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WAVE PROPAGATION IN A ROTATING RANDOMLY VARYING GENERALIZED GRANULAR THERMOELASTIC MEDIUM

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Abstract. *In the present paper the problem of wave propagation in a random granular generalized thermo-elastic medium has been studied subject to the assumption that the frame of reference is rotating with an angular velocity* $\vec{\Omega}$. *The analysis is based on the dynamics of granular medium as propounded by N. Oshima. The smooth perturbation technique relevant to stochastic linearized operator differential equation has been employed. An integro-differential equation is governed by the mean field quantity, a 7-vector. All the field parameters are functions of space vector and time. A general dispersion equation for waves propagating in the random granular generalized thermal elastic medium has been obtained. The compression and shear wave propagation have been studied. The change of phase speed and attenuation of waves have been computed in different cases. Effects of non-random granular elastic medium, randomness and rotation of the frame of reference are discernible from analysis of the dispersion equation.*

1 INTRODUCTION

The propagation of elastic waves, among others, in a random medium was treated by Keller [1] by using the method of smooth perturbation. Keller's smooth perturbation technique requires an inversion (L_0^{-1}) of the deterministic differential operator L_0 to be performed in order to enable the evaluation of the mean field quantity. Accordingly, in every application of this technique, the computation of an appropriate Green's tensor becomes unavoidable.

In this paper we shall adopt Keller's method to analyze the mean generalized thermo-elastic wave in a slightly random, inhomogeneous granular elastic medium under a rotating frame of reference Schöenberg and Censor [2]. By this we mean that the density, the *Lamé* constants, the thermal expansion coefficient, the specific heat, the thermal diffusivity and parameters defining granular character of the elastic medium are all random functions of positions. The granularity has been defined under the dynamics of granular medium described by Oshima [3] as a Cosserat-continuum. The medium under consideration is a discontinuous one being composed of numerous large and small grains. Unlike a continuous body each element or grain not only translates but also rotates about its centre of gravity. The later motion affects the equation of motion by producing internal friction. It is assumed that the medium contains so many grains that they will never be separated from each other during the deformation and the grain has perfect elasticity. In our case the medium is considered under generalized thermo-elastic coupling. Ahmed [4] discussed Rayleigh waves in a thermo-elastic granular medium under initial stress. In a subsequent paper Ahmed [5] investigated propagation of Stoneley waves in a non-homogeneous orthotropic granular elastic medium. A statistical approach to wave propagation in granular medium will be found in Hudson [6]. Backman et. al. [7] used methods of statistical mechanics to model the arrangements and behaviour of granular material while Kitamura [8] used Markov process to study the mechanics of granular materials. In a two part article, Suiker, Borst and Chang [9] discussed the micro-mechanical modeling of granular medium. A second gradient micropolar constitutive theory was developed and plane wave propagation in an infinite non-random medium was discussed.

2 THE PROBLEM

Let $\vec{u}(u,v,w)$ be the displacement vector in a random granular generalized thermo-elastic medium, θ the generalized temperature and $\vec{\xi}(\xi,\eta,\zeta)$ the angular displacement vector of grains. Then the displacement equation of motion can be written as,

$$
\begin{aligned}
\rho(x)\ddot{\vec{u}}(\vec{x},t) + 2\vec{\Omega}\times\dot{\vec{u}}(\vec{x},t) + \vec{\Omega}\times(\vec{\Omega}\times\vec{u}(\vec{x},t)) \\
= \left(\lambda(\vec{x})+\mu(\vec{x})\right)\vec{\nabla}\left(\vec{\nabla}.\vec{u}(\vec{x},t)\right) + \mu(\vec{x})\nabla^2\vec{u}(\vec{x},t) + \vec{\nabla}\lambda(\vec{x})\left(\vec{\nabla}.\vec{u}(\vec{x},t)\right) \\
+ \left(\vec{\nabla}\mu(\vec{x})\right)\times\left(\vec{\nabla}\times\vec{u}(\vec{x},t)\right) + 2\left(\vec{\nabla}\mu(\vec{x}).\vec{\nabla}\right)\vec{u}(\vec{x},t) + A(\overrightarrow{x})\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{\xi}(\vec{x},t)\right) \\
+ \frac{\partial}{\partial t}\left(\vec{\nabla}A(\vec{x})\times\vec{\xi}(\vec{x},t)\right) - \vec{\nabla}\left(m(\vec{x})\{\theta(\vec{x},t)+t_1(\vec{x})\dot{\theta}(\vec{x},t)\}\right) \\
+ \overrightarrow{F}. \qquad (1)
\end{aligned}
$$

The rotation equation of motion of the granular elastic medium can be taken as

$$B(\vec{x})\nabla^2\left(\vec{\nabla}\times\vec{u}(\vec{x},t)\right)+\left(\vec{\nabla}B(\vec{x}).\vec{\nabla}\right)\left(\vec{\nabla}\times\vec{u}(\vec{x},t)\right)$$

$$+\left[-2A(\vec{x})\frac{\partial\vec{\xi}\,(\vec{x},t)}{\partial t}+B(\vec{x})\nabla^2\vec{\xi}(\vec{x},t)+\left(\vec{\nabla}B(\vec{x}).\vec{\nabla}\right)\vec{\xi}(\vec{x},t)\right]+\vec{\psi}=0. \quad (2)$$

Also the heat conduction equation can be taken as

$$\gamma(\vec{x})\left\{\dot{\theta}(\vec{x},t)+t_0\ddot{\theta}(\vec{x},t)\right\}-\vec{\nabla}.\left\{\nu(\vec{x})\vec{\nabla}[\theta(\vec{x},t)]\right\}$$

$$+\theta_0 m(\vec{x})\vec{\nabla}.\left\{\dot{\vec{u}}(\vec{x},t)+\delta_{lk}t_0\ddot{\vec{u}}(\vec{x},t)\right\}=\Theta \quad (3)$$

where $\dot{\theta}=\frac{\partial\theta(\vec{x},t)}{\partial t}$, $\ddot{\theta}=\frac{\partial^2\theta(\vec{x},t)}{\partial t^2}$ etc.

Also, λ,μ are Lame' parameters, ρ is density. Here $B(\vec{x})$ represents the new, i.e, the third elastic parameter and $A(\vec{x})$ is the coefficient of friction between the individual grains. $m(\vec{x})$ is the thermomechanical coupling parameter, ν is the thermal diffusivity and n,γ are defined to be

$$n=(\lambda+\mu)\alpha\,,\ \gamma=\rho C_s$$

where α, C_s are the thermal expansion coefficient and the specific heat respectively. $\vec{u},\vec{\xi},\theta$ are functions of space vector $\vec{x}(x,y,z)$ and time t. The functions, $\vec{F},\vec{\psi},\Theta$ are constant linear body force, angular body force, and heat source respectively. All the parameters including the generalized thermal parameters t_0 and t_1 and A,B are functions of $\vec{x}$. The entire frame of reference is assumed to be rotating with uniform angular velocity $\vec{\Omega}=\Omega\vec{n}$ where $\vec{n}$ is a unit vector representing the direction of the axis of rotation. The displacement equation of motion in the rotating frame of reference has two additional terms [2]:

(i) Centripetal acceleration $\vec{\Omega}\times(\vec{\Omega}\times\vec{u}(\vec{x},t))$ due to the time varying motion only,

(ii) The Coriolis acceleration $2\vec{\Omega}\times\dot{\vec{u}}(\vec{x},t)$,

Let us take for equations (1) -- (3)

$$\left\{\vec{u}(\vec{x},t),\vec{\xi}(\vec{x},t),\theta(\vec{x},t)\right\}=e^{-i\omega t}\left\{\acute{\vec{u}}(\vec{x}),\acute{\vec{\xi}}(\vec{x}),\ \acute{\theta}(\vec{x})\right\}. \quad (4)$$

Dropping the dashes we write the system of equations (1) – (3) in the linear differential operator form as

$$LV=f \quad (5)$$

where
$$L(\vec{x})=\begin{pmatrix} M(\vec{x}) & P(\vec{x}) & K(\vec{x}) \\ N(\vec{x}) & Q(\vec{x}) & 0 \\ R(\vec{x}) & 0 & S(\vec{x}) \end{pmatrix}$$

$$V(\vec{x})=\begin{pmatrix} \vec{u}(\vec{x}) \\ \vec{\xi}(\vec{x}) \\ \theta(\vec{x}) \end{pmatrix}$$

$$f(\vec{x})=\begin{pmatrix} -\vec{F}(\vec{x}) \\ -\vec{\psi}(\vec{x}) \\ -\Theta(\vec{x}) \end{pmatrix}$$

The linear differential operators $\{M,P,K,N,Q,U,R,W,S\}$ are defined below

$$M = \rho\left[\omega^2 - \vec{\Omega} \times \left(\vec{\Omega} \times\right) + 2i\omega\left(\vec{\Omega} \times\right)\right] + (\lambda + \mu)\vec{\nabla}\left(\vec{\nabla}.\right) + \mu(\nabla^2) + \left(\vec{\nabla}\lambda\right)\left(\vec{\nabla}.\right) + \left(\vec{\nabla}\mu\right) \times \left(\vec{\nabla} \times\right) + 2\left(\vec{\nabla}\mu\right).\vec{\nabla}$$
$$P = -i\omega[A\left(\vec{\nabla} \times\right) + \left(\vec{\nabla}A \times\right)]$$
$$K = -\vec{\nabla}\{m(1 - i\omega t_1)\}$$
$$N = B\nabla^2\left(\vec{\nabla} \times\right) + \left(\vec{\nabla}B.\vec{\nabla}\right)(\vec{\nabla} \times)$$
$$Q = 2iA\omega + B\nabla^2 + (\vec{\nabla}B.\vec{\nabla})$$
$$U = 0$$
$$R = \theta_0 m(i\omega + t_0\omega^2\delta_{lk})(\vec{\nabla}.)$$
$$W = 0$$
$$S = \gamma(i\omega + \omega^2 t_0) + \vec{\nabla}.(\nu\vec{\nabla})$$

It is now assumed that the physical parameters characterizing the medium under consideration are random functions of the space variable $\vec{x}$ the statistics of which are known, viz, the statistical mean, auto- and cross-correlation functions and the variance. Following Keller [1] the linear operator L is now represented in the perturbed form as

$$L = L_0 + \varepsilon L_1(\vec{x}) + \epsilon^2 L_2(\vec{x}). \tag{6}$$

where ε is the scale of fluctuation of the inhomogeneities of the medium and L_0 and (L_1, L_2) are the constant and the randomly fluctuating parts of L respectively. Under the circumstances it can be shown that the mean field quantity $< V(\vec{x}) >$ satisfies the integro−differential equation

$$[L_0 + \varepsilon < L_1 > + \epsilon^2 < L_2 > + \epsilon^2 < L_1 > L_0^{-1} < L_1 > - \epsilon^2 < L_1 L_0^{-1} L_1 >] < V(\vec{x}) >= f \tag{7}$$

where ε measures the scale of fluctuation of random inhomogeneities of the medium. It is now important to compute the inverse of the non-random linear differential operator L_0 i.e L_0^{-1} which is defined as

$$L_0\, G_{ij}(\vec{x}, \vec{x}') = \delta(\vec{x}, \vec{x}')\delta_{lj}.$$

3 SOLUTION

Towards solving the problem of mean wave propagation it is important to define non- random and random parts of the parameters characterizing the randomly fluctuating inhomogeneous medium. Following Karal and Keller [10, 11] and Chow [12], assuming small random fluctuation, the fluctuating parameters are expressed as

$$(\lambda, \mu, \rho, A, B, m, m^*, \gamma, \nu)(\vec{x}) = (\lambda_0, \mu_0, \rho_0, A_0, B_0, m_0, m_0^*, \gamma_0, \nu_0) + \varepsilon(\lambda_1, \mu_1, \rho_1, A_1, B_1, m_1, m_1^*, \gamma_1, \nu_1)(\vec{x}).$$

Here $m^*(\vec{x}) = m(\vec{x})t_1(\vec{x})$ such that $m_1^* = m_1(\vec{x})t_1(\vec{x})$,

Also $< \lambda_1(\vec{x}) > = < \mu_1(\vec{x}) > =< \rho_1(\vec{x}) > = < A_1(\vec{x}) > = < B_1(\vec{x}) > =< \gamma_1(\vec{x}) > =< \nu_1(\vec{x}) > = 0$

and specially

$< m_1(\vec{x}) > = m_2 \neq 0$, (constant), $< m_1^*(\vec{x}) > = m_3 \neq 0$, (constant).

These last two assumptions indicate that only the case of weak thermo-elasticity is being considered.

Using the perturbation operator equation (6) and putting $f = 0$, the field equation (5) can now be set in the form

$$[\, L_0 + \varepsilon L_1(\vec{x}) + \epsilon^2 L_2(\vec{x})\ ]V(\vec{x}) = 0. \tag{8}$$

such that

$$L_0 = \begin{pmatrix} M_0 & P_0 & 0 \\ N_0 & Q_0 & 0 \\ 0 & 0 & S_0 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} M_1 & P_1 & K_1 \\ N_1 & Q_1 & 0 \\ R_1 & 0 & S_1 \end{pmatrix}, \quad L_2=0$$

where

$M_0 = \rho_0\varpi^2 + (\lambda_0+\mu_0)\vec{\nabla}(\vec{\nabla}.) + \mu_0(\nabla^2)$

$P_0 = -i\omega[A_0(\vec{\nabla}\times)\ ,\quad K_0 = 0$

$N_0 = B_0\nabla^2(\vec{\nabla}\times)$

$Q_0 = 2A_0 i\omega + B_0\nabla^2$

$U_0 = 0,\quad R_0 = 0,\ W_0 = 0$

$S_0 = \gamma_0(i\omega + \omega^2 t_0) + \nu_0(\nabla^2)$

$M_1 = \rho_1\varpi^2 + (\lambda_1+\mu_1)\vec{\nabla}(\vec{\nabla}.) + \mu_1(\nabla^2) + (\vec{\nabla}\lambda_1)(\vec{\nabla}.) + (\vec{\nabla}\mu_1)\times(\vec{\nabla}\times) + 2(\vec{\nabla}\mu_1).\vec{\nabla}$

$P_1 = -i\omega[A_1(\vec{\nabla}\times) + (\vec{\nabla}A_1\times)]$

$K_1 = -\vec{\nabla}(m_1 - i\omega_1 m_1^*)$

$N_1 = B_1\nabla^2(\vec{\nabla}\times) + (\vec{\nabla}\mathrm{B}_1.\vec{\nabla})(\vec{\nabla}\ \times)$

$Q_1 = 2A_1 i\omega + (B_1\nabla^2.) + (\vec{\nabla}B_1.\vec{\nabla})$

$R_1 = \theta_0 m_1(i\omega + t_0\omega^2\delta_{lk})(\vec{\nabla}.)$

$U_1 = 0,\ W_1 = 0$

$S_1 = \gamma_1(i\omega + \omega^2 t_0) +\ \nu_1(\vec{\nabla}^2) + (\vec{\nabla}\nu_1.\vec{\nabla})$

and $\varpi^2 = \omega^2 - \vec{\Omega}\times(\vec{\Omega}\times) + 2i\omega(\vec{\Omega}\times)$

Next let us assume for the mean field equation (7)

$$<\ V(\vec{x}) >= \begin{pmatrix} \vec{A} \\ \vec{B} \\ C \end{pmatrix} e^{i\vec{k}.\vec{x}}. \tag{9}$$

The components of Green's tensor corresponding to $L_0 < V >= 0$, were computed earlier in the form

$$G_{lj} = \begin{pmatrix} G_0 & G_1 & 0 \\ G_2 & G_3 & 0 \\ 0 & 0 & G_4 \end{pmatrix} \tag{10}$$

where

$$G_4(r) = \frac{-1}{4\pi r}e^{i\beta r},\ r = |\vec{x} - \vec{x}| \tag{11}$$

$$\beta = \sqrt{\left\{\frac{\omega\gamma_{0(i+t_0\omega)}}{\nu_0}\right\}} = \sqrt{\{\tfrac{\omega\gamma_0}{2\nu_0}\}}[\{\sqrt{(1+t_0^2\omega^2)} + t_0\omega\ \}^{\frac{1}{2}} + i\{\sqrt{1+t_0^2\omega^2} - t_0\omega\ \}^{\frac{1}{2}}]$$
$$= \beta_1 + i\beta_2.$$

Here $\vec{x}$ is the field point and $\vec{x}$' is the source point where $\quad \vec{r} = |\vec{x} - \vec{x}|\ ,\ d\vec{r} = -d\vec{x}'.$

Now writing

$$e^{i\vec{k}.\vec{x}} f_1\,\vec{A} =< M_1G_0M_1 + M_1G_1N_1\ + P_1G_2M_1 + P_1G_3N_1 + K_1G_4R_1 > (\vec{A}\,e^{i\vec{k}.\vec{x}})$$

$e^{i\vec{k}.\vec{x}} f_2\, \vec{B} = < M_1 G_0 P_1 + M_1 G_1 Q_1 \; + P_1 G_2 P_1 + P_1 G_3 Q_1 > (\vec{B} e^{i\vec{k}.\vec{x}})$
$e^{i\vec{k}.\vec{x}} f_3\, C = < M_1 G_0 K_1 + P_1 G_2 K_1 \; = K_1 G_4 S_1 > (C e^{i\vec{k}.\vec{x}})$
$e^{i\vec{k}.\vec{x}} g_1\, \vec{A} = < N_1 G_0 M_1 + N_1 G_1 N_1 \; + Q_1 G_2 M_1 + Q_1 G_3 N_1 > (\vec{A}\, e^{i\vec{k}.\vec{x}})$
$e^{i\vec{k}.\vec{x}} g_2\, \vec{B} = < N_1 G_0 P_1 + N_1 G_1 Q_1 \; + Q_1 G_2 P_1 + Q_1 G_3 Q_1 > (\; \vec{B}\, e^{i\vec{k}.\vec{x}})$
$e^{i\vec{k}.\vec{x}}\, g_3\, C = < N_1 G_0 K_1 + Q_1 G_2 K_1 \; > (C e^{i\vec{k}.\vec{x}})$
$e^{i\vec{k}.\vec{x}} h_1\, \vec{A} = < R_1 G_0 M_1 + R_1 G_1 N_1 \; + S_1 G_4 R_1 > (\vec{A}\, e^{i\vec{k}.\vec{x}})$
$e^{i\vec{k}.\vec{x}} h_2\, \vec{B} = < R_1 G_0 P_1 + R_1 G_1 Q_1 \; > (\vec{B} e^{i\vec{k}.\vec{x}})$
$e^{i\vec{k}.\vec{x}} h_3\, C = < R_1 G_0 K_1 + S_1 G_4 S_1 \; > (C e^{i\vec{k}.\vec{x}}),$ (12)

Substituting these values in (7), setting f=0 and neglecting ε^3 and ε^4 terms we get the following equation:

$$(\lambda_0 + \mu_0)(\vec{k}.\vec{A})\vec{k} + (\mu_0 k^2 - \rho_0 \varpi^2)\vec{A} + \frac{i\omega A_0 B_0 k^2}{B_0 k^2 - 2i\omega A_0} \vec{k} \times (\vec{k} \times \vec{A}) + \epsilon^2 \frac{i\theta_0 m_2 (\vec{k}.\vec{A})((\omega - i\omega^2 t_0 \delta_{lk})(m_2 - \mathrm{i}\omega m_3)}{\gamma_0\,(\mathrm{i}\omega + \omega^2 t_0) - \nu_0 k^2} \vec{k} - \epsilon^2 i\theta_0 m_2 (\vec{k}.\vec{A})(m_2 - \mathrm{i}\omega m_3)(i\omega + \omega^2 t_0 \delta_{lk}) \int \vec{\nabla} G_4(r) e^{-i\vec{k}.\vec{r}} d\vec{r} - \epsilon^2 \frac{\omega A_0}{B_0\, k^2 - 2A_0 i\omega} \vec{k} \times \int g_1 \vec{A} e^{-i\vec{k}.\vec{r}}\, d\vec{r} - \epsilon^2 \frac{i\omega A_0 B_0\, k^2}{(B_0\, k^2 - 2A_0 i\omega)^2} \vec{k} \times \int g_2 (\vec{k} \times \vec{A}) e^{-i\vec{k}.\vec{r}} d\vec{r} - \epsilon^2 \int \left(\; f_1 \vec{A} - f_2 \frac{iB_0 k^2 (\vec{k} \times \vec{A})}{B_0\, k^2 - 2A_0 i\omega} \right) e^{-i\vec{k}.\vec{r}}\, d\vec{r}. \tag{13}$$

This is the general dispersion equation for the wave propagation in the random weakly thermal generalized granular elastic medium under the rotating frame of reference. The presence of ϖ^2 indicates dependence of $\vec{\Omega}$ on the propagation of waves. No ϵ order terms appears in the equation, indicating that the effects of random granular elastic character as also of thermal field are small to order ϵ^2only.

The ϵ^2order terms except one, are integrals involving numerous auto-and cross-correlation functions between various elastic, thermal and granular parameters of the interacting medium. These terms in effect represent the effect of randomness of the inhomogeneities of the medium. It would however be extremely laborious to undertake an analysis of the general dispersion equation. If however, we decide to omit all correlation functions, then the dispersion equation (13) reduces to

$$(\lambda_0 + \mu_0)(\vec{k}.\vec{A})\vec{k} + (\mu_0 k^2 - \rho_0 \varpi^2)\vec{A} + \frac{i\omega\, A_0\, B_0\, k^2}{B_0\, k^2 - 2A_0 i\omega} \left[(\vec{k}.\vec{A})\vec{k} - k^2 \vec{A} \,\right] + \epsilon^2 \frac{i\theta_0 m_2 (\,\vec{k}\,.\;\vec{A})(\omega - i\omega^2 t_0 \delta_{lk})(m_2 - i\omega m_3)}{\gamma_{0(i\omega + \omega^2 t_0)} - \nu_0 k^2} \vec{k} \; = 0 \tag{14}$$

We are left with only one ϵ^2 level term representing effects of randomness due to generalized thermal field alone as m_2, m_3 are included. No integral terms appear in the simplified equation. The effects of non-random granular elastic parameters and rotation of the frame of reference are also discernible from this equation.

CASE I: Analysis of equation (14)

For compression waves we get from equation (14)

$$(\lambda_0 + 2\mu_0)k^2\vec{A} = \rho_0\big[\omega^2\vec{A} - (\vec{\Omega}.\vec{A})\vec{\Omega} + \Omega^2\vec{A} + 2i\omega(\vec{\Omega}\times\vec{A})\big] - \epsilon^2\frac{i\theta_0 m_2(\omega - i\omega^2 t_0\delta_{lk})(m_2 - i\omega m_3)}{\gamma_{0(i\omega+\omega^2 t_0)-\nu_0 k^2}}k^2\vec{A}. \quad (15)$$

This means that for compression waves even non-random granular character of the medium is not discernible at all but rotation of the frame and random generalized thermal field are quite effective.

Taking $\vec{\Omega} = (0,0,\Omega)$, $\vec{A} = (A_1, A_2, A_3)$, we get two fourth degree equations in k:

$$\nu_0(\lambda_0 + 2\mu_0)k^4 - \{(\lambda_0 + 2\mu_0)(i\omega + \omega^2 t_0)\gamma_0 + \rho_0\nu_0\omega^2 + \epsilon^2 i\theta_0 m_2(\omega - i\omega^2 t_0\delta_{lk})(m_2 - i\omega m_3)\}k^2 + \rho_0\gamma_0\omega^2(i\omega + \omega^2 t_0) = 0 \quad (16)$$

and

$$\nu_0(\lambda_0 + 2\mu_0)k^4 - \{(\lambda_0 + 2\mu_0)(i\omega + \omega^2 t_0)\gamma_0 + \rho_0\nu_0(\omega^2 + \Omega^2) + \epsilon^2 i\theta_0 m_2(\omega - i\omega^2 t_0\delta_{lk})(m_2 - i\omega m_3) \pm 2\omega\Omega\rho_0\nu_0\}k^2 + \rho_0\gamma_0(\omega^2 + \Omega^2)(i\omega + \omega^2 t_0) \pm 2\omega\Omega\rho_0\gamma_0(i\omega + \omega^2 t_0) = 0. \quad (17)$$

Equation (16) is independent of angular velocity $\vec{\Omega}$ but dependent on generalised thermal parameters θ_0, t_0 and thermo-mechanical coupling parameters m_2 and m_3.

Equation (17) is dependent on angular velocity $\vec{\Omega}$ and generalized thermal parameters θ_0, t_0.

CASE II:

Rejecting term to the order of ϵ^2, one gets from (14), the non-random dispersion equation:

$$(\lambda_0 + \mu_0)(\vec{k}.\vec{A})\vec{k} + (\mu_0 k^2 - \rho_0\varpi^2)\vec{A} + \frac{i\omega A_0 B_0 k^2}{B_0 k^2 - 2A_0 i\omega}\big[(\vec{k}.\vec{A})\vec{k} - k^2\vec{A}\big] = 0. \quad (18)$$

This turns out to be the dispersion equation for waves dependent on $\vec{\Omega}$ and propagating in the non-random, non-thermal granular elastic medium.

For compression waves one gets the dispersion equation as

$$[(\lambda_0 + 2\mu_0)k^2 - \rho_0(\omega^2 + \Omega^2)]\vec{A} + \rho_0(\vec{\Omega}.\vec{A})\vec{\Omega} - 2i\omega\rho_0(\vec{\Omega}\times\vec{A}) = 0. \quad (19)$$

This means for compression waves dispersion equation becomes independent of granular character of the medium but is strongly dependent on rotation of the frame.

Setting $\vec{\Omega} = (0,0,\Omega)$, $\vec{A} = (A_1, A_2, A_3)$, one gets from (19) the following three equations.

$$(\lambda_0 + 2\mu_0)k^2 = \rho_0\omega^2 \quad (20)$$

and

$$[(\lambda_0 + 2\mu_0)k^2 - \rho_0(\omega^2 + \Omega^2)] = \pm 2\omega\Omega\rho_0. \quad (21)$$

From equation (20) we get

$$k = \omega\sqrt{\frac{\rho_0}{(\lambda_0 + 2\mu_0)}} = k_c, (say).$$

Here k depends on λ_0, μ_0, ρ_0 but independent of angular velocity $\vec{\Omega}$ of the frame of reference.

Again from (21) we get

$$k^2 = \frac{\rho_0(\omega^2 + \Omega^2) \pm 2\omega\Omega\rho_0}{(\lambda_0 + 2\mu_0)}.$$

In these cases k strongly depends on angular velocity $\vec{\Omega}$.

CASE III:

The integrals in the general dispersion equation (13) consist of a large number of terms involving various auto-and cross-correlation functions between the thermal and granular elastic parameters. Consequently the task of analyzing the equation for determining the effect of thermal field in general and for low and high frequencies becomes extremely cumbrous and laborious. Besides these, it may be considered more important to study the effect of random variation of granular character of the medium on the phenomena of elastic wave propagation in the generalized thermal field. With this end in view we first decide to consider the case where

$$< m(\vec{x})m(\vec{x}') >= R_{mm}(r) \neq 0,$$
$$< A(\vec{x})A(\vec{x}') >= R_{AA}(r) \neq 0,$$
$$< B(\vec{x})B(\vec{x}') >= R_{BB}(r) \neq 0,$$
$$< \rho(\vec{x})\rho(\vec{x}') >= R_{\rho\rho}(r) \neq 0$$

but

$$R_{\lambda\lambda}(r) = 0 = R_{\mu\mu}(r)$$

and all cross correlation functions like $R_{\rho\mu}(r)$, etc., are zero.

It is clear that even under this simplification a large number of terms involving the non vanishing correlation functions remain to be considered for the proposed analysis. Obviously the analysis will still be lengthy and hence further simplification becomes imperative. Simplified case where $R_{mm}(r) \neq 0$ but all other correlation functions vanish is discussed bellow.

CASE IV:

The simplified case of $R_{mm}(r) \neq 0$ but all other $R_{ij}(r) = 0$.
The equation (13) now simplifies to

$$(\lambda_0 + \mu_0)(\vec{k}.\vec{A})\vec{k} + \mu_0 k^2\vec{A} - \rho_0[\omega^2\vec{A} - (\vec{\Omega}.\vec{A})\vec{\Omega} + \Omega^2\vec{A} - 2i\omega(\vec{\Omega} \times \vec{A})] + \frac{i\omega\, A_0 B_0 k^2}{B_0 k^2 - 2A_0 i\omega}[(\vec{k}.\vec{A})\vec{k} - k^2\vec{A}] + \epsilon^2 \frac{i\theta_0 m_2(\vec{k}.\vec{A})(\omega - i\omega^2 t_0\delta_{lk})(m_2 - i\omega m_3)}{\gamma_{0(i\omega+\omega^2 t_0) - \nu_0 k^2}}\vec{k} - \epsilon^2\theta_0 m_2(\vec{k}.\vec{A})\vec{k}(m_2 - i\omega m_3)(i\omega + t_0\omega^2\delta_{lk})\int_0^\infty (R_{mm} - m_2^2)G_4(r)e^{-i\vec{k}.\vec{r}}\, d\vec{r} = 0. \quad (22)$$

For **compression waves** equation (22) reduces to

$$(\lambda_0 + 2\mu_0)k^2\vec{A} - \rho_0[\omega^2\vec{A} - (\vec{\Omega}.\vec{A})\vec{\Omega} + \Omega^2\vec{A} - 2i\omega(\vec{\Omega} \times \vec{A})] + \epsilon^2 k^2\vec{A}\frac{i\theta_0 m_2(\omega - i\omega^2 t_0\delta_{lk})(m_2 - i\omega m_3)}{\gamma_{0(i\omega+\omega^2 t_0) - \nu_0 k^2}} - \epsilon^2\theta_0 m_2 k^2\vec{A}(m_2 - i\omega m_3)(i\omega + t_0\omega^2\delta_{lk})\int_0^\infty (R_{mm} - m_2^2)G_4(r)e^{-i\vec{k}.\vec{r}}\, d\vec{r} = 0. \quad (23)$$

It is observed that for compression waves the term representing effect of granular character of the medium again disappears. It is concluded that propagation of compression waves is independent of granularity of the medium but dependent on rotation vector$\vec{\Omega}$ and generalized thermo elastic peremeters θ_0, t_0. On the other hand the thermal effects are discernible to term to the order of ϵ^2 in this case. Both the non random and random parts of thermal parameters influence the propagation of compression waves in the medium. It may be pointed out that the

mean and auto- correlation function of the thermo-mechanical coupling parameter greatly influence the mean wave propagation. The effect, however, is small to the order of ϵ^2 only. The equation (23) will be studied numerically after performing the integration involved in the equation in a later section.
For propagation of **shear waves** the equation (22) reduces to

$$\mu_0 k^2\vec{A} - \rho_0\left[\omega^2\vec{A} - \left(\vec{\Omega}.\vec{A}\right)\vec{\Omega} + \Omega^2\vec{A} - 2i\omega\left(\vec{\Omega}\times\vec{A}\right)\right]$$

$$-i\omega\, A_0 B_0 k^4\left(\frac{B_0\, k^2 + 2iA_0\omega}{B_0^2 k^4 + 4\omega^2 A_0^2}\right)\vec{A} = 0. \tag{24}$$

It is observed that the randomness has no effect on wave propagation in this case. Terms to the order ϵ^2 disappear. Moreover the propagation of shear waves is completely independent of thermal character of the medium. Thermal parameters including the thermo-mechanical coupling parameters do not have any impact on mean wave propagation in this case even to the ϵ^2order terms. However the granularity of the medium effectively determines the shear wave propagation in the medium.

<u>Discussion on equation (23)</u>: <u>Compression waves</u> :

Equation (23) can be written as

$$(D + \epsilon^2 D_5)\vec{A} + \rho_0\left[\left(\vec{\Omega}.\vec{A}\right)\vec{\Omega} - 2i\omega\left(\vec{\Omega}\times\vec{A}\right)\right] = 0. \tag{25}$$

where

$$D_5 = D_3 - D_4,$$

$$D = (\lambda_0 + 2\mu_0)k^2 - \rho_0(\omega^2 + \Omega^2),$$

$$D_3 = \frac{i\theta_0 m_2(\omega - i\omega^2 t_0\delta_{lk})(m_2 - i\omega m_3)}{\gamma_{0(i\omega+\omega^2 t_0) - \nu_0 k^2}}k^2,$$

$$D_4 = \theta_0 m_2 k^2(m_2 - i\omega m_3)(i\omega + t_0\omega^2\delta_{lk})\int_0^\infty (R_{mm} - m_2^2)\, G_4(r)e^{-i\vec{k}.\vec{r}}\, d\vec{r}.$$

From equation (25) we get

$(D + \epsilon^2 D_5)(A_1, A_2, A_3) + \rho_0[(0,0,\Omega^2 A_3) + 2i\omega(\Omega A_2, -\Omega A_1, 0)] = 0.$

From this we can obtain the following two equations

$$(\lambda_0 + 2\mu_0)k^2 - \rho_0\omega^2 + \epsilon^2(D_3 - D_4) = 0 \tag{26}$$

and

$$(\lambda_0 + 2\mu_0)k^2 = \rho_0[(\omega^2 + \Omega^2) + 2\omega\Omega] - \epsilon^2(D_3 - D_4) \tag{27}$$

These two dispersion equations (26) and (27) may be analyzed in a separate section.

Discussion on equation (24): Shear waves:

Equation (24) can be written as

$$D_7\vec{A} + \rho_0\left[\left(\vec{\Omega}.\vec{A}\right)\vec{\Omega} - 2i\omega\left(\vec{\Omega}\times\vec{A}\right)\right] = 0 \tag{28}$$

where, $D_7 = \mu_0 k^2 - \rho_0(\omega^2 + \Omega^2) + D_{6,}$

$D_6 = -i\omega\, A_0 B_0 k^4\left(\frac{B_0\, k^2 + 2iA_0\omega}{B_0^2 k^4 + 4\omega^2 A_0^2}\right).$

From equation (28) we get

$$D_7(A_1, A_2, A_3) + \rho_0[(0,0,\Omega^2 A_3) + 2i\omega(\Omega A_2, -\Omega A_1, 0)] = 0.$$

From this we get the following two equations

$$\mu_0 k^2 = \rho_0\omega^2 - D_6 \tag{29}$$

and

$$\mu_0 k^2 = \rho_0[(\omega^2 + \Omega^2) + 2\omega\Omega] - D_6 \tag{30}$$

The pair of shear waves dispersion equation (29) and (30) are independent of randomness and consequently not of much interest; however these equation may be analyzed in a subsequent section.

4 ANALYSIS OF RESULTS

The mean wave propagation under specified circumstances is characterized by the two physical situations: the compression wave propagation is dependent of granularity but the shear wave propagation is dependent only on granularity but not on the thermal field. The shear wave propagation is actually independent of randomness to the order of ϵ^2. The physical implication may be found out only by carrying out a detailed numerical computational work firstly based on equations (22) and (23) and then on the general dispersion equation (13). These studies will be carried out latter in separate sections. In all cases the correlation functions will be chosen exponentially decaying forms such as [Chow]:
$R_{mm} = m_2^2 e^{-br}, b > 0.$
In equation (25) it has been assumed that

$$D_3 = \frac{i\theta_0 m_2(\omega - i\omega^2 t_0 \delta_{lk})(m_2 - i\omega m_3)}{\gamma_{0(i\omega+\omega^2 t_0)-\nu_0 k^2}} k^2.$$

Neglecting $\omega^3, \omega^4, \omega^5$ terms we get

$$D_3 = \omega^2 \theta_0 \left\{ \frac{m_2^2 \gamma_0}{\nu_0^2 k^2} - \frac{m_2}{\nu_0}(t_0 m_2 \delta_{lk} + m_3) - \frac{2\gamma_0 t_0 m_2}{\nu_0 k^2} \right\} - i\frac{\theta_0 m_2^2 \omega}{\nu_0}.$$

From above we note that D_3 is independent of A_0, B_0 but dependent on generalized thermal parameters θ_0, t_0 and thermo-mechanical coupling parameters m_2, m_3. Then following Chow; it can be shown that D_3 causes dissipation of waves and increase in phase speed.

$$D_4 = \theta_0 m_2 k^2 (m_2 - i\omega m_3)(i\omega + t_0 \omega^2 \delta_{lk}) \int_0^\infty (R_{mm} - m_2^2\, G_4(r) e^{-i\vec{k}.\vec{r}}\, d\vec{r}.$$

Setting
$R_{mm}(r) = m_2^2 e^{-br}, b > o$ and using $\int\int\int F(r) e^{-i\vec{k}.\vec{r}}\, d\vec{r} = \frac{4\pi}{k}\int_0^\infty r\, F(\, r)\, sinkrdr$

$$D_4 = \frac{-b}{k^2}\theta_0 (m_2 - i\omega m_3) m_2^3 (i\omega + t_0\omega^2 \delta_{lk})(2i\beta_1 - 2\beta_2 - b)$$

where $\beta = \beta_1 + i\beta_2$,

$$\beta_1 = \sqrt{\{\frac{\omega\gamma_0}{2\nu_0}\}}\{t_0\omega\ + \sqrt{(1 + t_0^2\omega^2)}\ \}^{\frac{1}{2}},$$

$$\beta_2 = \sqrt{\{\frac{\omega\gamma_0}{2\nu_0}\}}\{-\, t_0\omega\ + \sqrt{(1 + t_0^2\omega^2)}\ \}^{\frac{1}{2}}.$$

From above expression of D_4 we note that D_4 is independent of granular character of the medium but dependent on generalized thermal parameters. The dispersion equation (23) for compression waves becomes

$$(\lambda_0 + 2\mu_0)k^2\vec{A} - \rho_0\big[\omega^2\vec{A} - \big(\vec{\Omega}.\vec{A}\big)\vec{\Omega} + \Omega^2\vec{A} - 2i\omega\big(\vec{\Omega}\times\vec{A}\big)\big]$$
$$+\, \epsilon^2 k^2 \vec{A}\Big[\frac{i\theta_0 m_2(\omega - i\omega^2 t_0\delta_{lk})(m_2 - i\omega m_3)}{\gamma_{0(i\omega+\omega^2 t_0)-\nu_0 k^2}}$$
$$+\epsilon^2\frac{b}{k^2}\theta_0(m_2 - i\omega m_3)m_2^3(i\omega + t_0\omega^2\delta_{lk})(2i\beta_1 - 2\beta_2 - b)\Big]\vec{A} = 0.$$

From the last term it may be concluded that the phase velocity and attenuation depend uponb^{-1}, the correlation length for the thermo-mechanical coupling parameter.

Eliminating A_1, A_2, A_3 from the resulting three equations the dispersion equation can be found out. Substituting

$$k = k_c - i\delta_c, \quad k_c = \sqrt{\frac{\rho_0(\omega^2+\Omega^2)}{\lambda_0+2\mu_0}},$$

5 CONCLUSIONS

1. The general dispersion equation (13) is independent of ϵ-order terms. This indicates that the effects of random variation of parameters are indeed small to the order of ϵ^2only.
2. The simplified equation (14) shows effect of random generalized thermal parameters, non-random granular parameters and angular velocity $\vec{\Omega}$ of the frame of reference.
3. In CASE I equation (16) indicates the dependence of k^2on $\vec{\Omega}$ and m_2, m_3.
4. In CASE II from equations (20) and (21) we note that k is independent of generalized thermal parameters t_0, t_1 and reference temperature θ_0.
5. In CASE IV from equation (23) we see that propagation of compression waves is independent of granularity of the medium but dependent on generalized thermal parameters. Also the mean and auto-correlation functions of the thermo-mechanical coupling parameters greatly influence the wave propagation.
6. In CASE IV from equation (24) we note that propagation of shear waves is completely independent of thermal character of the medium. Thermal parameters including the thermo-mechanical coupling parameters and generalized thermal parameters do not have any impact on the mean wave propagation.

6 REFERENCES

[1]. J. B. Keller, 1964, Stochastic equations and wave propagation in random media, Proc. Symp. App. Math., vol. 16, 145-170.

[2]. M. Schöenberg and D.Censor, 1973, Elastic waves in rotating media, Quart. Appl. Math, 31, 115-125.

[3]. N. Oshima, 1955. Dynamics of granular media, Memoirs of the unifying study of the basic problems in engineering sciences by means of geometry, (General Ed. K. Kondo), Bulletin of Earthquake Res. Inst., Tokyo, vol. 1, Div. D, p.563.

[4]. S. M. Ahmed, 2000, Rayleigh waves in a thermoelastic granular medium under initial stress, Int. J. Math. & Math. Scs. , vol. 23, No. 9, 627-637.

[5]. S. M. Ahmed, 2005, Stoneley waves in a non-homogeneous orthotropic granular elastic medium under the influence of gravity, Int. J. Math. & Math. Scs., vol. 19, p. 3145-3155.

[6]. J. A. Hudson, 1968, The scattering of elastic waves in granular media, Quart. J. Mech. & App. Math., vol. XXI, 4, p.487.

[7]. B. F. Beckman, 1983. Statistical mechanics of granular materials, Advances in the Mechanics & the Flow of Granular Materials, Ed. M. Shahinpoor, p.259, Gulf Publishing Co., Houston.

[8]. R. Kitamura, 1983. Application of Markov process to mechanics of soil and granular materials, Advances in the Mechanics & the Flow of Granular Materials, Ed. M. Shahinpoor, Gulf Publishing Co., Houston.
[9]. A. S. J. Suiker, R. De Borst, C. S. Chang, 2001, Micro-mechanical modeling of granular material, Parts I, II, Acta Mechanica, 149, 161-180, 181-200.
[10]. F. C. Karal, J. B. Keller, 1964, Elastic Electromagnetic and other waves in Random Media, J. Math. Phys., 5, 4, 537-547
[11]. J. B. Keller, F. C. Karal, 1966, Effective dielectric constant, permeability and Conductivity of a random medium and the velocity and attenuation coefficient of Coherent waves, J. Math. Physics, 7, 4, 661-670.
[12]. P. L. Chow, 1973, Thermoelastic wave propagation in a random medium and related problems, Int. J. Engg. Sc., vol. 11, 953-971, Pergamon Press.