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CHAOS IN A FRACTIONAL ORDER MAGNETO-FLEXIBLE ROD

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Abstract. *The present paper investigates the existence of chaos in a fractional order magneto-flexible rod (FOMFR). This is a system subjected to magnetic forces that can bend it while simultaneously subjected to external excitations produces complex and nonlinear dynamic behavior. The system is analyzed based on numerical simulation involving both, integer order calculus (IOC) and fractional order calculus (FOC) approaches. The time histories and pseudo phase portraits have been presented. Using the maximal Lyapunov exponent criteria based on Wolf's algorithm, we show that the FOMFR exhibits chaos. The results obtained can be used as a source for conduct others numerical and physical experiments in order to obtain a possible controller design to suppress the chaos of the FOMFR.*

1 INTRODUCTION

The theory of fractional calculus dates back to the birth of the theory of differential calculus, but its inherent complexity delayed the application of its associated concepts. In fact, fractional calculus is a natural extension of classical mathematics. Since the inception of the theory of differential and integral calculus, mathematicians such as Euler and Liouville developed their ideas about the calculation of non-integer order derivatives and integrals.

Perhaps the subject would be more aptly called “integration and differentiation of arbitrary order.”

The basic aspects of the theory of fractional calculus are outlined in [14]. Insofar as it concerns the application of its concepts, we can cite research in different areas such as viscoelastic damping [1], robotics and control [7-9], signal processing [15], electric circuits [10].

As for the adoption of this concept in other scientific areas, several researchers have been inspired to examine this new possibility. Some work has been carried out in the field of dynamical systems theory, but the proposed models and algorithms are still in the preliminary stage.

With these ideas in mind, this work introduces the fundamentals of fractional order calculus (FOC) in order to investigate, by means of numerical simulations, the modified motion equation of an elastic wide plate induced by two electromagnets and submitted to external periodic excitations.

2 FUNDAMENTALS OF FRACTIONAL CALCULUS

Fractional order calculus can represent systems with high-order dynamics and complex nonlinear phenomena using few coefficients, since the arbitrary order of the derivatives provides an additional degree of freedom to fit a specific behavior.

Numerous mathematicians have contributed to the history of fractional calculus by attempting to solve a fundamental problem to the best of their understanding [14].

Lacroix expressed the nth derivative (for $n \leq m$) in terms of Legrende’s symbol Γ for the generalized factorial. Recalling that

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\, dt \tag{1}$$

and starting, for instance, with the function $y = x^m$, Lacroix expressed it as follows:

$$\frac{d^n y}{d x^n} = \frac{m!}{(m-n)!}\, x^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}\, x^{m-n} \tag{2}$$

It was Liouville who engaged in the first major study of fractional calculus. Liouville’s first definition of a derivative of arbitrary order ν involved an infinite series. Here, the series must be convergent for some ν. Liouville’s second definition succeeded in giving a fractional derivative of x^{-a} whenever both x and are positive. Based on the definite integral related to Euler’s gamma integral, the integral formula can be calculated for x^{-a}. Note that in the integral

$$\int_0^\infty u^{a-1} e^{-xu}\, du \tag{3}$$

if we change the variables $t = x\,u$, then

$$\int_0^\infty u^{a-1} e^{-xu}\, du = \int \left(\frac{t}{x}\right)^{a-1} e^{-t} \frac{1}{x} dt =$$

$$\int \frac{t^{a-1}}{x^a x^{-1}} e^{-t} \frac{1}{x} dt = \frac{1}{x^a} \int_0^\infty t^{a-1} e^{-t}\, dt \tag{4}$$

Thus,

$$\int_0^\infty u^{a-1} e^{-xu}\, du = \frac{1}{x^a} \int_0^\infty t^{a-1} e^{-t}\, dt \tag{5}$$

However, in accordance with equation (1), this yields the integral formula

$$x^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty u^{a-1} e^{-xu}\, du \tag{6}$$

Consequently, by assuming that $\frac{d^\nu}{dx^\nu}\left(e^{ax}\right) = a^\nu e^{ax}$ for any $\nu > 0$, then

$$\frac{d^\nu x^{-a}}{d x^\nu} = \frac{\Gamma(a+\nu)}{\Gamma(a)} x^{-a-\nu} = \left(-1\right)^\nu \frac{\Gamma(a+\nu)}{\Gamma(a)} x^{-a-\nu} \tag{7}$$

The $(-1)^\nu$ term in the latter equation suggests the need to expand the theory to include complex numbers.

In 1884 Laurent published what is now recognized as the definitive paper on the fundaments of fractional calculus. Using Cauchy's integral formula for complex valued analytical functions and a simple change of notation to employ a positive υ rather than a negative υ will now yield Laurent's definition of integration of arbitrary order υ>0 :

$${}_c D_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x \left(x-t\right)^{\nu-1} f(t)\, dt \tag{8}$$

The appropriate definition of differentiation of arbitrary order is to integrate it up to a point from which the desired result can be obtained by conventional differentiation.

Let $\nu = m - \rho$ where, for convenience, m is considered the smallest integer larger than υ and $0 < \rho \le 1$.

Observe that,

$${}_c D_x^{\nu} f(x) = {}_c D_x^{m-\rho} f(x) \tag{9}$$

Thus

$${}_c D_x^{m-\rho} f(x) = \frac{d^m}{dx^m}\left[{}_c D_x^{-\rho} f(x)\right] \tag{10}$$

and consequently,

$$\frac{d^m}{dx^m}\left[{}_c D_x^{-\rho} f(x)\right] = \frac{d^m}{dx^m}\left[\frac{1}{\Gamma(\rho)} \int_c^x \left(x-t\right)^{\rho-1} f(t)\, dt\right] \tag{11}$$

In order to present Caputo's fractional derivative, let m be the smallest integer that exceeds α, thus enabling Caputo's fractional derivative of order $\alpha > 0$ to be defined as follows:

$$D_*^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau$$

$m-1 < \alpha < m$ (12)

Each researcher sought a definition and therefore different approaches, which has led to various definitions of differentiation and antidifferentiation of non-integer orders that are provenly equivalent. Although all these definitions may be equivalent, from one specific standpoint, i.e., for a specific application, some definitions seem more attractive.

Although all these definitions may be equivalent, from one specific standpoint, i.e., for a specific application, some definitions seem more attractive.

The two most commonly used definitions are the Caputo and Riemann-Liouville definitions. In this paper, we emphasize our simulations in Riemann-Liouville definitions, i.e.,

$$_cD_x^{\nu} f(x) = \frac{d^m}{dx^m}\left[\frac{1}{\Gamma(\rho)} \int_0^x (x-t)^{\rho-1} f(t)dt\right] \tag{13}$$

where, for convenience, m is considered the smallest integer larger than υ and $0 < \rho \leq 1$.

3 MECHANICAL SYSTEM-THEORETICAL MODEL

Many technical devices such as motors, generators, transformers, and fusion reactors are known to employ flexible rods in magnetic fields.

A system involving a flexible rod subjected to magnetic forces that can bend it while simultaneously subjected to external excitations produces complex and nonlinear dynamic behavior [3-5], which may present different types of solutions for its different movement-related responses [2]. This fact motivated us to analyze such a mechanical system based on modeling and numerical simulation involving both, (IOC) and (FOC), approaches.

A continuum model based on linear elastic and nonlinear magnetic forces was developed, which can be reduced to an oscillator model with a single degree of freedom using the Lagrangian formalism [6] and the Galerkin's method [4-6].

The mechanical system whose theoretical model is developed is shown in Figure 1. A flexible rod is clamped in a rigid base. The electromagnets pull the beam in opposite directions and intensity of the field is strong enough to deflect the rod from one side to another.

The electromagnets generate a magnetic field that induces a magnetization **M** per unit volume in the solid.

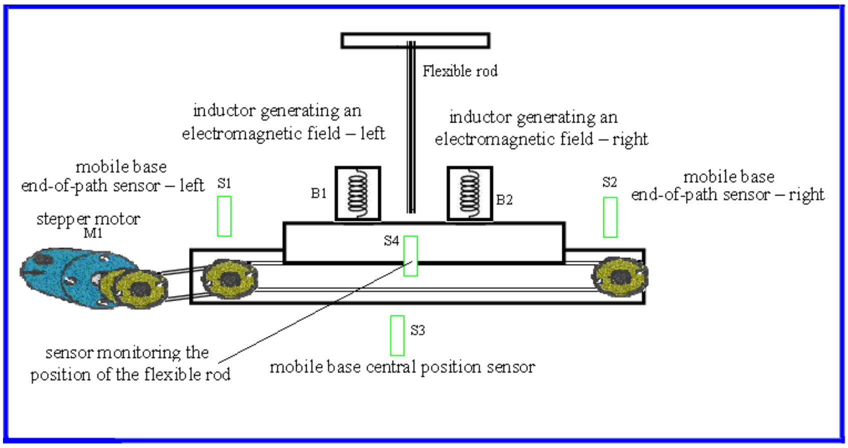

Figure 1: Mechanical system.

The rod can be modelled as a soft magnetic material where **M** is proportional to the local magnetic field in the solid [12]; i.e.,

$$\vec{M} = \left[\frac{\chi}{(\chi+1)}\right]\frac{\vec{B}}{\mu_0} \tag{14}$$

μ_0 is the magnetic permeability of a vacuum;

χ is the magnetic susceptibility

The field **B** can be written in terms of the field $\mathbf{B^0}$ produced by external magnets, and a field produced by the magnetization itself, $\mathbf{B^1}$. If self-forces on the rod are neglected then the external magnets produce both a force and moment distribution on the beam given by [12,13]

$$\vec{F} = \vec{M}\cdot\nabla B^0 \tag{15}$$

$$\vec{C} = \vec{M}\times\vec{B}^0 \tag{16}$$

These forces can be derived from a magnetic potential and therefore they are conservative. Thus,

$$W = -\frac{1}{2}\int \vec{M}\cdot\vec{B}^0\, dv \tag{17}$$

The non-linearities included in the analysis reflect the inhomogeneous nature of the magnetic field $\mathbf{B^0}$ [13] and the magnetic force and couple.

If the x, y components of $\mathbf{B_0}$ are introduced, defined by $B_{0x}=B_0\cos\alpha$ and $B_{0y}=B_0\sin\alpha$ and the local slope of the rod with the x axis by θ, then the magnetic energy potential then takes the form [2]

$$W = -\frac{\chi}{4\mu_0\mu_r}\int_0^L (B_1 + B_2\sin 2\theta + B_3\cos 2\theta)ds \tag{18}$$

with:

$$B_1 = (\mu_r+1)(B_{0x}^2 + B_{0y}^2); \tag{19}$$

$$B_2 = 2(\mu_r-1)B_{0x}B_{0y}; \tag{20}$$

$$B_3 = (\mu_r-1)(B_{0x}^2 - B_{0y}^2). \tag{21}$$

Here the integration is carried out over the original length of the rod and B_1, B_2 and B_3 are functions of the rod displacement. The nonlinear elastic forces are small even for the large displacements of the rod tip. Moreover, if a single mode approximation is made for the beam deformation, the elastic energy can be write in the form

$$P = \tfrac{1}{2} K y^2 + (\text{higher order terms}) \tag{22}$$

Thus, the potential for the elastic and magnetic forces will be

$$V = W + P \tag{23}$$

With the usage Lagrangian's formalism we can write

$$L = T - (W + P) \tag{24}$$

where T , W and P are kinetic, magnetic and elastic energies respectively.

Beside this, in Galerkin's method a suitable set of (orthogonal) basis function $\phi_j(x)$, which satisfy the boundary conditions, is chosen and the unknown displacement, expressed as

$$v(x,t)=\sum_{j=1}^{\infty} a_j(t)\phi_j(x) \tag{25}$$

A typical choice for the ϕ_j in vibration problems are the normal modes of the associated linear problem. One then substitutes expression (25) into the equation of motion and takes the inner product (i.e., integrates over the rod length) with ϕ_k , k = 1, 2, 3, ..., thus obtaining na infinite set of second order ordinary differential equations for the unknown modal coefficients $a_j(t)$. Since it is considered that the lowest mode is dominant in the motion of interest here, a single mode approximation

$$v(x,t)=a(t)\phi(x) \tag{26}$$

can be chosen where

$\phi_j(x)$ is required to satisfy

$$\phi(0)=\phi'(0)=\phi''(L)=0 \qquad D\phi''(L)+k\phi'(L)=0$$

$$\int_0^L \phi^2\,dx=1 \tag{27}$$

Finally, we can write the equation of motion this system with the dimensionless form [2],

$$\ddot{x}+\delta\dot{x}-x+x^3=F\cos\omega t \tag{28}$$

where:

$\delta>0$ is the damping constant , F is the forcing strength and ω is the forcing frequency.

In this article we investigated, by means of numerical simulations using Matlab/Simulink™ , the modified magneto-elastic dynamic equations (29) , i.e.,

$$x^{\lambda+1}+\delta x^{\lambda}-x+x^3=F\cos\omega t \tag{29}$$

We presented the time histories and pseudo phase portraits. Using the maximal Lyapunov exponent criteria based on Wolf's algorithm [17], we showed that the FOMFR exhibits chaos for some values of λ , ω, F and for δ = 0.2.

4 SOME SIMULATION RESULTS

We would like to illustrate some simulations results for non-integer order equation (29) in comparison to integer order for F = 0.18N , ω = 1rad/s and λ = [0.8(red line); 1.0(blue line); 1.2(green line)].

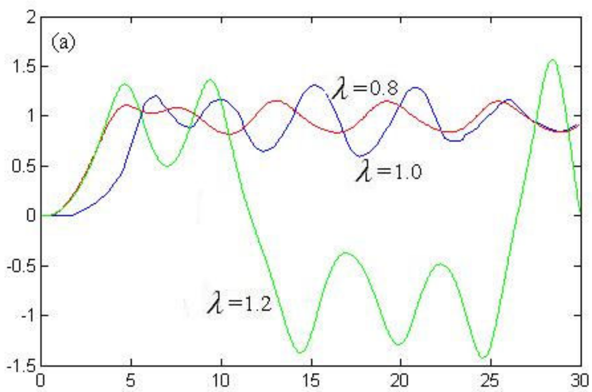

Figure 2 : x vs. t

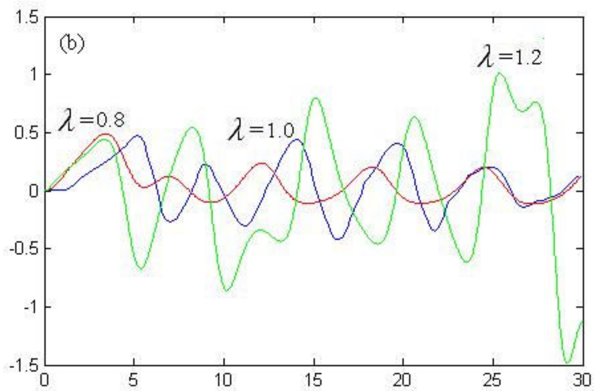

Figure 3: x^{λ} vs. t

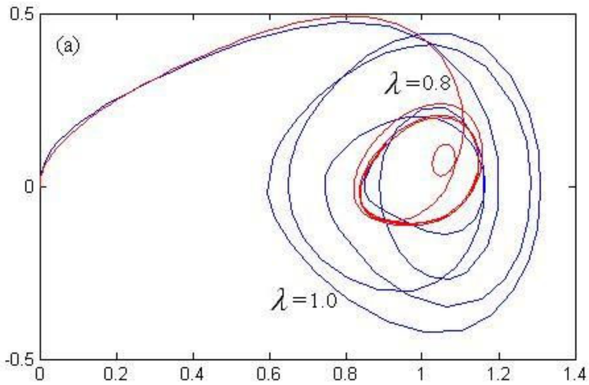

Figure 4 : Phase plane(x^{λ} vs. x), λ = {0.8, 1}

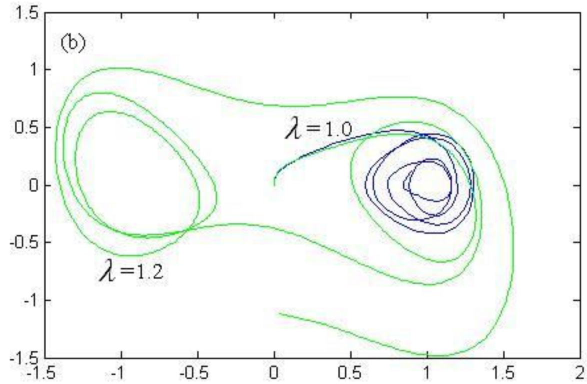

Figure 5: Phase plane (x^{λ} vs. x), λ = {1, 1.2}

We have illustrated too some simulations results for non-integer order equation (29) in comparison to integer order for F = 4N, ω = 1.12 rad/s and λ = [0.8(red line); 1.0(blue line); 1.2(green line)].

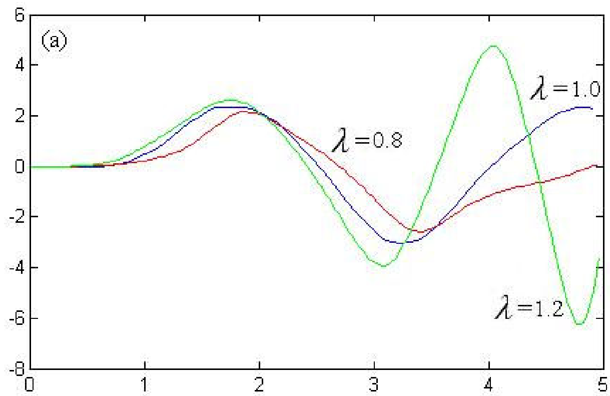

Figure 8: x vs. t;

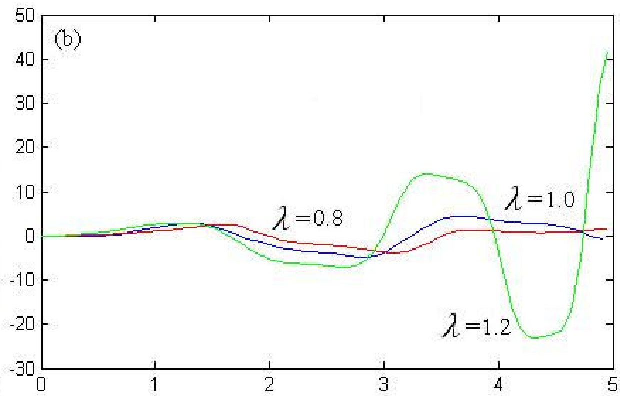

Figure 9: x^{λ} vs. t

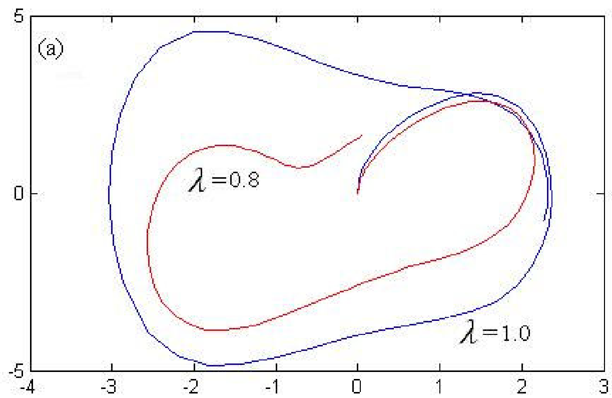

Figure 10: Phase plane, (x^{λ} vs. x), λ = {0.8, 1}

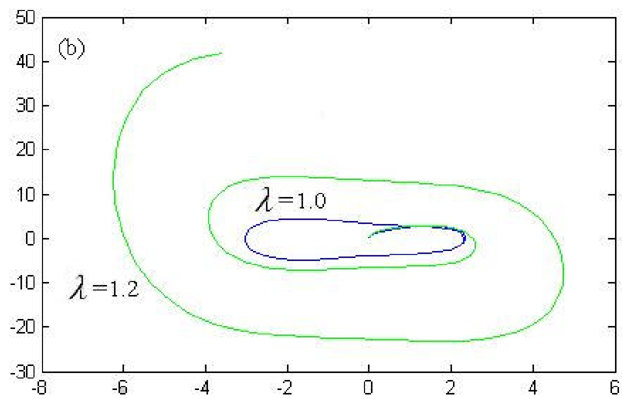

Figure 11: Phase plane , (x^{λ} vs. x), λ= {1, 1.2}

We applied the Wolf's algorithm for the determination of the first (or largest) Lyapunov exponent in FOMFR . We considered $0{,}7 < \lambda < 1{,}3$ in both cases, i.e., Case 1 (Fig. 12) and Case 2 (Fig. 13). These simulations were applied with Runge-Kutta solver, sampling time of data 0,005s and 40000 numbers of points in the file. In all cases the embedding dimension considered was 2 and the embedding delay was calculated individually.

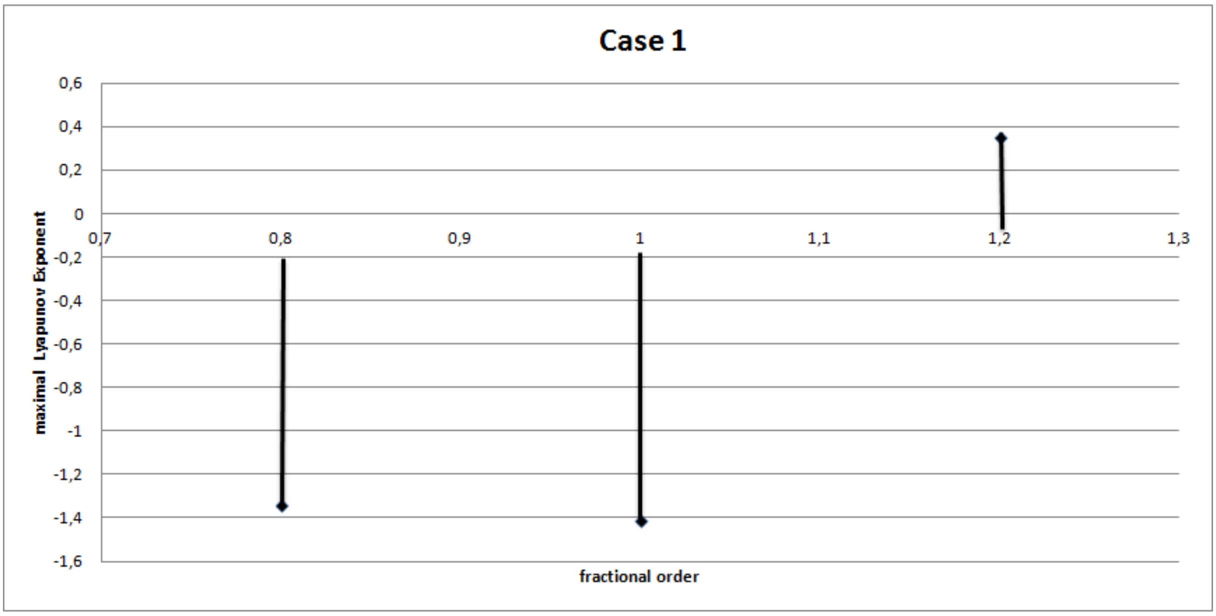

Figure 12: Maximal Lyapunov Exponent for F = 0.18 N , ω = 1rad/s and λ = 0.8; 1.0 and 1.2.

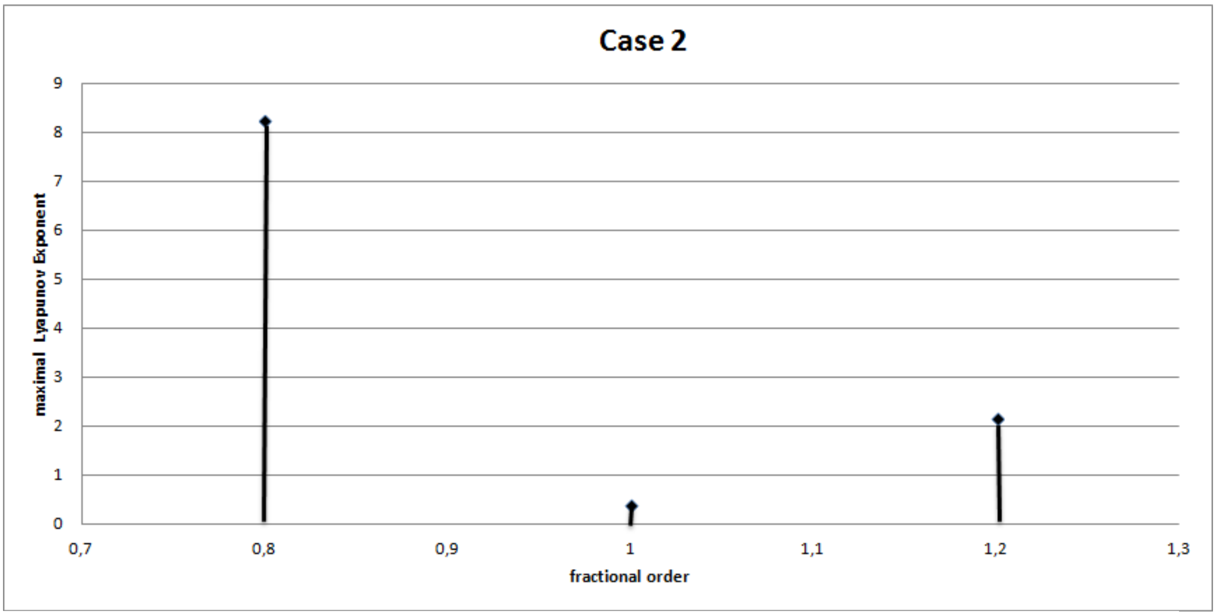

Figure 13: Maximal Lyapunov Exponent for F = 4 N, ω = 1.12 rad/s and λ = 0.8; 1.0 and 1.2.

Others maximal Lyapunov exponent parameters were minimal and maximal initial distance, maximal initial angle difference and evolution time per length element.

We found that chaos exist in the FOMFR behaves chaotic where the largest Lyapunov exponents are positive. One can note that the FOMFR behaves chaotic when λ = 1,2 in the Case 1 (Fig. 11). On the other hand, the FOMFR exhibits chaos for all values of λ investigated in the Case 2 (Fig.12).

5 CONCLUSIONS

In the present paper, we deal with a fractional order magneto-flexible rod (FOMFR). The system was analyzed based on numerical simulation involving both, integer order calculus (IOC) and fractional order calculus (FOC) approaches. The FOMFR system behaves chaotic where the largest Lyapunov exponents are positive, respectively, for λ = 1,2 in the Case 1(Fig.12) and for all values of λ in the range investigated for the Case 2 (Fig.13). The results obtained can be used as a source for conduct others numerical and physical experiments in order to obtain a possible controller design to suppress the chaos of the FOMFR.

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