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EFFECT OF RADIAL HYDROSTATIC LOADS AND BOUNDARY CONDITIONS ON THE NATURAL FREQUENCIES OF THIN-WALLED CIRCULAR CYLINDRICAL SHELLS

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Abstract. *Cylindrical shells are widely used in industrial applications. During operation, these cylindrical shells are subjected to radial as well as axial loads, which alter the natural frequencies of these structures. These loads might have a fluctuating component along with a mean value. To avoid the phenomenon of resonance due to the fluctuating component, the variation of the natural frequencies with the mean load needs to be taken into account. In this paper, the variation of the natural frequencies associated with the various circumferential wavenumbers with the mean radial hydrostatic loads load has been studied. This study also helps in identifying the circumferential wavenumber associated with the lowest natural frequency for a given radial load. Static buckling load for the cylindrical shells where the lowest natural frequency becomes zero is also obtained as a special case. The study has been performed for four different types of boundary conditions viz. pined-pined, pined-free, clamped-free, and clamped-clamped. Flugge-Lur'e-Byrne shell theory in conjunction with the appropriately obtained averaged equations of motion (nonlinear PDEs) has been used for the present analysis. These nonlinear PDEs are first linearized about the steady solutions corresponding to the mean load and then converted into system of ODEs using Galerkin projections with appropriate mode shapes. These ODEs are used to obtain the natural frequencies for different circumferential and axial wavenumbers. Our studies show that the circumferential wavenumber corresponding to the lowest natural frequency is strongly affected by the radial load especially for lower slenderness ratios for all boundary conditions. Increase in the magnitude of an inward radial load decreases the natural frequencies but results in an increase in the circumferential wavenumber corresponding to the first fundamental mode. The situation is reversed for an outward radial load but the effect is less pronounced. In this paper, further study has been made on the variation of natural frequency with hydrostatic pressure. These results have been verified against FEA results obtained using ABAQUS.*

1 INTRODUCTION

Cylindrical shells are widely used in industrial applications. During operation, these cylindrical shells are subjected to radial as well as axial loads, which alter their natural frequencies. Vibration analysis of thin walled circular cylindrical shells have been studied since 19th century. Rayleigh [1] and Love [2] have first studied the vibrations of circular cylindrical shell using Love approximation theory. Arnold and Warburton [3] have studied the vibration of circular cylindrical shell with simply supported ends theoretically and experimentally. They have used energy method using strain-displacement relations given by Timoshenko. They also obtained the variation of natural frequencies with different geometric parameters (i.e. radius, thickness and length). Warburton [4] has presented numerical study of the vibration of a circular cylindrical shell using Flügge equations of motion. There are several other works [5, 6, 7, 8, 9] on vibrations of cylindrical shells. Fung [10] has studied the effect of internal pressure on natural frequencies and corresponding circumferential wavenumbers. He has observed that when internal pressure increases, natural frequency also increases, but corresponding circumferential wavenumber decreases. In this analysis, some important linear and nonlinear terms are neglected without giving any proper explanations. Lakis and Païdoussis [11] have analyzed free vibration of vertical thin circular cylindrical shell, which is partially filled with liquid, using finite element methods. Sanders equations [6] of motion for shell has been used by them for this analysis . Free vibration of a clamped-free circular cylindrical shell partially filled with liquid has been studied by Chiba *et al.* [12].

In the present study, we have investigated the effect of internal and external uniform pressure and linearly varying pressure with axial direction (e.g hydrostatic pressure) on natural frequency and corresponding circumferential wavenumber. These cylindrical shells are subjected to radial as well as axial loads, which alter the natural frequencies of these structures. Also, these loads might have fluctuating components along with corresponding mean values. To avoid the phenomenon of resonance due to the fluctuating component, the variation of the natural frequencies with the mean load needs to be taken into account. In this paper, we have studied the variation of the natural frequencies associated with the corresponding circumferential wavenumbers with the mean radial load. This study also helps in identifying the circumferential wavenumber associated with the lowest natural frequency for a given radial load. Static buckling load for the cylindrical shells wherein the lowest natural frequency becomes zero is also obtained as a special case. The study has been performed for three different types of boundary conditions viz. pined-pined, pined-free and clamped-free. Flugge-Lure-Byrne shell theory in conjunction with the appropriately obtained averaged equations of motion (nonlinear PDEs) has been used for the present analysis. These nonlinear PDEs are first linearized about the steady solutions corresponding to the mean load and then converted into system of ODEs using Galerkin projections with appropriate mode shapes. These ODEs are used to obtain the natural frequencies for different circumferential and axial wavenumbers. Our studies show that the circumferential wavenumber corresponding to the lowest natural frequency is strongly affected by the radial load especially for lower slenderness ratios for all boundary conditions. Increment in the magnitude of an inward radial load decreases the natural frequencies, but results in an increment in the circumferential wavenumber corresponding to the first fundamental mode. The situation is reversed for an outward radial load but the effect is less pronounced. These results have been verified against FEA using ABAQUS.

2 Basic Formulations

We consider a cylindrical shell of radius R and thickness h as shown in Figure 1, h being much smaller than R (i.e., $h \ll R$). The middle surface of the shell is located at a distance $h/2$ from both the inner and outer surfaces of the cylinder. The components of the displacement of a representative point with coordinates (x, θ, ρ), along the axial, tangential and radial directions are u_1, u_2 and u_3, respectively. Note that $\rho = R + z$, where z is the distance of the point from the middle surface. Equations of motion in cylindrical coordinates for thin walled circular

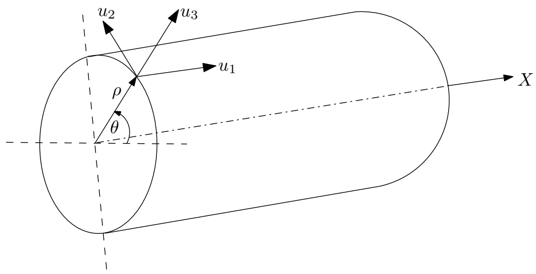

Figure 1: Schematic of a cylindrical shell showing the displacements of a point in the shell. The cylindrical coordinates $(\rho(= R + z), \theta, x)$ are as shown.

cylindrical shell, which have been obtained from free body diagram, are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{\theta x}}{\rho \partial \theta} + \frac{\partial \sigma_{zx}}{\partial z} + \frac{\sigma_{zx}}{\rho} + X = 0 \,, \tag{1a}$$

$$\frac{\partial \sigma_{\theta\theta}}{\rho \partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{\partial \sigma x \theta}{\partial x} + \frac{\sigma_{z\theta}}{\rho} + \frac{\sigma_{\theta x}}{\rho}\frac{\partial u_3}{\partial x} + \frac{\sigma_{\theta\theta}}{\rho}\frac{\partial u_3}{R\partial \theta} + \frac{\sigma_{\theta z}}{\rho}\left(1 + \frac{\partial u_3}{\partial z}\right) + Y = 0 \,, \tag{1b}$$

$$\begin{aligned}
&\frac{\partial}{\partial z}\left[\sigma_{zx}\frac{\partial u_3}{\partial x} + \sigma_{z\theta}\frac{\partial u_3}{R\partial \theta} + \sigma_{zz}\left(1 + \frac{\partial u_3}{\partial z}\right)\right] + \frac{\partial}{\rho \partial \theta}\left[\sigma_{\theta x}\frac{\partial u_3}{\partial x} + \sigma_{\theta\theta}\frac{\partial u_3}{R\partial \theta} + \sigma_{\theta z}\left(1 + \frac{\partial u_3}{\partial z}\right)\right] \\
&+ \frac{\partial}{\partial x}\left[\sigma_{xx}\frac{\partial u_3}{\partial x} + \sigma_{x\theta}\frac{\partial u_3}{R\partial \theta} + \sigma_{xz}\left(1 + \frac{\partial u_3}{\partial z}\right)\right] + \frac{1}{\rho}\left[\sigma_{zx}\frac{\partial u_3}{\partial x} + \sigma_{z\theta}\frac{\partial u_3}{R\partial \theta} + \sigma_{zz}\left(1 + \frac{\partial u_3}{\partial z}\right)\right] \\
&- \frac{\sigma_{\theta\theta}}{\rho} + Z = 0 \,.
\end{aligned} \tag{1c}$$

These equations (1a)-(1c) are obtained after neglecting all nonlinear terms containing inplane displacements i.e. u_1 and u_2. Flügge-Luré-Byrne nonlinear shell theory has been used in the present analysis. Displacement fields (i.e. displacement of a generic point in terms of displacement of a point on the middle surface in radial, tangential and circumferential directions) for Flügge-Luré-Byrne theory are given as [6]

$$u_1(x, \theta, z, t) = u(x, \theta, t) - z\frac{\partial w(x, \theta, t)}{\partial x} \,, \tag{2a}$$

$$u_2(x, \theta, z, t) = v(x, \theta, t) - \frac{z}{R}\left(\frac{\partial w(x, \theta, t)}{\partial \theta} - v(x, \theta, t)\right) \,, \tag{2b}$$

$$u_3(x, \theta, z, t) = w(x, \theta, t) \,. \tag{2c}$$

where $u(x, \theta, t)$, $v(x, \theta, t)$ and $w(x, \theta, t)$ are the displacement components of a point on the middle surface along axial, circumferential and radial directions respectively. We substitute equations (2) in equations (1a)-(1c).

The wall of the cylindrical shell is very thin, so the variations of all quantities across thickness is very small, hence we take average of all quantities across thickness for simplification. In this way we reduce three dimension problems to two dimension. The modified equations of motion in terms of stress resultants are,

$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial N_{\theta x}}{\partial \theta} - \rho_m h \frac{\partial^2 u}{\partial t^2} = 0 \,, \tag{3a}$$

$$\frac{1}{R}\frac{\partial N_\theta}{\partial \theta} + \frac{\partial N_{x\theta}}{\partial x} + \frac{N_{\theta x}}{R}\frac{\partial w}{\partial x} + \frac{N_\theta}{R^2}\frac{\partial w}{\partial \theta} + \frac{Q_\theta}{R} - \rho_m h \frac{\partial^2 v}{\partial t^2} = 0 \,, \tag{3b}$$

$$\begin{aligned} &\frac{1}{R}\frac{\partial Q_\theta}{\partial \theta} + \frac{\partial Q_x}{\partial x} - \frac{N_\theta}{R} + \frac{1}{R}\frac{\partial N_{\theta x}}{\partial \theta}\frac{\partial w}{\partial x} + \frac{N_{\theta x}}{R}\frac{\partial^2 w}{\partial x \partial \theta} + \frac{1}{R^2}\frac{\partial N_\theta}{\partial \theta}\frac{\partial w}{\partial \theta} + \frac{N_\theta}{R^2}\frac{\partial^2 w}{\partial \theta^2} \\ &+ \frac{\partial N_x}{\partial x}\frac{\partial w}{\partial x} + N_x \frac{\partial^2 w}{\partial x^2} + \frac{1}{R}\frac{\partial N_{x\theta}}{\partial x}\frac{\partial w}{\partial \theta} + \frac{N_{x\theta}}{R}\frac{\partial^2 w}{\partial x \partial \theta} + P - \rho_m h \frac{\partial^2 w}{\partial t^2} = 0 \,, \end{aligned} \tag{3c}$$

where $(N_x \;, N_{x\theta} \;, N_{\theta x} \;, N_\theta \;, Q_x \;, Q_\theta)$ are the stress resultants per unit length and ρ_m is density of material. Now substituting expressions of stress resultants (N_x, N_θ, $N_{x\theta}$, $N_{\theta x}$, Q_x and Q_θ), we get equations of motion in terms of displacements $(u \;, v \;, w)$ of the point on the middle surface. The expression for stress resultants are given

$$N_x = \int_{-h/2}^{h/2} \sigma_{xx}\left(1 + \frac{z}{R}\right) dz \,, \quad N_\theta = \int_{-h/2}^{h/2} \sigma_{\theta\theta}\, dz \,, \quad N_{x\theta} = \int_{-h/2}^{h/2} \sigma_{x\theta}\left(1 + \frac{z}{R}\right) dz \,, \tag{4a}$$

$$Q_x = \int_{-h/2}^{h/2} \sigma_{xz}\left(1 + \frac{z}{R}\right) dz \,, \quad Q_\theta = \int_{-h/2}^{h/2} \sigma_{\theta z}\, dz \,, \tag{4b}$$

and the resultant moments per unit length are

$$M_x = \int_{-h/2}^{h/2} \sigma_{xx}\left(1 + \frac{z}{R}\right) zdz \,, \quad M_\theta = \int_{-h/2}^{h/2} \sigma_{\theta\theta}\, zdz \,, \quad M_{x\theta} = \int_{-h/2}^{h/2} \sigma_{x\theta}\left(1 + \frac{z}{R}\right) zdz \,. \tag{5}$$

The resultant transverse shear stresses Q_x and Q_θ are related to the resultant moments through the relations

$$Q_x = \frac{M_x}{\partial x} + \frac{\partial M_{\theta x}}{R \partial \theta} \,, \quad \text{and} \quad Q_\theta = \frac{M_{x\theta}}{\partial x} + \frac{\partial M_\theta}{R \partial \theta} \,. \tag{6}$$

The relations between the second Kirchhoff stress and strain components for linearly elastic, homogeneous and isotropic material, under plane stress condition, are

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\epsilon_{xx} + \nu \epsilon_{\theta\theta}) \,, \quad \sigma_{\theta\theta} = \frac{E}{1-\nu^2}(\epsilon_{\theta\theta} + \nu \epsilon_{xx}) \,, \quad \tau_{x\theta} = \frac{E}{2(1+\nu)}\gamma_{x\theta} \,. \tag{7}$$

3 Vibration of thin walled circular cylindrical shell for different boundary conditions subjected to hydrostatic pressure

3.1 Simply supported at both ends

In this case we have neglected damping effect for simplification. Here we have assumed that applied pressure is axi-symmetric, so steady state solutions have to be axi-symmetric, i.e.

independent of circumferential coordinates (θ). During calculating steady state solutions, we use boundary conditions for simply supported at both ends as given

$$at \;\; x = 0 \, , \begin{cases} w(x) = 0 \, , \\ u(x) = 0 \, , \\ v(x) = 0 \, , \\ M_x = 0 \, . \end{cases} \quad , \quad \text{and} \quad at \;\; x = L \, , \begin{cases} w(x) = 0 \, , \\ v(x) = 0 \, , \\ N_{x\theta} = 0 \, , \\ M_x = 0 \, . \end{cases}$$

This way, we get steady state solutions along radial and axial directions. The steady state displacement along circumferential direction is assumed to be zero, since all points move along radial direction only, i.e. all points experience zero circumferential displacement. We linearize equations of motion about obtained steady state solutions. The displacement fields become $u(x, \theta, t) = u_0(x) + \epsilon\tilde{u}(x, \theta, t)$, $v(x, \theta, t) = v_0(x) + \epsilon\tilde{v}(x, \theta, t)$, and $w(x, \theta, t) = w_0(x) + \epsilon\tilde{w}(x, \theta, t)$, where $u_0(x)$, $v_0(x)$ and $w_0(x)$ are the steady state displacements along axial, circumferential and radial directions respectively. The displacements $\tilde{u}(x, \theta, t)$, $\tilde{v}(x, \theta, t)$ and $\tilde{w}(x, \theta, t)$ are considered small compared to the corresponding steady state displacements. We linearize these three nonlinear PDEs to get corresponding linear PDEs. Galerkin projection method is then applied with appropriate mode shape functions for axial, circumferential and radial displacements. The mode shapes for simply supported boundary conditions at both ends are given as

$$\tilde{u} = U_{mn}(t)\cos(\frac{m\pi}{L}x)\cos(n\theta) \, ,$$
$$\tilde{v} = V_{mn}(t)\sin(\frac{m\pi}{L}x)\sin(n\theta) \, ,$$
$$\tilde{w} = W_{mn}(t)\sin(\frac{m\pi}{L}x)\cos(n\theta) \, ,$$

where m is the axial wavenumber and n is the circumferential wavenumber. $U_{mn}(t)$, $V_{mn}(t)$ and $W_{mn}(t)$ are time varying amplitudes of the axial, circumferential and radial displacements, respectively. It is easy to check that the shape functions satisfy the necessary boundary conditions. Applying Gelerkin projection method, we have the following system of ODEs

$$\begin{bmatrix} \ddot{U}_{mn}(t) \\ \ddot{V}_{mn}(t) \\ \ddot{W}_{mn}(t) \end{bmatrix} = [A] \begin{bmatrix} U_{mn}(t) \\ V_{mn}(t) \\ W_{mn}(t) \end{bmatrix} , \tag{8}$$

where $[A]$ is a 3×3 stiffness matrix obtained using MAPLE.

Eigenvalues of this stiffness matrix $[A]$ give natural frequencies of circular cylindrical shells with simply supported at both ends.

3.2 One end is pinned and other is free

Boundary conditions for pined-free circular cylindrical shell are

$$at \;\; x = 0 \, , \begin{cases} w(x) = 0 \, , \\ u(x) = 0 \, , \\ v(x) = 0 \, , \\ M_x = 0 \, . \end{cases} \quad \text{and} \quad at \;\; x = L \, , \begin{cases} Q_x = 0 \, , \\ N_{x\theta} = 0 \, , \\ M_x = 0 \, . \end{cases}$$

Using boundary conditions given for pined-free case, we obtain steady state solutions. Same procedure is employed as in simply supported condition (3.1), and linearized PDEs are obtained. The mode shapes used in Galerkin projection method are assumed to be

$$\tilde{u} = U_{mn}(t)\cos(\frac{\beta_m}{L}x)\cos(n\theta)\,,$$

$$\tilde{v} = V_{mn}(t)\sin(\frac{\beta_m}{L}x)\sin(n\theta)\,,$$

$$\tilde{w} = W_{mn}(t)\sin(\frac{\beta_m}{L}x)\cos(n\theta)\,.$$

Here values of β_m are taken from cantilever beam mode shape function for simplification. Applying Gelerkin projection method, we finally get three equations of motion as

$$\begin{bmatrix}\ddot{U}_{mn}(t)\\ \ddot{V}_{mn}(t)\\ \ddot{W}_{mn}(t)\end{bmatrix} = [A]\begin{bmatrix}U_{mn}(t)\\ V_{mn}(t)\\ W_{mn}(t)\end{bmatrix}\,, \tag{9}$$

where $[A]$ is again a 3×3 stiffness matrix.

3.3 One end is fixed and other is free

Boundary conditions for clamped and free ends are ,

$$At\ \ x=0\,,\begin{cases}w(x)=0\,,\\ u(x)=0\,,\\ v(x)=0\,,\\ \frac{\partial w(x)}{\partial x}=0\,.\end{cases} \quad \text{and} \quad At\ \ x=L\,,\begin{cases}Q_x=0\,,\\ N_{x\theta}=0\,,\\ M_x=0\,.\end{cases}$$

The mode shapes used in Galerkin projection in this case are the same as used in the case of pinned-free.

4 Results and Discussions

Here cylindrical shell subjected to hydrostatic pressure in that case uniform pressure term P in equation (3) is replace by term

$$\rho_f g(a-x)\mathcal{H}(a-x)\,,$$

where ρ_f is fluid density, g is acceleration due to gravity, a is liquid hight measured from bottom, x is axial coordinate measured from bottom and $\mathcal{H}$ is heaviside function.

As we have seen, vibration analysis of thin walled circular cylindrical shells with boundary conditions pinned-pinned, pinned-free and fix-free have been studied in present paper. In figure 2, variations of natural frequencies with circumferential wavenumber are given for three different boundary conditions viz pinned-pinned, pinned-free, fixed-free. The vibration analysis has been done for three different L/R ratios. Analytical results are compared with FEA results using ABAQUS. As shown in figure 2, for the case of pinned-pinned analytical and FEA results have good agreement, but for pined-free and fixed-free analytical results have slightly higher values than those obtained from FEA for short cylinder. Pinned -pinned and fixed-free

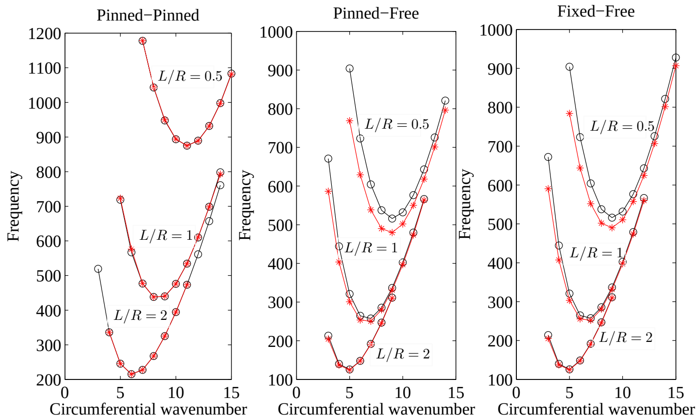

Figure 2: Comparison of natural frequencies obatined analytically and obtained FEA using ABAQUS for different L/R ratio

conditions have same value of circumferential wavenumber corresponding to their lowest natural frequencies, but natural frequencies of fixed-free boundary condition have slightly higher values as compared to the corresponding values of pinned-free boundary condition. It has been observed that, for short cylinder, circumferential wavenumber corresponding to lowest natural frequency is large for all boundary conditions, i.e if length of cylinder increases, circumferential wavenumber corresponding to lowest frequency decreases and vice-versa.

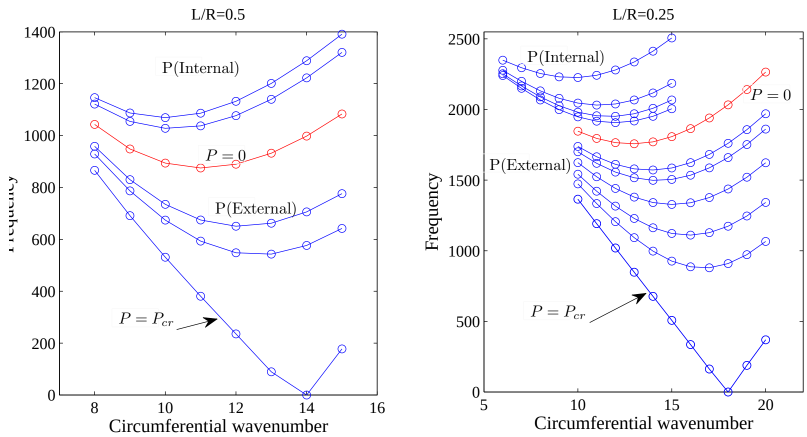

Figure 3: Natural frequency of circular cylindrical shell with simply supported at both ends at different radial loads (eg internal and external radial pressure). Here $L = 1m$, $R = 2m$, and $h = 0.01m$ and $L = 0.5m$, $R = 2m$, and $h = 0.01m$

The variations of natural frequencies with circumferential wavenumber under uniform radial pressure (both inward and outward case) have been shown in Fig.3 for pinned-pinned boundary conditions. When cylindrical shells are subjected to inward radial load (i.e. externally) natural frequencies decrease and corresponding circumferential wavenumbers increase. It has been

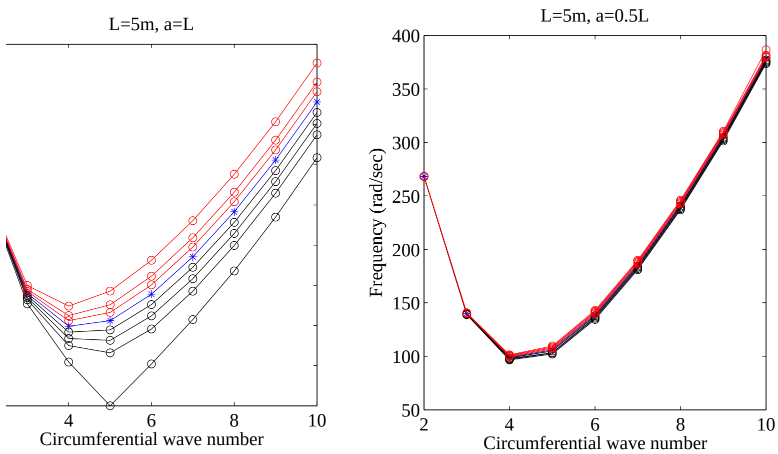

Figure 4: Variations of natural frequencies of circular cylindrical shell of fixed-free BCs with circumferential wave number and density of liquid (eg internal and external hydrostatic pressure). Blue star line showing variation natural frequencies with circumferential wave number without subjected any pressure. Above blue star line all lines (red lines) represent variation of frequencies with internally applied hydrostatic pressure and below blue star line all lines (black lines) represent variation due to externally applied hydrostatic pressure

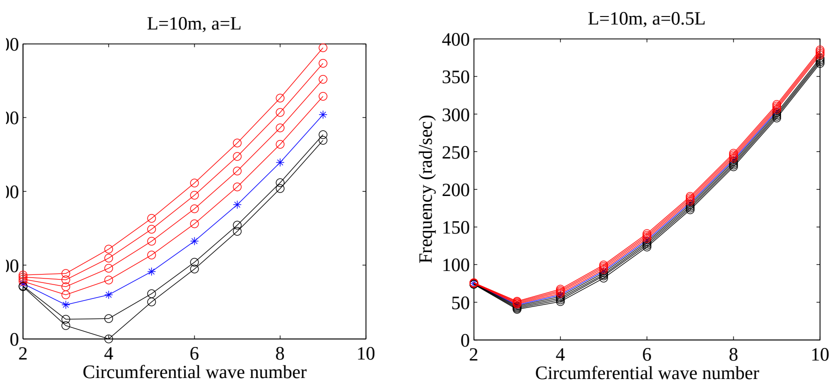

Figure 5: Variations of natural frequencies of circular cylindrical shell of fixed-free BCs with circumferential wave number and density of liquid (eg internal and external hydrostatic pressure). Blue star line showing variation natural frequencies with circumferential wave number without subjected any pressure. Above blue star line all lines (red lines) represent variation of frequencies with internally applied hydrostatic pressure and below blue star line all lines (black lines) represent variation due to externally applied hydrostatic pressure

shown in Fig.3 that, when inward radial pressure reaches the critical value, the lowest natural frequency becomes zero. The circumferential wavenumber corresponding to zero natural frequency is called first buckling mode. If we apply outward radial load, natural frequencies increase and corresponding circumferential wavenumbers decrease. The effect of radial load on

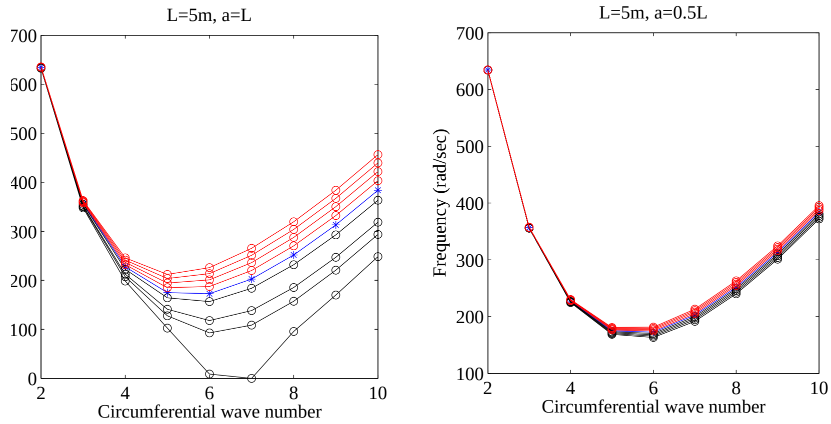

Figure 6: Variations of natural frequencies of circular cylindrical shell of pinned-pinned BCs with circumferential wave number and density of liquid (eg internal and external hydrostatic pressure). Blue star line showing variation natural frequencies with circumferential wave number without subjected any pressure. Above blue star line all lines (red lines) represent variation of frequencies with internally applied hydrostatic pressure and below blue star line all lines (black lines) represent variation due to externally applied hydrostatic pressure

short cylinders are more pronounced than long cylinders. This study can be used in understanding the resonance phenomena of circular cylindrical shells subjected to radial pressure (internal or external). Same results have been shown in figures Fig.4, Fig. 5 and Fig.6. As cylindrical shells are subjected to internally applied hydrostatic pressure natural frequency increases and associated circumferential wavenumber decreases and vice versa. Natural frequencies associated with low circumferential wavenumbers have less effect of hydrostatic pressure as compared to frequencies associated with high circumferential wavenumbers.

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