coordinates. Due to this, the analytical expressions describing these fields are suitable to study the response when flying at any speed.

Over a long period of time, the statistical properties of air pulsations can be considered as practically unchanged. Therefore, the wind load can be treated as a stationary random function. As a function describing the distribution of power with respect to the spatial frequency k, of which a function $S(k)$ consists of three components along the axes is selected.

The theory of isotropic turbulence implies [8] that it is enough two correlation functions $f_u(r)$ and $f_w(r)$, downstream and across-stream, and consequently two spectral densities $S_u(k)$ (Figure 1) and $S_w(k)$ to describe all of the constituent of random wind gusts. A parameter r is the distance between two points. The most commonly encountered models of turbulent atmosphere are the model by Dryden [5] and the model by Karman [4]. Dryden proposed the following empirical expressions to define the correlation functions:

$$f_u(r) = e^{-r/L},\qquad f_w(r) = (1 - 0.5\,r/L)e^{-r/L}. \tag{1}$$

Hence the spectral densities of the longitudinal and transverse components for the one-dimensional turbulence can be expressed as follows:

$$S_u(k) = 4\sigma_u^2 L \frac{1}{1+(2\pi kL)^2},\qquad S_w(k) = 2\sigma_w^2 L \frac{1+3(2\pi kL)^2}{\left[1+(2\pi kL)^2\right]^2}. \tag{2}$$

Here σ_u^2 and σ_w^2 are the variances of the turbulent velocity in the longitudinal and transverse directions, respectively, and the parameter L is the scale of turbulence that is the value proportional to the average size of atmospheric disturbances.

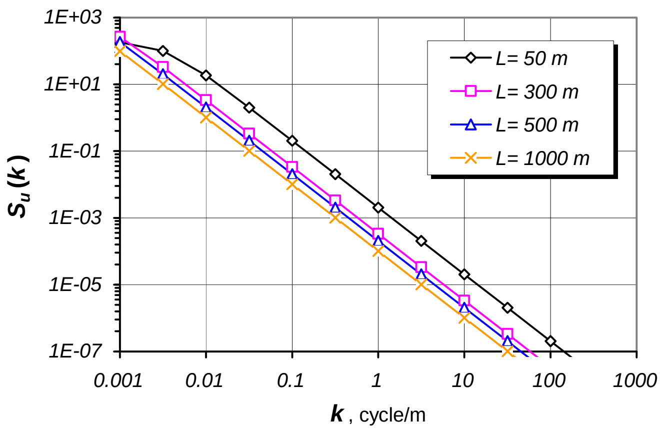

Figure 1 : Spectral density of atmospheric turbulence by Dryden.

2.2 Stochastic vibrations of SDOF system on fixed base

Having received the turbulent air energy, the structure gets an extra excitation which leads to increased levels of the stress state in the structure elements. The load-bearing structure can be represented simplistically as a linear elastic system. An external input signal of the system is atmospheric turbulence and its output is also a random process whose characteristics are to be determined (for example, displacements or velocities of points).

As the single one-degree-of-freedom (SDOF) system may be considered a linear system consisted of a linear oscillator under exposure to an external random load having the properties of atmospheric turbulence, as shown on Figure 2. While studying the response to atmospheric fluctuations, the system can be assumed to be in a dynamically balanced state [8]. It is assumed that all its mass *m* is concentrated at a point and is more significantly the mass of the holding structure. The load $q(t)$ generated by head wind gusts is a realization of a stationary random process $Q(t)$ and leads to longitudinal vibrations at the center of gravity $y(t)$. The equation of motion for such a system is given by

$$m\ddot{y} + c\dot{y} + ky = q(t) = a\,\dot{u}(t)\,, \tag{3}$$

A variance of the output signal $y(t)$ is

$$\bar{y}^2 = \int_0^\infty S_y(\omega)d\omega = \int_0^\infty |G(i\omega)|^2 S_u(\omega)d\omega\,. \tag{4}$$

A transfer function $G(i\omega)$ for the system Eq. (3) is given by

$$G(i\omega) = K_u \frac{i\omega}{(-\omega^2 + 2\beta\omega_0 i\omega + \omega_0^2)}\,, \tag{5}$$

where ω_0 is a natural frequency obtained from $\omega_0^2 = k/m$, β is a damping coefficient, $K_u = a/m$ is constant.

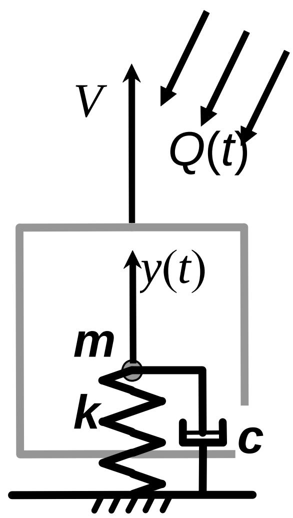

Figure 2: SDOF system under exposure of random external force $Q(t)$.

A spectral density of the output can be obtained as follows

$$S_y(\omega) = |G(i\omega)|^2 S_u(\omega) = K_u^2 \frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + (2\beta\omega_0\omega)^2} S_u(\omega)\,. \tag{6}$$

Substituting the corner frequency instead of the spatial frequency, the Eq. (2) becomes

$$S_u(\omega)=\frac{\sigma_u^2}{\pi}\frac{2\lambda}{1+\omega^2\lambda^2}$$
$$S_w(\omega)=\frac{\sigma_w^2}{\pi}\frac{L}{v}\frac{1+3\omega^2\lambda^2}{\left[1+\omega^2\lambda^2\right]^2} \tag{7}$$

Here $\lambda = L/v$, and v is an air speed.
Then, the variance of the longitudinal displacement of the center of mass $y(t)$ can be found

$$D_y=K_u^2\frac{\sigma_u^2}{\pi}\lambda\int_0^\infty\frac{1+3\omega^2\lambda^2}{\left[1+\omega^2\lambda^2\right]^2}\cdot\frac{\omega^2}{\left(\omega_0{}^2-\omega^2\right)^2+\left(2\beta\omega_0\omega\right)^2}d\omega\,. \tag{8}$$

After integration the expression (8) for D_y is as follows

$$D_y\approx\frac{K_u^2\sigma_u^2}{2\beta\lambda}\cdot\omega_0^{-3}=\frac{4\pi^3K_u^2\sigma_u^2}{\beta\lambda}\cdot f_0^{-3}=A_u\cdot f_0^{-3}\,. \tag{9}$$

Here $A_u=\dfrac{4\pi^3K_u^2\sigma_u^2}{\beta\lambda}$.

Thus the variance of displacement of the mass is a function of the *natural frequency* of the SDOF system as follows

$$\bar{y}^2(f_0)\approx A_u\cdot f_0^{-p} \tag{10}$$

The Eqs. (9) and (10) above implies that the variance of displacement is inversely proportional to the natural frequency of the oscillator. To show it, a variance $\bar{y}^2=D_y(f_0)$ obtained at different frequencies $f_0=\omega_0/(2\pi)$ and a function $F(f_0)=A\cdot f_0^{-p}$ with $p=3$ could be draw on the single graph to comparable scale together, see Figure 3.

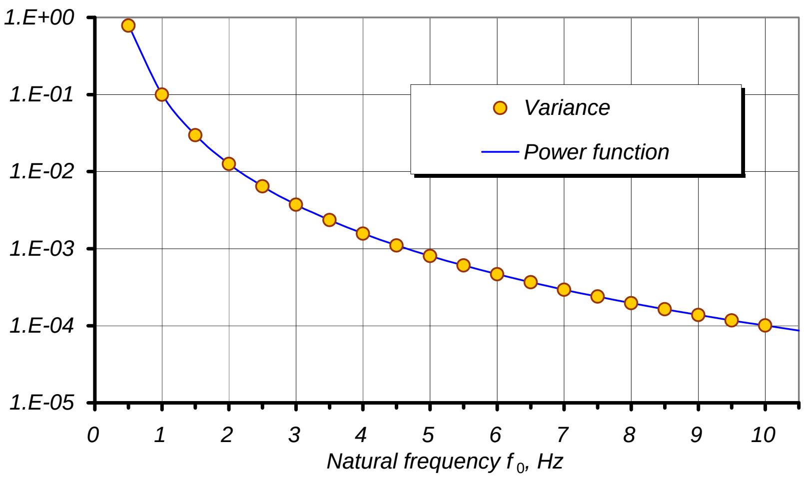

Figure 3: Variance of displacement as a function of the natural frequency (for turbulence model by Dryden).

Similarly the expressions for the one-dimensional turbulence model by Karman [4] can be obtained as follows:

$$S_u(\omega) = 2\frac{\sigma_u^2}{\pi}\frac{\lambda}{\left[1+\omega^2(1{,}339\lambda)^2\right]^{5/6}}$$
$$S_w(\omega) = \frac{\sigma_w^2}{\pi}\lambda\frac{1+\frac{8}{3}(1{,}339\lambda\omega)^2}{\left[1+(1{,}339\lambda\omega)^2\right]^{11/6}} \tag{11}$$

Here $p \approx 2.667$ for Eq. (10).

2.3 Stochastic vibrations of SDOF system on movable base

Turbulent air causes low-frequency vibrations of the aircraft structure, which are the external excitation regarding the set on its equipment, and vibrations arising from this equipment are the output process. Mode of deformation in structural elements which connect the equipment to the main structure is a function of the mutual displacements between an element and a ground supporting them.

A simple model of oscillations of inside equipments is shown as a linear elastic system in Figure 4. In the system, the base makes a displacements with acceleration $\ddot{x}(t)$ which are a realization of a stationary random process $X(t)$. The equation of motion of the mass m with respect to the variable $z = y - x$ (relative displacement) can be written as

$$m(\ddot{z}+\ddot{x}) + c\dot{z} + kz = 0\,. \tag{12}$$

Knowing the displacement z, tension in the spring k can be found.

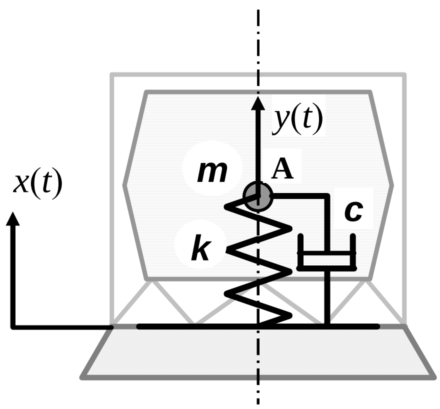

Figure 4: Elastic SDOF system at when exciting the base.

The Eq. (9) is simply transformed into

$$\ddot{z} + 2\beta_A\omega_A\dot{z} + (\omega_A)^2 z = -\ddot{x}\,. \tag{13}$$

Here β_A and ω_A are the damping ratio and the natural frequency respectively. The square of the transfer function of the system is given by

$$|G(i\omega)|^2 = \frac{\omega^4}{\left[(\omega_A)^2-\omega^2\right]^2 + (2\beta_A\omega_A\omega)^2}\,. \tag{14}$$

Thus, the spectral density of the relative displacement $z(t)$ is

$$S_z(\omega)=\frac{\omega^4}{\left[(\omega_A)^2-\omega^2\right]^2+(2\beta_A\omega_A\omega)^2}S_x(\omega). \tag{15}$$

To calculate the variance of the amplitude $z(t)$, the following expression is used

$$\bar{z}^2=D_z=\int_0^\infty S_z(\omega)d\omega=\int_0^\infty S_x(\omega)\cdot\frac{\omega^4 d\omega}{\left[(\omega_A)^2-\omega^2\right]^2+(2\beta_A\omega_A\omega)^2}. \tag{16}$$

From the previous example, the spectrum of the output $S_y(\omega)$ is considered here as the input spectral density function $S_x(\omega)$. Here, ω_0 and β are related to the structure, which is the exciting base. Figure 5 shows the values of the function $\bar{z}^2(f_A)$ for fixed values $f_0 = 2$ Hz and $f_0 = 9$ Hz.

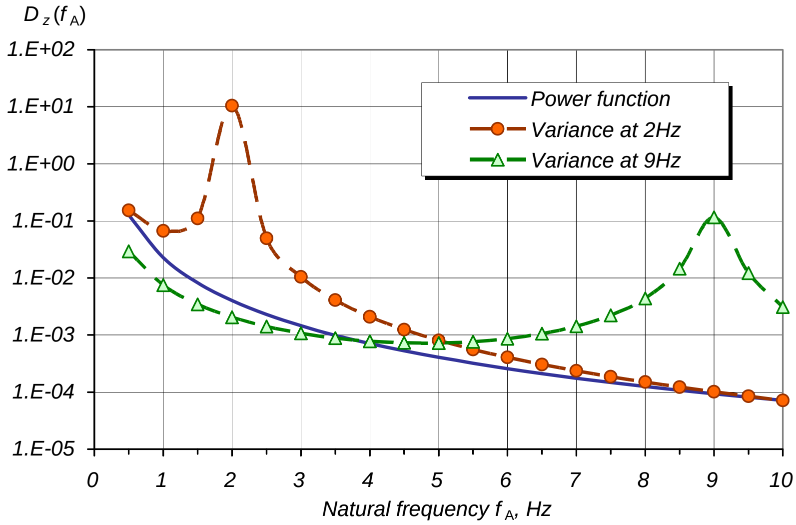

Figure 5: Variance of amplitude at $f_0 = 2$ Hz and $f_0 = 9$ Hz.

The chart shows that the relationship between the variance of the output process and the natural frequency is different from the power function like Eq. (10) in contrast to the previous case. Here is an ejection of the variance values near to the frequencies which close to the natural frequency of the base. The steepness of the ejection depends on the damping properties of the base. As it can be seen, a mismatch between the variance and the power function near the resonant frequency. Thus, the variance of displacement amplitude of the system is close to a power function at a certain distance from the resonance frequency.

Dynamic analysis of the systems presented in Figures 2 and 4 can be used for any mode of the real structure if the mass of the spring is negligibly small compared with the mass supported by this spring. The real structures are often well suited to this condition, and it is easy to apply the above calculation scheme based on the examination of SDOF systems. The expression for the variance of the displacement amplitude above is suitable for the variance of stresses generated in the structure as a result of these fluctuations, and differs from the expressions for the variance of stress by a constant factor.

2.4 Stochastic vibrations of MDOF systems

In a similar way a two degrees of freedom system shown on Figure 6,*a* could be considered. By separating the spatial and temporal variables, the displacements of all points can be represented as the product of the angle of rotation $\theta(t)$ as a function of time and some coordinate function $\varphi(x)$ (Figure 6,*b*)

$$y(x,t) = \theta(t) \cdot \varphi(x). \quad (17)$$

For the normal modes, the following coordinate functions can be taken

$$\varphi_1(x) = x - (l_1 + H_1), \quad (18)$$

$$\varphi_2(x) = x - (l_1 - H_2). \quad (19)$$

When considering the behavior of such the elastic system, the external load $q(t)$, which has stochastic properties discussed above, should be added in the right-hand side of the equations of motion. In accordance with the principle of superposition, a displacement of any point in the system is the sum of N displacements for each normal mode of vibration:

$$y(x,t) = \sum_{k=1}^{N} \xi_k(t)\varphi_k(x). \quad (20)$$

Here $\xi_k(t)$ is a stochastic function, which is the angle of rotation for the mode k and is simultaneously the generalized coordinate of the system, for which the random properties are to be determined.

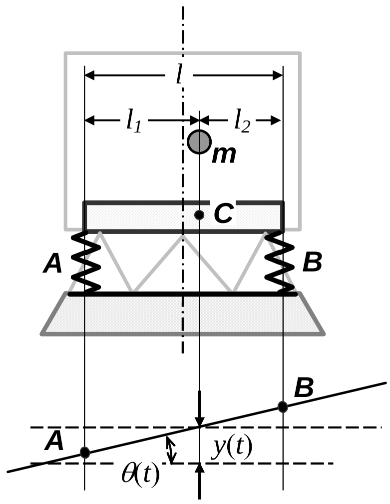

Figure 6: 2-DOF elastic system.

After substituting Eq. (20) in the equation of motion, the system decomposes into a couple of independent differential equations in generalized coordinates $\xi_k(t)$:

$$\ddot{\xi}_k + 2\beta_k\dot{\xi} + \omega_k^2\xi_k = p_k(t), \ (k = 1, 2). \quad (21)$$

To account damping nature of vibrations the term proportional to the logarithmic decrement also is introduced in these equations. The random function is a generalized external load, which is obtained by

$$p_k(t) = \frac{\int_0^l q(t)\varphi_k(x)dx}{\int_0^l \gamma\varphi_k^2(x)dx}. \quad (22)$$

The A–B body length over which the integration is done can be taken approximately equal to the sum $l = l_1 + l_2$; here γ is linear weight of the body.

Using the terms introduced above, the following expressions for the generalized forces can be obtained

$$p_1(t) = K_u \frac{[l/2 - (l_1 + H_1)]}{[l^2/2 - l(l_1 + H_1) + (l_1 + H_1)^2]} \dot{u}(t) = K_u \eta_1 \dot{u}(t), \tag{23}$$

$$p_2(t) = K_u \frac{[l/2 - (l_1 - H_2)]}{[l^2/2 - l(l_1 - H_2) + (l_1 - H_2)^2]} \dot{u}(t) = K_u \eta_2 \dot{u}(t). \tag{24}$$

Similar to the SDOF system, the cross spectral density of the generalized forces is given by

$$S_{p_j p_k}(\omega) = K_u^2 \eta_j \eta_k S_u(\omega). \tag{25}$$

As mentioned above, the spectral density of generalized coordinates can be expressed in terms of the spectral density of generalized forces and the frequency response of the system

$$S_{\xi_j \xi_k}(\omega) = G_j(i\omega) G_k(-i\omega) S_{p_j p_k} = K_u^2 \frac{\eta_j \eta_k \omega^2}{(\omega_j^2 - \omega^2 - 2\beta_j \omega_j i\omega)(\omega_k^2 - \omega^2 + 2\beta_k \omega_k i\omega)} S_u(\omega). \tag{26}$$

Variances of the generalized coordinates can be found from the following expression

$$D_{jk} = K_{jk}(0) = \int_0^\infty S_{\xi_j \xi_k}(\omega) d\omega = K_u^2 \eta_j \eta_k \frac{2\sigma_u^2}{\pi} \lambda \times$$

$$\times \int_0^\infty \frac{1}{[1 + (\omega\lambda)^2]} \frac{\omega^2}{(\omega_j^2 - \omega^2 - 2\beta_j \omega_j i\omega)(\omega_k^2 - \omega^2 + 2\beta_k \omega_k i\omega)} d\omega. \tag{27}$$

If the natural frequencies is sufficiently widely spaced and the relation $\beta^2 << \left|1 - \omega_j / \omega_k\right|$ is valid, then the off-diagonal correlation matrix elements K_{jk} can be neglected because of their smallness [5], i.e. $K_{jk} << K_{kk}$. Hence $D_{12} = D_{21} = 0$. The damping coefficients β are also assumed to be equal. The integration of Eq. (27) yield results (the longitudinal turbulence is taken as an external load) in follows expressions

$$\begin{aligned} D_{11} &\cong 0.5 \frac{K_u^2 \sigma_u^2 \eta_1^2}{\beta\lambda} \frac{1}{\omega_1^3}, \\ D_{22} &\cong 0.5 \frac{K_u^2 \sigma_u^2 \eta_2^2}{\beta\lambda} \frac{1}{\omega_2^3}. \end{aligned} \tag{28}$$

Thus, the stochastic solution of the motion equations set is expressed as follows

$$\bar{y}^2(x) \cong D_{11}\varphi_1(x) + D_{22}\varphi_2(x). \tag{29}$$

Substituting the values x in Eqs. (18) and (19), the variance of point displacements of the system can be obtained. For example, the mean squares of the displacement of A or B can be determined from the following equations

$$\bar{y}_A^2(x) = 0.5\frac{K_u^2\sigma_u^2}{\beta\lambda}\left[\eta_1^2\frac{(l_1+H_1)^2}{\omega_1^3}+\eta_2^2\frac{(-l_1+H_2)^2}{\omega_2^3}\right],$$
$$\bar{y}_B^2(x) = 0.5\frac{K_u^2\sigma_u^2}{\beta\lambda}\left[\eta_1^2\frac{(l_2-H_1)^2}{\omega_1^3}+\eta_2^2\frac{(l_2+H_2)^2}{\omega_2^3}\right]. \qquad (30)$$

In accordance with the theory of stochastic processes [3], the stochastic solution of the vector random process, which is the response of a multi-degree-of-freedom (MDOF) system to random effects, is obtained by summing the variances of amplitudes over all the normal coordinates:

$$\bar{y}^2 = \sum_{j=1}^{N}\sum_{k=1}^{N} D_{jk} = \sum_{j=1}^{N}\sum_{k=1}^{N} K_{jk}(0). \qquad (31)$$

The cross-correlation functions K_{jk} for the j–th and k–th modes are obtained by the following equation

$$K_{jk}(0) = \int_0^\infty |G(i\omega)|^2 S_{Q_jQ_k}(\omega)d\omega, \qquad (32)$$

where S_{QjQk} is a cross-spectral density of generalized forces.

The transfer function (or squared absolute value) $G(i\omega)$ is defined as follows

$$|G_{jk}(i\omega)|^2 = G_j(i\omega)G_k(-i\omega) = \frac{1}{(\omega_j^2-\omega^2+2\beta_j\omega_j i\omega)(\omega_k^2-\omega^2+2\beta_k\omega_k i\omega)}. \qquad (33)$$

When the lower natural frequencies are greatly different from each other, the cross-correlation of two degrees of freedom are usually ignored, because the cross correlation energy fraction of the total energy in similar systems does not exceed 2% as shown in [6]. Therefore, if the components of vector stochastic process are not correlated or weakly correlated, the variance of process is the sum of the variances for each mode, and the Eq. (31) can be simplified

$$\bar{y}^2 = \sum_{n=1}^{N} D_n = \sum_{n=1}^{N}\int_0^\infty \frac{S_{Q_n}(\omega)}{(\omega_n^2-\omega^2)^2+(2\beta_n\omega_n\omega)^2}d\omega, \qquad (34)$$

As shown above, for a SDOF system exposed to stationary random excitation, there is a simple expression relating the variance of amplitude and the natural frequency of system. Since each component of the vector random process behaves like an independent SDOF system, then its variance D_n is defined in the same form

$$D_n = a_n f_n^{-p}, \qquad (35)$$

where f_n is the n–th mode's natural frequency, a_n is a coefficient characterizing the contribution of the n–th mode in the total sum, p is a index of power and is nearly equal to 3.

Thus, the variance of displacement is approximately equal to:

$$\bar{y}^2 = \sum_{n=1}^{N} D_n = \sum_{n=1}^{N} a_n f_n^{-p}. \qquad (36)$$

The studies of different structure models have shown [9] that the displacements relating to the highest vibration modes are the terms having a higher order of smallness and have a very little effect on the total displacement. For example, the study of a uniform beam, simply supported and clamped, shows that the sum of terms (36), beginning from the second term, does not exceed 50% of the total sum, and for load-carrying structures, whose weight contribution does not exceed 10% of the total weight of system, the contribution of the first term of Eq. (36) is usually greater than 95%. Taking into account this fact, the expression (36) can be further simplified, leaving therein only the first term corresponding to the lowest natural frequency, with the factor whose value may range from 1.05 to 1.5, depending on the design concept.

2.5 Comparative analysis of structural layouts

According to the obtained relationships, a criterion for evaluation of design based on changing the natural frequency can be formulated, as well as a technique to estimate the effect of the dynamic properties upon the low-frequency loads can be proposed. To estimate the effect the study of alternative designs or several versions of a single design, in which the elastic and mass properties and/or the composition of structural elements is varying, is required.

The solution using the criterion above can include several steps. First one or more initial structural layouts are selected and the models are built to determine its natural frequencies. Then, the lower natural frequencies are found, simultaneously disclosing technical faults possible during building the model. To calculate the impact of each natural frequency to low-frequency loads, the frequency found for each option *n* can being inserted the following formula:

$$\begin{aligned} K_n &= \left(f_0 / f_n\right)^p \\ D_n &= \left(K_n - 1\right)\cdot 100\% \end{aligned} \tag{37}$$

Here f_0, f_n are the lower natural frequencies of the initial and the n-th option respectively, p is a power index equal approximately 1.5 (see above). The estimates calculated indicate how low-frequency loads depend on the variation of the natural frequency of system.

To easy represent and best understand the consequences, the analysis results are placed in a table. To evaluate the design changing "cost", the weight added or withdrawn during the alteration of design can be entered in the table as well.

By comparing the changing of loads and weights, the merits and demerits of each option can be evaluated to choose the most suitable one and a way of further improving of the design can be outlined.

3 CONCLUSIONS

The study of the probabilistic behavior of linear systems brought to a simple expression characterizing the relation between the lowest natural frequency of system and the variance of the amplitude of motion, and, therefore, the levels of maximum stress in the power structure. The quality of a design concept can be evaluated using this criterion as well.

The most rational design solutions can be found using the criterion of comparative assessment on the early stages of development without carrying out expensive computing of loads and stresses.

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