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MODAL INTERACTIONS IN STRINGS VIBRATING AGAINST AN OBSTACLE: RELEVANCE TO MUSICAL INSTRUMENTS LIKE SITAR AND TANPURA

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Abstract. *Vibrations of a string against a smooth unilateral parabolic obstacle is investigated in this work. The string is assumed to wrap and unwrap around the obstacle smoothly. This situation is akin to string vibrations in Indian musical instruments like Sitar/Tanpura for which the sound is far less inharmonic than that of the Western stringed instruments like the Guitar. Another distinguishing sound characteristics of these instruments is the modulations in the intensity of sound. We explore the effect of the finite-size of the obstacle on the vibrations characteristics of the string. As the string wraps and unwraps around the obstacle, the effective length of the string changes making it a moving boundary problem. We first present a mathematical model based on the Hamilton's principle wherein we obtain a partial differential equation for the string displacement with appropriate boundary conditions at the moving as well as the fixed end of the string. The complexity of the moving boundary is converted into nonlinearity of the governing partial differential equation with a fixed boundary by a dynamic rescaling of the length. Finite-dimensional ODE approximations are obtained using a Galerkin approximation consistent with the boundary conditions. To validate the results from this model, we also present results from an alternate formulation based on the Euler-Lagrange equations with assumed modes which are different from those used for the Galerkin projection. Linear analysis shows that the presence of the obstacle effects the natural frequencies by determining the free length of the string but has no effect on the harmonicity of the frequencies. Fully nonlinear analysis with multiple modes shows the possibility of modal interactions resulting in the modulations in both the frequency content as well as the amplitudes of the individual modes. Hence, the obstacle is critical in determining the sound characteristics of the Indian stringed musical instruments.*

1 INTRODUCTION

Sounds of musical instruments like Tanpura[1], Sitar, Veena are very attractive in many senses. Acoustics of these musical instruments have been reported in [1, 2, 5, 6]. Mentioned musical instruments have a common constructional feature that they have a curved bridge instead of a sharp bridge which is commonly used in Western stringed instruments. The form of the bridge for Veena and Tanpura are well described in [1]. In these instruments the string keeps changing its effective length during vibration due to the presence of the the curved bridge. Therefore, the problem becomes a moving boundary problem for the string vibrating against an one sided obstacle. Amerio [3] investigated vibration of a string against a rigid wall, where he assumed that the impact between the wall and string is elastic. He formulated the problem considering the conservation of energy. Schatzman [4] studied vibrations of a string constrained by a concave obstacle considering no loss of energy in impact with the obstacle. Existence and uniqueness of the solution were proved in his work. Burridge et al. [7] addressed Sitar string vibration problem considering inelastic impact of the string with rigid parabolic bridge. They considered kinetic energy loss due to inelastic impact of the string with the bridge. Bamberger and Schatzman [8] investigated vibration of the string against convex and plane obstacles. Haraux and Cabannes [9] considered straight fixed obstacle with no loss of energy in impact. They showed periodic nature of the solution. In another work Cabannes et al. [11] studied vibrations of string against arbitrary obstacle. Alsahlani and Mukherjee [13] modeled vibrations of string against a smooth obstacle placed at one end of the string. They considered loss of kinetic energy while the string wraps around the obstacle. Vyasarayani et al. [12] modeled vibrations in string as modeled by Burridge et al. [7]. But Vyasarayani et al. considered smooth wrapping and unwrapping of the string around the bridge which was not there in the model proposed by Burridge et al. [7].

In the present work we have investigated vibrations of a string against a smooth unilateral obstacle which was placed at one end of the string. We have used the same model which was used by Vyasarayani et al.[12]. We have investigated natural frequencies of the system and nonlinear modal interactions in string vibrations.

2 MATHEMATICAL MODEL

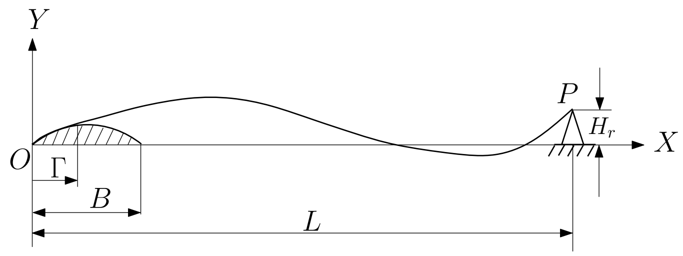

Figure 1: String model

As shown in figure 1, an ideal string is vibrating against a smooth parabolic obstacle/bridge placed at O. The string is under tension T, and has uniform density ρ per unit length. The major assumptions are (1) the string perfectly wraps and unwraps around the bridge while going down and up during its motion respectively, (2) the string remains tangent to the bridge surface at the point of separation($X = \Gamma$), (3) the string has no bending stiffness. The geometry of the bridge

[1]Tanpura is also known as Tambura.

is defined by

$$Y_B(X) = A_P X(B - X), \tag{1}$$

where B is the length of the bridge. Lagrangian of an element of the string is given by

$$d\mathcal{L} = \frac{1}{2}\rho\left(\frac{\partial Y(X,t)}{\partial t}\right)^2 - \frac{1}{2}T\left(\frac{\partial Y(X,t)}{\partial X}\right)^2, \tag{2}$$

where $Y(X,t)$ is the transverse displacement of a string element from X axis. Geometric constraint for this problem can be identified as

$$Y(X,t) \geq Y_B(X). \tag{3}$$

Using Hamilton's principle we get

$$\delta\int_{t_1}^{t_2}\left[\int_0^{\Gamma_-(t)}\left\{\frac{1}{2}\rho\left(\frac{\partial Y(X,t)}{\partial t}\right)^2 - \frac{1}{2}T\left(\frac{\partial Y(X,t)}{\partial X}\right)^2 + \lambda(X)G(X,t)\right\}dX\right]dt$$
$$+\delta\int_{t_1}^{t_2}\left[\int_{\Gamma_+(t)}^{L}\left\{\frac{1}{2}\rho\left(\frac{\partial Y(X,t)}{\partial t}\right)^2 - \frac{1}{2}T\left(\frac{\partial Y(X,t)}{\partial X}\right)^2\right\}dX\right]dt = 0, \tag{4}$$

where $\Gamma(t)$ is the wrapped length with the obstacle along X axis. $\lambda(X)$ is the distributed constraint force, and $G(X,t)$ is defined as

$$G(X,t) = Y(X,t) - Y_B(X) \tag{5}$$

using(3). Applying Leibniz's integration rule in (4) we get

$$\int_{t_1}^{t_2}\left[\int_0^{\Gamma_-(t)}\delta\left\{\frac{1}{2}\rho\left(\frac{\partial Y_2(X,t)}{\partial t}\right)^2 - \frac{1}{2}T\left(\frac{\partial Y_2(X,t)}{\partial X}\right)^2 + \lambda(X)G(X,t)\right\}dX\right]dt$$
$$+\int_{t_1}^{t_2}\left[\int_{\Gamma_+(t)}^{L}\delta\left\{\frac{1}{2}\rho\left(\frac{\partial Y_2(X,t)}{\partial t}\right)^2 - \frac{1}{2}T\left(\frac{\partial Y_2(X,t)}{\partial X}\right)^2\right\}dX\right]dt$$
$$+\int_{t_1}^{t_2}T_1(\Gamma_-(t),t)\delta\Gamma_-(t)dt - \int_{t_1}^{t_2}T_2(\Gamma_+(t),t)\delta\Gamma_+(t)dt = 0, \tag{6}$$

where $T_1\left(\Gamma_-(t),t\right) = \left[\frac{1}{2}\rho\left(\frac{\partial Y_2(X,t)}{\partial t}\right)^2 - \frac{1}{2}T\left(\frac{\partial Y_2(X,t)}{\partial X}\right)^2 + \lambda(X)G(X,t)\right]\Bigg|_{X=\Gamma_-(t)}$ and $T_2\left(\Gamma_+(t),t\right) = \left[\frac{1}{2}\rho\left(\frac{\partial Y_2(X,t)}{\partial t}\right)^2 - \frac{1}{2}T\left(\frac{\partial Y_2(X,t)}{\partial X}\right)^2\right]\Bigg|_{X=\Gamma_+(t)}$. Before the further simplification of (6), we first derive some relations which will be useful afterwards. At the point $X = \Gamma$, displacement $Y(X,t)$ and slope $Y'(X,t)$ are continuous. Therefore, we can write

$$(a)\,\delta\Gamma_-(t) = \delta\Gamma_+(t), \quad (b)\,\delta Y(\Gamma_-(t),t) = \delta Y(\Gamma_+(t),t), \quad (c)\,\frac{\partial Y(\Gamma_-(t),t)}{\partial X} = \frac{\partial Y(\Gamma_+(t),t)}{\partial X}. \tag{7}$$

After further simplification, (6) reduces down to

$$\int_0^{\Gamma_-(t)} \rho \, \frac{\partial Y_2(X,t)}{\partial t} \delta Y_2(X,t)\Big|_{t=t_1}^{t_2} dX - \int_{t_1}^{t_2} T \left(\frac{\partial Y_2(X,t)}{\partial X} \right) \delta Y_2(X,t) \Big|_{X=0}^{\Gamma_-(t)} dt$$
$$+ \int_{t_1}^{t_2} \int_0^{\Gamma_-(t)} \left[T \frac{\partial^2 Y_2(X,t)}{\partial X^2} - \rho \frac{\partial^2 Y_2(X,t)}{\partial t^2} + \lambda(X) \right] \delta Y_2(X,t) dX dt$$
$$+ \int_{\Gamma_+(t)}^{L} \rho \, \frac{\partial Y_2(X,t)}{\partial t} \delta Y_2(X,t) \Big|_{t=t_1}^{t_2} dX$$
$$+ \int_{t_1}^{t_2} \int_{\Gamma_+(t)}^{L} \left[T \frac{\partial^2 Y_2(X,t)}{\partial X^2} - \rho \frac{\partial^2 Y_2(X,t)}{\partial t^2} \right] \delta Y_2(X,t) dX dt$$
$$+ \int_{t_1}^{t_2} \left[T_1(\Gamma_-(t),t) - T_2(\Gamma_+(t),t) \right] \delta \Gamma_+(t) dt = 0. \tag{8}$$

The first and the fourth term in (8) are always zero since the variations of Y_2 in initial and final times are zero. The string wraps and unwraps the bridge perfectly, which makes the string to remain stationary in its wrapped length. Therefore, velocity $\dfrac{\partial Y_2(X,t)}{\partial t}$ at $\Gamma_-(t)$ is zero. From (5) we get $G(\Gamma_-(t),t) = 0$. With the use of these two facts and (7c), the last term in (8) can be simplified to $\dfrac{\partial Y_2(\Gamma_+(t),t)}{\partial t} \delta \Gamma_+(t)$. From (8) equations of motion and boundary conditions can be identified as

$$T \frac{\partial^2 Y(X,t)}{\partial X^2} - \rho \frac{\partial^2 Y(X,t)}{\partial t^2} + \lambda(X) = 0, \qquad 0 \le X \le \Gamma_-(t), \tag{9}$$

$$T \frac{\partial^2 Y(X,t)}{\partial X^2} - \rho \frac{\partial^2 Y(X,t)}{\partial t^2} = 0, \qquad \Gamma_+(t) \le X \le L, \tag{10}$$

$$Y(0,t) = 0, \qquad Y(\Gamma_-(t),t) = Y_B(\Gamma_-(t)), \tag{11}$$

$$Y(\Gamma_+(t),t) = Y_B(\Gamma_+(t)), \qquad Y(L,t) = H_r, \tag{12}$$

$$\frac{\partial Y(\Gamma_+(t),t)}{\partial t} = 0. \tag{13}$$

We previously assumed that the string perfectly wraps and unwraps the bridge during its motion. Therefore, the solution of the wrapped portion is nothing but the geometry of the bridge. So, we are interested to solve equation of motion (10) with boundary conditions (12). The so called transversality condition (string remains tangent to the bridge at the point $X = \Gamma$) can be written as

$$\frac{\partial Y(\Gamma(t),t)}{\partial X} = A_p(B - 2\Gamma(t)). \tag{14}$$

Now we define the nondimensional parameters

$$\bar{x} = \frac{X}{L}, \;\; \bar{y}(\bar{x},\bar{\tau}) = \frac{Y(X,t)}{h}, \;\; \gamma = \frac{\Gamma(t)}{L}, \;\; b = \frac{B}{L}, \;\; \alpha = A_p \frac{L^2}{h}, \;\; K = \frac{EI}{TL^2}, \;\; \bar{\tau} = t\sqrt{\frac{T}{\rho L^2}}, \tag{15}$$

where h is any length comparable to the height of the bridge $(A_p B^2/4)$. Using the above nondimensional parameters we rewrite (10) in its nondimensional form as

$$\frac{\partial^2 \bar{y}(\bar{x},\bar{\tau})}{\partial \bar{x}^2} - \frac{\partial^2 \bar{y}(\bar{x},\bar{\tau})}{\partial \bar{\tau}^2} = 0, \qquad \gamma \le \bar{x} \le 1. \tag{16}$$

Note that now onward we will use nondimensional $\Gamma_+(t)$ as γ. Using geometry of the bridge (1) and nondimensional parameters (15), boundary conditions (12) can be written as

$$\bar{y}(\gamma,\bar{\tau}) = \alpha\gamma(b-\gamma), \qquad \bar{y}(1,\bar{\tau}) = \alpha h_r, \tag{17}$$

where $\alpha h_r = \dfrac{H_r}{h}$. Nondimensional form of (13) and transversality condition (14) are given by

$$\left.\frac{\partial \bar{y}(\bar{x},\bar{\tau})}{\partial \bar{\tau}}\right|_{\bar{x}=\gamma} = 0, \tag{18}$$

$$\left.\frac{\partial \bar{y}(\bar{x},\bar{\tau})}{\partial \bar{x}}\right|_{\bar{x}=\gamma} = \alpha(b-2\gamma). \tag{19}$$

Now we rescale the spacial domain with $x = \dfrac{\bar{x}-\gamma}{1-\gamma}$ and time domain with $\tau = \bar{\tau}$ and transverse displacement $\bar{y}(\bar{x},\bar{\tau})$ can be written as $\bar{y}(\bar{x},\bar{\tau}) = y(x,\tau)$. With this rescaling from (16) we get

$$\left[\frac{(x-1)^2\dot{\gamma}^2-1}{(1-\gamma)^2}\right]\frac{\partial^2 y}{\partial x^2} + \left[\frac{(x-1)}{(1-\gamma)}\left(\frac{2\dot{\gamma}^2}{(1-\gamma)} + \ddot{\gamma}\right)\right]\frac{\partial y}{\partial x} + \left[\frac{2(x-1)\dot{\gamma}}{(1-\gamma)}\right]\frac{\partial^2 y}{\partial x\partial\tau} + \frac{\partial^2 y}{\partial \tau^2} = 0. \tag{20}$$

where $y(x,\tau) = y$. Boundary conditions (17), (18) can be rewritten as

$$y(0,\tau) = \alpha\gamma(b-\gamma), \qquad y(1,\tau) = \alpha h_r, \tag{21}$$

$$(1-\gamma)\frac{\partial y(0,\tau)}{\partial \tau} - \dot{\gamma}\frac{\partial y(0,\tau)}{\partial x} = 0. \tag{22}$$

The rescaled form of the transversality condition (19) is

$$\left.\frac{\partial y(x,\tau)}{\partial x}\right|_{x=0} = \alpha(b-2\gamma)(1-\gamma). \tag{23}$$

Equation of motion (20) is a nonlinear equation where γ is an internal variable which is related to the dependent variable y by the relation (23).

3 LINEARIZATION OF THE EQUATION OF MOTION

We will linearize the nonlinear equation of motion (20) about static configuration of the string. We assume

$$y(x,t) = y_{st}(x) + \epsilon\tilde{y}(x,\tau), \tag{24}$$

and

$$\gamma = \gamma_{st} + \epsilon\tilde{\gamma}(\tau), \tag{25}$$

where ϵ is a very small positive number, $y_{st}(x)$ is the static configuration of the string, and γ_{st} is the distance of the contact point of string and obstacle for static string configuration, measured from the origin along X axis. We substitute (24) and (25) into (20), expand it in a Taylor series about $\epsilon = 0$ and then we collect ϵ^1 order terms which constitute the linearized equation of motion

$$\frac{\partial^2 \tilde{y}(x,\tau)}{\partial \tau^2} - \frac{1}{(1-\gamma_{st})^2}\frac{\partial^2 \tilde{y}(x,t)}{\partial x^2} - \frac{(1-x)}{(1-\gamma_{st})}\frac{\mathrm{d}y_{st}(x)}{\mathrm{d}x}\frac{\mathrm{d}^2\tilde{\gamma}}{\mathrm{d}\tau^2} - \frac{2\tilde{\gamma}}{(1-\gamma_{st})^3}\frac{\mathrm{d}^2 y_{st}(x)}{\mathrm{d}x^2} = 0. \tag{26}$$

Now, we substitute (24) and (25) into boundary condition (21). Then separating different order terms of ϵ and equating them to zero, we get

$$y_{st}(0,\tau) = \alpha\gamma_{st}(b-\gamma_{st}), \qquad \tilde{y}(0,\tau) = \alpha(b-2\gamma_{st})\tilde{\gamma}(\tau), \qquad y_{st}(1,\tau) = \alpha h_r, \qquad \tilde{y}(1,\tau) = 0. \tag{27}$$

Similarly, linearizing (22) we get

$$(1-\gamma_{st})\frac{\partial \tilde{y}(0,\tau)}{\partial \tau} - \dot{\tilde{\gamma}}\frac{\partial y_{st}(0,\tau)}{\partial x} = 0. \tag{28}$$

Using (27) we can choose the static configuration of the string as

$$y_{st}(x) = \alpha\gamma_{st}(b-\gamma_{st})(1-x) + \alpha h_r x, \tag{29}$$

which gives

$$y(x,\tau) = \alpha\gamma_{st}(b-\gamma_{st})(1-x) + \alpha h_r x + \epsilon\tilde{y}(x,\tau). \tag{30}$$

We substitute (30) into (26) and get

$$\frac{\partial^2 \tilde{y}(x,\tau)}{\partial \tau^2} + \alpha\left[\gamma_{st}(b-\gamma_{st}) - h_r\right]\frac{(1-x)}{(1-\gamma_{st})}\frac{\mathrm{d}^2\tilde{\gamma}}{\mathrm{d}\tau^2} - \frac{1}{(1-\gamma_{st})^2}\frac{\partial^2 \tilde{y}(x,t)}{\partial x^2} = 0. \tag{31}$$

(30) has to satisfy the transvarsality condition (23). We substitute (30) and (25) into (23) which gives

$$-\alpha\left[\gamma_{st}(b-\gamma_{st}) - h_r\right] + \epsilon\left.\frac{\partial \tilde{y}}{\partial x}\right|_{x=0} = \alpha(b-2\gamma_{st})(1-\gamma_{st}) - \epsilon\alpha\tilde{\gamma}(b-4\gamma_{st}+2) + \epsilon^2 2\alpha\tilde{\gamma}^2. \tag{32}$$

Now separating the coefficient of ϵ^0 from (32) and equating it to zero, we get

$$\gamma_{st}^2 - 2\gamma_{st} + b - h_r = 0, \tag{33}$$

which upon solving (33) gives us[4]

$$\gamma_{st} = 1 - \sqrt{1 + h_r - b}. \tag{34}$$

Similarly, collecting the coefficients of ϵ^1 from (32) and equating them to zero produces

$$\left.\frac{\partial \tilde{y}}{\partial x}\right|_{x=0} = -\alpha\tilde{\gamma}(b - 4\gamma_{st} + 2). \tag{35}$$

Now we will use Galerkin projection technique to obtain the reduced order equation for (31). We assume a solution

$$\tilde{y} = \alpha(b-2\gamma_{st})\tilde{\gamma}(1-x) + \sum_{i=1}^{N}\beta_i(t)\sin(i\pi x) \tag{36}$$

for the linearized equation of motion (31). Now, we substitute (36) into (35) which gives us

$$\tilde{\gamma} = -\frac{1}{2\alpha(1-\gamma_{st})}\sum_{i=1}^{N} i\pi\beta_i(t). \tag{37}$$

[4]The other root $1 + \sqrt{1 + h_r - b} > 1$ is not feasible.

We substitute (36) and (37) into (31), which gives

$$\sum_{i=1}^{N}\ddot{\beta}_i(t)\sin(i\pi x)-\frac{(b-2\gamma_{st}+\gamma_{st}^2-h_r)\,(1-x)}{2(1-\gamma_{st})^2}\sum_{i=1}^{N}i\pi\ddot{\beta}_i(t)$$
$$+\frac{1}{(1-\gamma_{st})}\sum_{i=1}^{N}i^2\pi^2\beta_i(t)\sin(i\pi x)=0. \tag{38}$$

The second term of (38) becomes zero with the use of (33). Multiplying (38) with $\sin(j\pi x)$ and integrating over $[0,\ 1]$, we can write the jth equation as

$$\ddot{\beta}_j(t)+\frac{j^2\pi^2}{(1-\gamma_{st})^2}\beta_j(t)=0. \tag{39}$$

From (39) we can say that the natural frequencies ($\frac{j\pi}{1-\gamma_{st}}$ for $j=1,\ 2,\ ...N$) are harmonic. Also, we have obtained same natural frequencies using an alternate method.

4 STUDY OF THE MATHEMATICAL MODEL FOR LARGE AMPLITUDE OSCILLATIONS

In the previous section, we investigated the natural frequencies of the system with linearizing the mathematical model about its static configuration. In this section, we will reduce (20) to a set of nonlinear ODE's using Galerkin projection method and numerially investigate their response. We assume a solution

$$y(x,\tau)=\alpha\gamma(b-\gamma)(1-x)+\alpha h_r x+\sum_{i=1}^{N}\beta_i(\tau)\sin(i\pi x) \tag{40}$$

to (20). (40) satisfies the boundary conditions (21). Satisfying the transversality condition (23) and zero velocity condition at the point of separation ($x=0$) with (40), we get the relation

$$\gamma^2-2\gamma+b-h_r-\frac{1}{\alpha}\sum_{i=1}^{N}i\pi\beta_i=0. \tag{41}$$

We calculate $\ddot{\gamma}$ from (41) and substitute it with (40) into (20). Then multiplying with $\sin(j\pi x)$ and integrating over $[0,\ 1]$, we get the jth equation as

$$\sum_{i=1}^{N}\left[K_{2ij}-\frac{i\pi}{2\alpha(1-\gamma)^2}\left[\sum_{m=1}^{N}m\pi\beta_m K_{4mj}-\alpha(b-2\gamma+\gamma^2-h_r)K_{3j}\right]\right]\ddot{\beta}_i(\tau)$$
$$+\frac{\dot{\gamma}^2}{(1-\gamma)^2}\left[\sum_{i=1}^{N}i\pi\left(3K_{4ij}-i\pi K_{1ij}\right)\beta_i+\alpha\left(2-3b+2\gamma-\gamma^2+3h_r\right)K_{3j}\right]$$
$$+\frac{2\dot{\gamma}}{(1-\gamma)}\sum_{i=1}^{N}i\pi\dot{\beta}_i K_{4ij}+\frac{1}{(1-\gamma)^2}\sum_{i=1}^{N}i^2\pi^2\beta_i K_{2ij}=0, \tag{42}$$

where $j=1,\ 2,\ ...N$ and the K's are defined as

$$K_{1ij}\ =\ \int_0^1(x-1)^2\sin(i\pi x)\sin(j\pi x)dx,$$

$$
\begin{aligned}
K_{2ij} &= \int_0^1 \sin(i\pi x)\sin(j\pi x)dx, \\
K_{3j} &= \int_0^1 (x-1)\sin(j\pi x)dx, \\
K_{4ij} &= \int_0^1 (x-1)\cos(i\pi x)\sin(j\pi x)dx.
\end{aligned}
$$

Equation (42) contains γ and $\dot{\gamma}$ which can be substituted in terms of β_i's and $\dot{\beta}_i$'s using (41). We solve (42) numerically using Matlab ODE solver ode45. Here we have taken $N = 3$ for the qualitative study of (42), with zero initial conditions except $\beta_1(0) = 2$. We have taken $b = 0.05$, $\alpha = 4/b^2$ (according to [12]) and $h_r = 0$. Amplitude plot for second mode is shown in figure 2. We can easily observe that the amplitude has some modulation with time. We observed similar modulations in amplitude for the other modes. Equation (42) also depends on

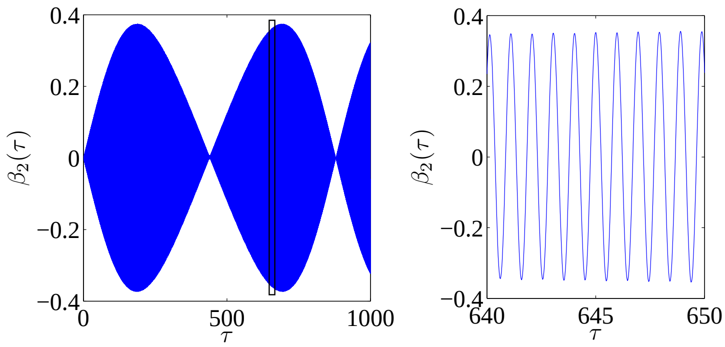

Figure 2: Left: Amplitude plot of $\beta_2(\tau)$. Right: A magnified view of the boxed portion from the left.

parameter h_r. We observed that the variation in the value of h_r just changes the frequency of the solutions but does not change the qualitative behavior of the solutions.

5 CONCLUSION

We have derived the governing equations for a string vibrating against a smooth unilateral obstacle, slightly modifying the model proposed in [12]. After rescaling the variables to make the string length constant, we have investigated the natural frequencies of the system in two different ways and have found that they are harmonic. These natural frequencies also depend on the static wraped length (γ_{st}), which can lead to a possible frequency modulation (for the fully nonlinear case). Next, from the numerical study of the fully nonlinear model, modal interactions is observed between different modes. Hence, the curved bridge plays a critical role for the Indian stringed instruments Sitar, Tanpura, Veena etc.

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