11th International Conference on Vibration Problems
Z. Dimitrovová *et al.* (eds.)
Lisbon, Portugal, 9-12 September 2013

DESIGN CRITERIA FOR A PENDULUM ABSORBER TO CONTROL HIGH BUILDING VIBRATIONS

Fernando dos S. Oliveira[1], José L. V. de Brito[1], Suzana M. Avila*[2]

[1]Programa de Pós Graduação em Estruturas e Construção Civil (PECC)
Universidade de Brasília (UnB)
Campus Universitário Darcy Ribeiro, Brasília
jlbrito@unb.br, fernandoengciv@hotmail.com

[2]Faculdade do Gama – FGA
Universidade de Brasília (UnB)
Área Especial de Indústria Projeção A – UnB - Brasília - CEP: 72.444-240 - Setor Leste (GAMA)
avilas@unb.br

Keywords: Structural control, passive control, structural dynamics, tuned mass damper.

Abstract. *The increasing development of structural analysis techniques, the appearance of more resistant materials, and the high cost of construction in big metropolis, caused an elevation on the height of buildings, making these structures considerably more vulnerable to actions of dynamic loads such as wind and earthquakes [1]. Due to the problems caused by the action of these dynamic loads, vibration control has become a relevant issue in Civil Engineering. One of the most used vibration control systems is the Tuned Mass Damper (TMD). It basically consist of a mass-spring-damper system attached to the main structure, the frequency of the damper is tuned to a particular frequency, with the goal of making the TMD vibrate out of phase with the main system, thus transferring the energy system to the damper. The present study purpose is to evaluate the efficiency of a pendulum TMD when the structure is subjected to dynamic loads. Parametric studies are performed to define design criteria through frequency response function minimization searching its minimum maximum amplitudes. This minimization is achieved by an optimization iterative process. The behavior of a ten storey shear building is analyzed, reducing it to one degree of freedom through modal analysis [2]. A pendulum TMD is attached to it considering the optimal parameters obtained in this work. A set of general dimensionless optimal parameters for a pendulum TMD are presented in this research, they can be employed to the design of a pendulum to control any tall building, subjected to dynamic loads, with different mass and damping ratios.*

1 INTRODUCTION

The increasing development of structural analysis techniques, the appearance of more resistant materials, and the high cost of construction in big metropolis, caused an elevation on the height of buildings, making these structures considerably more vulnerable to actions of dynamic loads such as wind and earthquakes. This kind of vibrations is undesirable, not only because of structural safety but also because of human comfort [1].

Thus concern about civil structures protection including its contents and occupants is a global reality. An alternative widely studied in the last years is the structural control. Originally developed in aerospace engineering this technology was extended for civil engineering problems to protect bridges and high buildings from excessive dynamic loads. Structural control fundamentally changes structure stiffness and damping properties, both adding external device or applying external forces. It is classified on passive, active, hybrid or semi-active control [1,2].

Widely studied in the last years, passive control consists in adding one or more devices to the structure to absorb or transfer part of the energy from to the main structure. Passive control typical mechanisms are: mass dampers that control structural response by transferring the energy between the main structure and an auxiliary mass; structural dampers that dissipate energy while deforming themselves and base isolation systems that uncouple stricture moving from seismic soil vibrations.

Tuned mass dampers (TMD) are passive control devices, they consist in its simplest form by a mass-spring-dashpot system installed on the structure. The beginning of TMD appliance to civil structures was at the sixties on high buildings, bridges, towers and industrial chimneys to control vibrations caused by wind forces. A TMD tuned to the first structure natural frequency reduces substantially the response associated to the first mode vibration while little reducing or even increasing the response associated to higher modes. Moreover, a single TMD is more sensitive to discrepancies on the first natural frequency and/or damping ratio considered on the design. These limitations can be overcome by adding more than one damper, each one of them tuned to a different vibration natural frequency [3,4].

A TMD is a device composed by a mass-spring-dashpot attached to the structure aiming to reduce structural vibration response. The damper frequency is tuned to a particular frequency of the structure, since it will vibrate out of phase with the structural movement. The energy acting on the primary system is transferred to the secondary system, thereby reducing its vibration [2].

One of the alternative geometries of the TMD is the pendulum shape [3]. The pendulum is attached to the structure and its movement excites the device transferring portion of the energy from one system to another, reducing this way structural member request of energy dissipation. This type of damper has its vibration period depending on the length of the cable, and can only be considered a linear device when the vibration amplitudes are small.

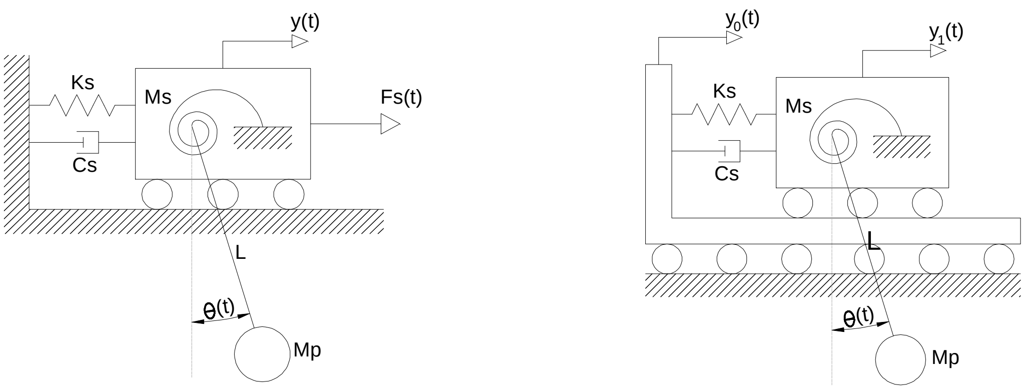
Lourenço [5] described the design, construction, implementation and performance of a prototype adaptative pendulum tuned mass damper (APTMD), demonstrating the performance improvements obtained when the tuned mass damper (TMD) parameters are optimized. In his study was considered the effect of adjusting the APTMD tuned frequency and damping ratio on a two storey test structure subjected to broadband and narrowband excitation.

The present study purpose is to evaluate the efficiency of a pendulum TMD when the structure is subjected to dynamic loads. Parametric studies are performed to define design criteria through frequency response function minimization searching its minimum maximum amplitudes. This minimization is achieved by an optimization iterative process. The behavior of a ten storey shear building is analyzed, reducing it to one degree of freedom through modal analysis [2]. A pendulum TMD is attached to it considering the optimal parameters obtained in this work. A set of general dimensionless optimal parameters for a pendulum TMD are presented in this research, they can be employed to the design of a pendulum to control any tall building, subjected to dynamic loads, with different mass and damping ratios.

2 MATHEMATICAL FORMULATION

2.1 Response frequency function to a system excited by harmonic force and base acceleration

Figure 1 shows a schematic description of a pendulum TMD attached to a main system composing a two degree of freedom model DOF, the main system is reduced to a 1 DOF model corresponding to the mode to be controlled [3]. Figure 1(a) presents the main system subjected to a force *Fs(t)* and Figure 1(b) the sytem subjected to a base acceleration.



(a) excitation due to a force *Fs(t)* (b) excitation due to a base acceleration

Figura 1 – Structure with a linear pendulum attached.

The equation of motion considering small displacement to Figure 1(a) system are:

$$\left(M_s + M_p\right)\ddot{y} + M_pL\ddot{\theta} + C_s\dot{y} + K_sy = F_s(t) \tag{1}$$

$$M_pL\ddot{y} + M_pL^2\ddot{\theta} + C_p\dot{\theta} + \left(K_p + M_pgL\right)\theta = 0 \tag{2}$$

and to Figure 1(b) system are:

$$\left(M_s + M_p\right)\ddot{y} + M_pL\ddot{\theta} + C_s\dot{y} + K_sy = -\left(M_s + M_p\right)\ddot{y}_0(t) \tag{3}$$

$$M_pL\ddot{y} + M_pL^2\ddot{\theta} + C_p\dot{\theta} + \left(K_p + M_pgL\right)\theta = -M_pL\ddot{y}_0(t) \tag{4}$$

Where M_s: main system modal mass; C_s: main system modal damping; K_s: main system modal stiffness; M_p: pendulum mass; C_p: pendulum damping; K_p: pendulum stiffness; L: cable length; g: gravity acceleration; $F_s(t)$: excitation modal force; $y(t)$: main system displacement; $\theta(t)$: pendulum angular displacement; $y(t)$: relative displacement of the main system to the base $y(t) = y_1(t) - y_0(t)$; $y_1(t)$: main system absolute displacement;$y_0(t)$: base displacement; $\ddot{y}_0(t)$: base acceleration. The pendulum natural frequency is $\omega_p = \sqrt{(K_p + M_p gL)/M_p L^2}$.

These equations can be rewritten using the following dimensionless terms:

$$\alpha = \frac{\omega_p}{\omega_s} \qquad \beta = \frac{\omega}{\omega_s} \qquad \delta = L/H \qquad \tau = \omega_s t \qquad \mu = M_p/M_s \qquad \eta = \frac{y}{H} \tag{5}$$

$$f_s(t) = F_s(t)/M_s {\omega_s}^2 H \tag{6}$$

$$\eta_0 = \frac{y_0}{H} \tag{7}$$

Substituting equations 5 and 6 into equations 1 and 2, and equations 5 and 7 into equations 3 and 4 the following equations are obtained:

$$(1 + \mu)\ddot{\eta} + \mu\delta\ddot{\theta} + 2\xi_s\dot{\eta} + \eta = f_s(t) \tag{8}$$

$$\ddot{\eta} + \delta\ddot{\theta} + 2\xi_p\alpha\delta\dot{\theta} + \alpha^2\delta\theta = 0 \tag{9}$$

to the system excited by an harmonic force and:

$$(1 + \mu)\ddot{\eta} + \mu\delta\ddot{\theta} + 2\xi_s\dot{\eta} + \eta = -(1 + \mu)\ddot{\eta}_0 \tag{10}$$

$$\ddot{\eta} + \delta\ddot{\theta} + 2\xi_p\alpha\delta\dot{\theta} + \alpha^2\delta\theta = -\ddot{\eta}_0 \tag{11}$$

to the system subjected to a base excitation

Where η: dimensionless ratio between relative displacement and structure highness; α: frequency ratio; β: forced frequency ratio; μ: mass ratio; ξ_s: mains system damping ratio; ξ_p: pendulum damping ratio; δ: ratio between cable length and structure highness $f_s(t)$: dimensionless excitation modal force; $\theta(t)$: pendulum angular displacement, and

$\dot{\eta} = d\eta/d\tau$; $\dot{\theta} = d\theta/d\tau$; $\ddot{\eta} = d^2\eta/d\tau^2$; $\ddot{\theta} = d^2\theta/d\tau^2$. Considering $f_s = e^{i\omega t} = e^{i\beta\tau}$, $\ddot{y}_0(t) = e^{i\omega t} = e^{i\beta\tau}$, substituting this values into equations 8, 9, 10 and 11, and solving the linear equation system it is obtained the dimensionless frequency response functions $H_\eta(\beta)$ e $H_\theta(\beta)$ to both excitation cases considered. These equations are shown on Tables 1 and 2.

Structure	$H_\eta(\beta) = \dfrac{-\beta^2 B_2 + i\beta B_1 + B_0}{\beta^4 A_4 - i\beta^3 A_3 - \beta^2 A_2 + i\beta A_1 + A_0}$ $B_0 = \alpha^2$ $\quad B_1 = 2\xi_P\alpha$ $\quad B_2 = 1$ $A_0 = \alpha^2$ $\quad A_1 = 2\xi_P\alpha + 2\xi_s\alpha^2$ $A_2 = 1 + 4\xi_P\alpha\xi_s + \alpha^2(1+\mu)$ $A_3 = 2\xi_s + 2\xi_P\alpha(1+\mu)$ $\quad A_4 = 1$
Pendulum	$H_\theta(\beta) = \dfrac{-\beta^2 B_2 + i\beta B_1 + B_0}{\beta^4 A_4 - i\beta^3 A_3 - \beta^2 A_2 + i\beta A_1 + A_0}$ $B_0 = 0$ $\quad B_1 = 0$ $\quad B_2 = 1$ $A_0 = \delta\alpha^2$ $\quad A_1 = -2\delta\xi_p\alpha - 2\delta\xi_s\alpha^2$ $A_2 = -\delta - 4\delta\xi_p\xi_s\alpha - \delta\alpha^2(1+\mu)$ $A_3 = -2\delta\xi_s - 2\delta\xi_p\alpha(1+\mu)$ $\quad A_4 = -\delta$

Table 1: Dimensionless frequency response when the structure is subjected to an harmonic loading.

Structure	$H_\eta(\beta) = \dfrac{-\beta^2 B_2 + i\beta B_1 + B_0}{\beta^4 A_4 - i\beta^3 A_3 - \beta^2 A_2 + i\beta A_1 + A_0}$ $B_0 = \alpha^2(1+\mu)$ $\quad B_1 = 2\xi_P\alpha(1+\mu)$ $\quad B_2 = 1$ $A_0 = -\alpha^2$ $\quad A_1 = -2\xi_P\alpha - 2\xi_s\alpha^2$ $A_2 = -1 - 4\xi_P\alpha\xi_s - \alpha^2(1+\mu)$ $A_3 = -2\xi_s - 2\xi_P\alpha(1+\mu)$ $\quad A_4 = -1$
Pendulum	$H_\theta(\beta) = \dfrac{-\beta^2 B_2 + i\beta B_1 + B_0}{\beta^4 A_4 - i\beta^3 A_3 - \beta^2 A_2 + i\beta A_1 + A_0}$ $B_0 = 1$ $\quad B_1 = 2\xi_s$ $\quad B_2 = 0$ $A_0 = -\delta\alpha^2$ $\quad A_1 = -2\delta\xi_p\alpha - 2\delta\xi_s\alpha^2$ $A_2 = -\delta - 4\delta\xi_p\xi_s\alpha - \delta\alpha^2(1+\mu)$ $A_3 = -2\delta(\xi_s + \xi_p\alpha\mu)$ $\quad A_4 = -\delta$

Table 2: Dimensionless frequency response when the structure is subjected to a base acceleration.

3 NUMERICAL STUDY

This work used Minmax numerical procedure to mimimize the maximum amplitude of the frequency response when the structure is subjected to an harmonic force.

Tsai e Lin [5] showed for undamped system cases that the reduction of resonance peak to its lowest value occurs using Den Hartog [7] values, as well as, through Minmax numerical search that are performed various parameter combinations in order to store those representing the lowest maximum amplitude.

Therefore, repeated attempts are performed varying each one of the parameters in the frequency response function, in every attempt the parameter range to be analyzed is fixed as well as the discretization of this range. When the numerical search is completed, a new range with values near of those that improve the TMD performance is considered to perform a new try.

The computational routine varies the system parameters (mass ratio, pendulum damping ratio, main structure damping ratio, frequency ratio, forced frequency ratio) and calculates the frequency response function value for each one of the analyzed cases. After it is stored the less response value found in all parameter combinations. The numerical search is ended when all parameters are combined, and the combination that produced the lowest response provides the optimal parameters.

Figure 2 compares the frequency response obtained to an undamped case using Den Hartog [7] parameters, the present work parameters and considering the uncontrolled case. It can be observed a similar behavior between the two controlled cases.

Vibration amplitudes for damped systems with $\xi_s \neq 0$ don´t have the same behavior of the undamped systems. Frequency response curves to different damping ratios don´t have fixed points, therefore closed solution for optimal parameters cannot be determined in the same way of the undamped case. However, optimal parameters can be found through a numerical search in a way to minimize the response peak.

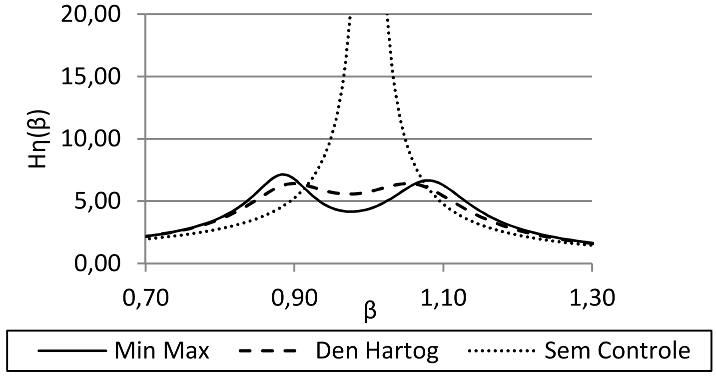


Figura 2 – Dimensionless frequency respons of the structure subjected to an harmonic force with Den hartog and Minmax parameters ($\mu = 0{,}05$ $\xi_s = 0{,}00$)

To determine optimal values to ξ_p, $H\eta$ e α to specific values of μ e ξ_s, it is performed a numerical iterative procedure where various values of ξ_p e α are combined on the response function equation in a way to define maximum amplitudes values. Once found and stored the maximum amplitudes, a new search is performed to find the lowest of them, in other words, the parameters that define the minimum maximal amplitudes are found. It is noteworthy that parameters presented on Tables 3 and 4 are general and therefore valid to any structure.

	ξs = 0,00			ξs = 0,02			ξs = 0,05			ξs = 0,10		
μ	ξp	α	Hη	ξp	α	Hη	ξp	α	Hη	ξp	α	Hη
0,0050	0,0426	0,9952	20,0962	0,0461	0,9930	11,7447	0,0495	0,9875	7,1093	0,0550	0,9740	4,2743
0,0100	0,0616	0,9900	14,1965	0,0632	0,9870	9,4717	0,0675	0,9807	6,2521	0,0700	0,9672	3,9774
0,0150	0,0749	0,9852	11,5941	0,0784	0,9810	8,2800	0,0821	0,9743	5,7197	0,0875	0,9580	3,7668
0,0200	0,0849	0,9805	10,0574	0,0898	0,9760	7,4581	0,0923	0,9683	5,3370	0,0987	0,9511	3,6085
0,0250	0,0954	0,9756	9,0053	0,0974	0,9710	6,8722	0,1024	0,9624	5,0400	0,1095	0,9440	3,4822
0,0300	0,1037	0,9709	8,2303	0,1050	0,9660	6,4228	0,1108	0,9568	4,8000	0,1176	0,9384	3,3735
0,0350	0,1122	0,9662	7,6295	0,1126	0,9610	6,0571	0,1216	0,9512	4,5996	0,1270	0,9314	3,2793
0,0400	0,1207	0,9615	7,1461	0,1202	0,9560	5,7510	0,1273	0,9458	4,4272	0,1323	0,9256	3,1947
0,0450	0,1278	0,9569	6,7459	0,1297	0,9510	5,4910	0,1367	0,9405	4,2785	0,1380	0,9199	3,1219
0,0500	0,1345	0,9523	6,4081	0,1373	0,9460	5,2691	0,1407	0,9353	4,1462	0,1439	0,9139	3,0566
0,0550	0,1399	0,9479	6,1177	0,1433	0,9413	5,0731	0,1478	0,9302	4,0288	0,1520	0,9082	2,9964
0,0600	0,1456	0,9434	5,8648	0,1464	0,9368	4,9020	0,1547	0,9251	3,9234	0,1567	0,9028	2,9413
0,0650	0,1514	0,9390	5,6419	0,1547	0,9320	4,7469	0,1590	0,9203	3,8273	0,1620	0,8973	2,8920
0,0700	0,1562	0,9346	5,4438	0,1590	0,9275	4,6085	0,1643	0,9154	3,7398	0,1705	0,8921	2,8446
0,0750	0,1628	0,9302	5,2657	0,1641	0,9230	4,4827	0,1690	0,9106	3,6595	0,1732	0,8868	2,8017
0,0800	0,1668	0,9259	5,1051	0,1687	0,9185	4,3679	0,1744	0,9059	3,5854	0,1813	0,8816	2,7611
0,0850	0,1723	0,9216	4,9589	0,1751	0,9139	4,2622	0,1781	0,9013	3,5167	0,1834	0,8766	2,7231
0,0900	0,1765	0,9174	4,8253	0,1783	0,9097	4,1653	0,1832	0,8966	3,4527	0,1927	0,8716	2,6875
0,0950	0,1810	0,9132	4,7025	0,1830	0,9053	4,0753	0,1892	0,8920	3,3929	0,1932	0,8668	2,6540
0,1000	0,1859	0,9090	4,5893	0,1875	0,9010	3,9915	0,1925	0,8875	3,3368	0,1972	0,8620	2,6222

Table 3 – Optimal pendulum parameters for a structure submitted to an harmonic force
($\xi_s = 0{,}00, \xi_s = 0{,}02, \xi_s = 0{,}05$ e $\xi_s = 0{,}10$)

	ξs = 0,00			ξs = 0,02			ξs = 0,05			ξs = 0,10		
μ	ξp	α	Hη	ξp	α	Hη	ξp	α	Hη	ξp	α	Hη
0,005	0,0438	0,994	20,1853	0,0452	0,9896	11,7928	0,0521	0,9828	7,1268	0,0544	0,965	4,2711
0,010	0,0619	0,9882	14,3966	0,0657	0,9828	9,5354	0,0688	0,9733	6,2864	0,0736	0,9524	3,9841
0,015	0,0737	0,9807	11,8361	0,0767	0,9755	8,3407	0,0798	0,9644	5,7725	0,0889	0,9416	3,7898
0,020	0,0874	0,9759	10,2393	0,0893	0,9688	7,5575	0,0949	0,9568	5,4022	0,101	0,9317	3,6425
0,025	0,098	0,9698	9,1959	0,1007	0,9622	6,9893	0,104	0,949	5,1191	0,1109	0,9222	3,5239
0,030	0,1042	0,9635	8,4149	0,1101	0,9557	6,5511	0,1123	0,9415	4,8914	0,1217	0,9135	3,425
0,035	0,1161	0,9578	7,8331	0,1157	0,949	6,1996	0,122	0,9344	4,7024	0,1296	0,9048	3,3406
0,040	0,1192	0,9518	7,3607	0,1256	0,9429	5,9092	0,1297	0,9273	4,5418	0,138	0,8966	3,2673
0,045	0,1279	0,9461	6,9717	0,1307	0,9365	5,6641	0,1374	0,9205	4,4033	0,1444	0,8884	3,2026
0,050	0,1346	0,9404	6,6461	0,1382	0,9304	5,4532	0,1431	0,9135	4,2821	0,1523	0,8807	3,145
0,055	0,1409	0,9347	6,3675	0,1461	0,9245	5,2703	0,1499	0,9069	4,1747	0,1584	0,873	3,0933
0,060	0,1462	0,929	6,1265	0,151	0,9184	5,1089	0,1563	0,9004	4,0787	0,1655	0,8657	3,0462
0,065	0,1523	0,9235	5,9139	0,1565	0,9124	4,9653	0,1622	0,894	3,9924	0,1704	0,8582	3,0036
0,070	0,1599	0,9181	5,726	0,1619	0,9066	4,8369	0,1699	0,8875	3,914	0,1778	0,8514	2,9643
0,075	0,1653	0,9127	5,5581	0,1693	0,901	4,7209	0,1735	0,8814	3,8424	0,1839	0,8446	2,9284
0,080	0,1703	0,9073	5,4068	0,1728	0,8951	4,6155	0,1794	0,8754	3,7769	0,1883	0,8375	2,895
0,085	0,1746	0,9018	5,2701	0,1788	0,8896	4,5193	0,1841	0,8693	3,7166	0,1943	0,8309	2,8641
0,090	0,18	0,8966	5,1455	0,1833	0,8839	4,4312	0,188	0,8631	3,6609	0,1997	0,8244	2,8355
0,095	0,1845	0,8913	5,0316	0,188	0,8784	4,3499	0,1948	0,8576	3,6092	0,2033	0,8175	2,8086
0,100	0,1893	0,8861	4,927	0,1934	0,8731	4,2748	0,1998	0,8516	3,5611	0,2098	0,8115	2,7835

Table 4 – Optimal pendulum parameters for a structure submitted to an harmonic base acceleration ($\xi_s = 0{,}00$, $\xi_s = 0{,}02$, $\xi_s = 0{,}05$ e $\xi_s = 0{,}10$)

Figures 3, 4 and 5 present grafically Table 3 results. In Figure 6 the dimensionless structure displacement (η) is compared using Den Hartog and Minmax parameters. Figures 7 and 8 compare optimal values of damping ratio (ξ_p), frequency ratio (α) obtained by Tsai and Lin (1993) with the results here obtained varying mass ratio and considering ($\xi_s = 0{,}02$). It is noticed good agreement between the two analysis.

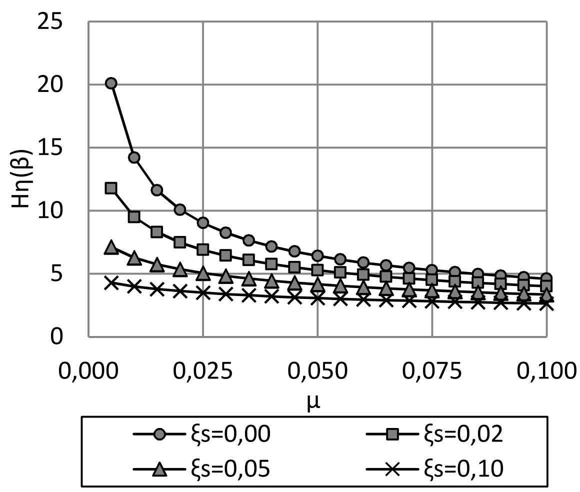


Figure 3 – Sistem maximum response using Minmax

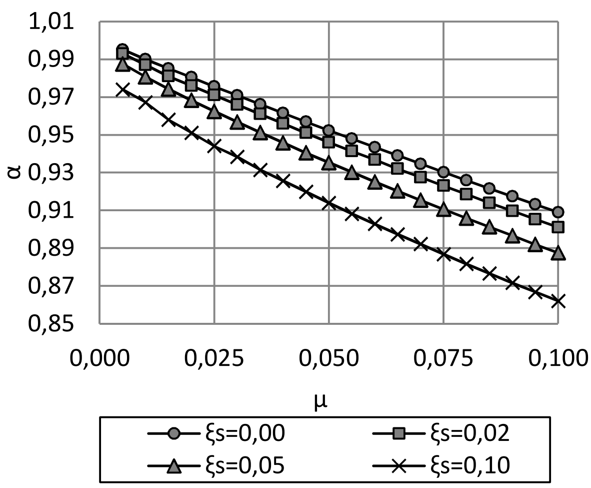


Figure 4 – Frequency ratio (harmonic force)

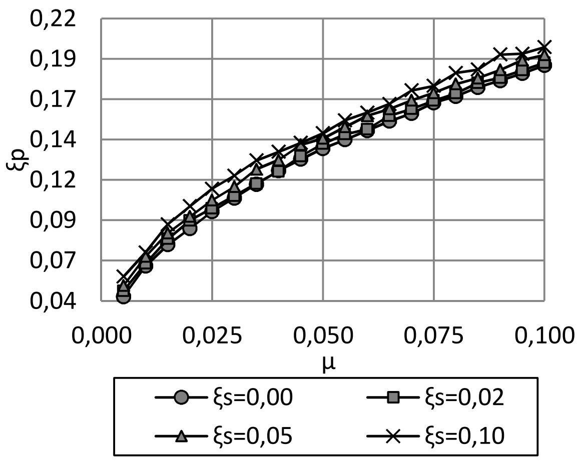


Figure 5 – Optimal damping (harmonic force

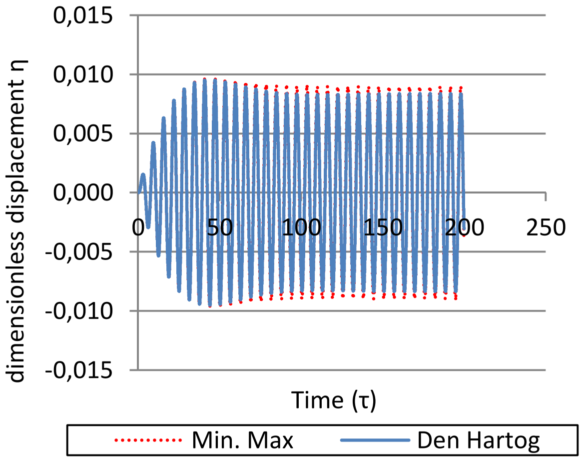


Figure 6 – Structure displacement (μ = 0,01 ξ_s = 0,02)

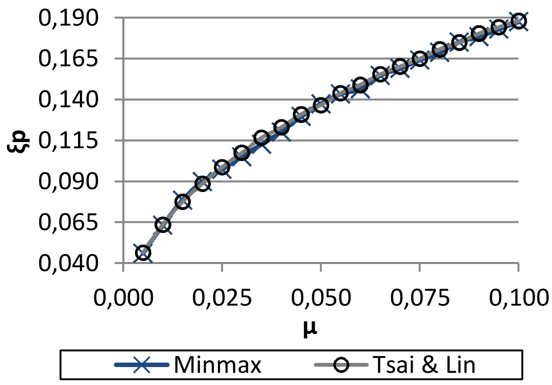


Figure 7 – Optimal damping ratio comparison.

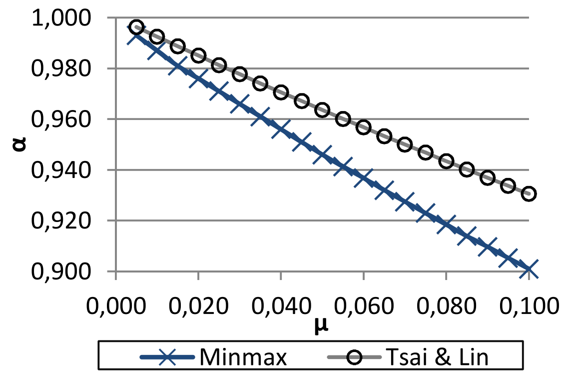


Figura 8 – Optimal mass ratio comparison

4 CONCLUSIONS

The TMD optimization aiming to reduce response function amplitude to an undamped structure subjected to an harmonic force was performed by Den Hartog. However, all types of structural systems have some level of damping, thus to find TMD optimal parameters, structural damping cannot be unvalued.

A parametric study was developed in this research to find optimal parameters to a damped structure submitted to an harmonic force. It was found an optimal value table to a damped structure : $\xi_s = 0{,}00$, $\xi_s = 0{,}02$, $\xi_s = 0{,}05$ e $\xi_s = 0{,}10$. It was observed that the frequency peaks tends to decrease with mass ratio increase. However , this parameter cannot increase too much because it implies a static load increase on the main system.

It was verificed that Minmax procedure is a efficient tool to pendulum TMD optimization. Using the suggested parameters in this work will improve considerably pendulum performance on reducing amplitude vibrations.

REFERENCES

[1] AVILA, S. M. Controle híbrido para atenuação de vibrações em edifícios. Tese de doutorado, Pontifícia Universidade Católica do Rio de Janeiro, Rio de Janeiro, Brasil, 2002.

[2] SOONG, T. T; DARGUSH, G. F. Passive energy dissipation systems in structural engineering. Chichester: John Wiley & Sons, 1997.

[3] OLIVEIRA F.S. Critérios de Projeto para Amortecedor tipo Pêndulo para Controle de Vibrações em Edifícios Altos. Dissertação de Mestrado – Departamento de Engenharia Civil, Universidade de Brasília, Brasília, 2012.

[4] CARNEIRO, R. B. Controle de vibrações em edifícios altos utilizando amortecedor de massa sintonizado múltiplo (AMSM). Dissertação de Mestrado – Departamento de Engenharia Civil, Universidade de Brasília, Brasília, 2004.

[5] LOURENCO, R. Design, Construction and Testing of an Adaptive Pendulum Tuned Mass Damper. Tese de Mestrado - Waterloo, Ontario, Canada, 2011.

[6] TSAI, H. C.; LIN, G. C. Optimum tuned-mass dampers for minimizing steady-state response of support-excited and damped systems. Department of Construction Engineering, National Taiwan Institute of Technology. Taipei, Taiwan. 2003

[7] DEN HARTOG, J. P. *Mechanical vibrations*. McGraw-Hill, New York, 1956.