

BAYESIAN STRUCTURAL SOURCE IDENTIFICATION USING GENERALIZED GAUSSIAN PRIORS

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Abstract. *The reconstruction of mechanical sources from vibration measurements is known to be an ill-posed inverse problem. One of the most widespread method to overcome this difficulty is the Tikhonov regularization. Such an approach consists in including in the formulation of the inverse problem some prior information on the nature of the sources to constrain the space of solutions. However, the reconstructed sources are generally too smooth to properly deal with localized sources. This paper aims at providing an identification methodology able to take advantage of prior information on the nature of excitation sources. More particularly, we will focus on the identification of localized sources. For this purpose, the bayesian framework is well adapted, since it offers a rigourous probabilistic approach to exploit our a priori knowledge on the nature of the sources to identify.*

The proposed bayesian formulation is based on the use of generalized gaussian priors, which provide a flexible way to introduce a priori information. Usually one use standard gaussian priors for both uncertainties and source distributions. Because noise and sources are usually of different kind (e.g. white noise on measurement and sparsity of sources distribution), we propose to introduce two different priors. The first one tuned to deal with uncertainties and the second one adapted to the nature of the sources exciting the structure. Practically, the inverse problem is solved from a Generalized Iteratively Reweighted Least-Squares algorithm. The proposed methodology is illustrated on various numerical and experimental examples (i.e. for different sources distribution). It is shown that using priors depending on sources distribution nature are needed to improve drastically the quality of the source identification.

1 INTRODUCTION

The source identification of mechanical sources from vibration measurements is an ill-posed inverse problem, meaning that the existence of a unique stable solution is not guaranteed. A classical approach to bypass this difficulty consists in including in the formulation of the inverse problem some prior information on the measurement noise and the spatial distribution of sources to constrain the space of solutions. This idea is at core of Tikhonov-like regularization methods [1]. In such a regularization procedure, however, the a priori on the spatial distribution of sources is global. Incidentally, this can lead to poor identification if actual sources combine both localized and distributed sources, since the a priori has to reflect a compromise between two contradictory distributions.

The present paper aims at remedying this problem using available information on the distribution and the nature of sources to identify. The proposed approach relies on the Bayesian inference, which offers a rigorous mathematical framework allowing combining both probabilistic and mechanical data [2]. To exploit our a priori knowledge, we separate a priori on the measurement noise and on the excitation sources like the sparsity or the regularity. In doing so, prior information can be accurately tuned for each term of the problem. From a mathematical point of view, the solution of the problem is defined as the maximum a posteriori estimate. Practically, one seeks the solution of the dual minimization problem, which is solved from a Generalized Iteratively Reweighted Least-Squares algorithm [3]. To assess the validity of such an approach, a numerical validation is proposed. Obtained results clearly show that using prior information significantly improves the quality of the identification.

2 BAYESIAN SOURCE IDENTIFICATION

2.1 Problem description

First, let consider the situation where the vibration field \mathbf{X} , measured over the surface of a structure, is caused by an unknown excitation field \mathbf{F} . If the structure is linear, its dynamic behavior is completely determined by the transfer functions matrix \mathbf{H} , relating the vibration field \mathbf{X} to the unknown field \mathbf{F} . Now, let suppose that the measured vibration field is corrupted by a measurement noise \mathbf{N} . In such a situation, the measured vibration field \mathbf{X} is obtained from the following direct formulation:

$$\mathbf{X} = \mathbf{H}\mathbf{F} + \mathbf{N} \quad (1)$$

The structural source identification problem consists in estimating the unknown excitation field \mathbf{F} acting on a structure from the vibration field \mathbf{X} measured on its surface. The Bayesian paradigm consists in considering all the parameters of the problem as random variables. Consequently, the uncertainty on each parameter is modeled by a probability distribution, describing the state of knowledge or the prior on this parameter. The Bayesian source identification formulation relies on the Bayes' rule:

$$p(\mathbf{F}|\mathbf{X}) \propto p(\mathbf{X}|\mathbf{F})p(\mathbf{F}) \quad (2)$$

where:

- $p(\mathbf{F}|\mathbf{X})$ is the posterior probability distribution, representing the probability of observing \mathbf{F} given a vibration field \mathbf{X} . In other words, it defines what it is known about the excitation field \mathbf{F} after making vibration measurements;

- $p(\mathbf{F}|\mathbf{X})$ is the likelihood function, representing the probability of measuring \mathbf{X} given an excitation field \mathbf{F} . It reflects the uncertainty related to the measurement of the vibration field \mathbf{X} ;
- $p(\mathbf{F})$ is the prior probability distribution, representing our knowledge on the unknown excitation field \mathbf{F} before measuring the vibration field \mathbf{X} .

In this paper, one search the most probable excitation field $\hat{\mathbf{F}}$ given a measured vibration field \mathbf{X} . From the Bayesian point of view, it consists in finding a point estimate of $\hat{\mathbf{F}}$ corresponding to a mode of the posterior probability distribution. Mathematically speaking, the solution of the identification problem is sought as the maximum a posteriori estimate, that is:

$$\hat{\mathbf{F}} = \arg \max_{\mathbf{F}} p(\mathbf{F}|\mathbf{X}) = \arg \max_{\mathbf{F}} p(\mathbf{X}|\mathbf{F})p(\mathbf{F}) \quad (3)$$

Practically, it is generally easier to find a solution of the following dual minimization problem:

$$\hat{\mathbf{F}} = \arg \min_{\mathbf{F}} -\log p(\mathbf{F}|\mathbf{X}) = \arg \min_{\mathbf{F}} -\log p(\mathbf{X}|\mathbf{F}) - \log p(\mathbf{F}) \quad (4)$$

2.2 Choice of the likelihood function and the prior probability distribution

2.2.1 Choice of the likelihood function

The likelihood function reflects the uncertainty related to the measurement of the vibration field \mathbf{X} . By definition, this uncertainty is mainly related to the measurement noise \mathbf{N} . Consequently, the likelihood function $p(\mathbf{X}|\mathbf{F})$ can be rewritten under the following form:

$$p(\mathbf{X}|\mathbf{F}) = p(\mathbf{X} - \mathbf{HF}|\mathbf{N}) \quad (5)$$

representing the probability of obtaining $\mathbf{X} - \mathbf{HF} = 0$ given the measurement noise \mathbf{N} .

If the noise is supposed to be due to multiple independent causes, then the likelihood function $p(\mathbf{X}|\mathbf{F})$ can be represented by a normal distribution with mean \mathbf{HF} and variance α^2 :

$$p(\mathbf{X}|\mathbf{F}) \propto \exp \left[-\frac{1}{2} \frac{\|\mathbf{X} - \mathbf{HF}\|_2^2}{\alpha^2} \right] \quad (6)$$

where $\|\bullet\|_2$ is the L_2 -norm. It can be noticed that the form of the probability distribution implicitly assumes that all the elements of the vector $\mathbf{X} - \mathbf{HF}$ are independent identically distributed variables with same standard deviation α .

2.2.2 Choice of the prior probability distribution

The prior probability distribution reflects the uncertainty related to the unknown excitation field \mathbf{F} . It can be seen as a measure of the a priori knowledge of the experimenter on the sources to identify. For practical reasons, one assumes that the excitation field \mathbf{F} follows a generalized Gaussian distribution with zero mean. The global prior probability distribution is written as:

$$p(\mathbf{F}) \propto \exp \left[-\frac{1}{q} \frac{\|\mathbf{LF}\|_q^q}{\beta^q} \right] \quad (7)$$

where:

- q is the shape parameter controlling the shape of the probability distribution. The shape parameter is defined in the interval $]0, +\infty[$;
- $\|\bullet\|_q$ is the L_q -norm or quasi-norm;
- β is a scaling factor which defines the dispersion of the distribution around the mean. It is therefore a generalized measure of the variance;
- \mathbf{L} is a differentiation operator that allows controlling the regularity of the solution.

2.3 Practical form of the identification problem

To get the practical form of the identification problem, introduce Eqs. 6 and 7 in Eq. 4. One obtains the formulation of the identification problem similar to a generalized Tikhonov regularization:

$$\hat{\mathbf{F}} = \arg \min_{\mathbf{F}} \frac{1}{2} \frac{\|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2}{\alpha^2} + \frac{1}{q} \frac{\|\mathbf{L}\mathbf{F}\|_q^q}{\beta^q} \quad (8)$$

Here one has three parameters to mathematically describe prior information on the measurement noise and the nature of the sources. However, parameters q and β play a similar role. Only one parameter can be used. By choosing to work with the scale parameter q , while setting β^q to a constant value $\underline{\beta}$, one finally gets:

$$\hat{\mathbf{F}} = \arg \min_{\mathbf{F}} \frac{1}{2} \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 + \frac{\lambda}{q} \|\mathbf{L}\mathbf{F}\|_q^q \quad (9)$$

where $\lambda = \alpha^2/\underline{\beta}$ is known as the regularization parameter. If $q = 2$, the proposed formulation is equivalent to the standard Tikhonov regularization.

3 SOURCE IDENTIFICATION

The solution of the minimization problem given by Eq. 9 is computed from a general version of the Iteratively Reweighted Least-Squares (IRLS) algorithm [3].

3.1 General principle

The core idea of the Generalized Iteratively Reweighted Least-Squares (GIRLS) algorithm is to replace, within an iterative scheme, the L_q -norm, that appears in Eq. 9, by a weighted L_2 -norm, so that the functional to minimize has an explicit expression at each iteration. For this purpose, one just has to notice that:

$$\forall x_n, \forall q, \frac{1}{q} \|x\|_q^q = \frac{1}{q} \sum_n |x_n|^q = \frac{1}{2} \sum_n w(x_n) |x_n|^2 \quad (10)$$

where $w(x_n) = \frac{2}{q} |x_n|^{q-2}$ is the weighting coefficient.

As part of an iterative scheme, one tries to find the solution $x^{(k+1)}$ at iteration $k+1$ from the solution $x^{(k)}$ at iteration k by setting $w(x_n) = w(x_n^{(k)})$, in order to find the equality 10 when the process has converged. Here, the direct application of this idea consists in replacing the minimization problem given by Eq. 9 by an equivalent iterative process, for which the excitation field $\hat{\mathbf{F}}^{(k+1)}$ at iteration $k+1$ is the solution of the following minimization problem:

$$\hat{\mathbf{F}}^{(k+1)} = \arg \min_{\mathbf{F}} \frac{1}{2} \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 + \frac{\lambda^{(k+1)}}{2} \|\mathbf{W}^{(k)}\mathbf{L}\mathbf{F}\|_2^2 \quad (11)$$

where $\mathbf{W}^{(k)}$ is a definite positive matrix of the form:

$$\mathbf{W}^{(k)} = \text{diag} \left[\frac{2}{q} T_\epsilon \left(\mathbf{L}\hat{\mathbf{F}}^{(k)} \right) \right] \quad (12)$$

with:

$$T_\epsilon(x^{(k)}) = \begin{cases} |x^{(k)}|^{q-2} & \text{if } x^{(k)} > \epsilon^{(k)} \\ |\epsilon^{(k)}|^{q-2} & \text{if } x^{(k)} \leq \epsilon^{(k)} \end{cases} \quad (13)$$

where $\epsilon^{(k)}$ is a small real positive number acting as damped parameter. It avoids infinite weights when $|x^{(k)}| \rightarrow 0$ and $q < 2$.

3.2 Choice of the tuning parameters, initial guess and stopping criterion

According to Eq. 11, the tuning parameters of the problem are the shape parameter q , the regularization parameter $\lambda^{(k+1)}$ and the damping parameter $\epsilon^{(k)}$.

To properly choose the shape parameter q , one has to keep in mind that this parameter control the general shape of the distribution. So, using $q \geq 2$ will give priority to smooth solutions, while using $q \leq 1$ will favor sparse solutions.

Regarding the choice of the regularization parameter $\lambda^{(k+1)}$ and the damping parameter $\epsilon^{(k)}$, automatic selection procedures have been implemented. For the regularization parameter, the L-curve principle is used [4], while for the damping parameter, its value is set so that a fixed percentage p_i of the smallest values of $|\mathbf{L}\hat{\mathbf{F}}^{(k)}|$ are below $\epsilon^{(k)}$. In the next of the paper, $p = 5\%$.

For iterative algorithm, another critical issue is the choice of the initial solution as well as that of the stopping criterion. Choosing a good initial guess is a key point of the convergence of the algorithm, since the functional to minimize is non-convex when $q \leq 1$. The question that arises here is: What is a good initial guess? Actually, it is a coarse solution of the problem, easy to calculate, but sufficiently close to the final solution to ensure the convergence of the iterative process. Such requirements are fulfilled by the solution of the standard Tikhonov regularization. Finally, it remains to define a stopping criterion for the GIRLS algorithm. In this paper, a stopping criterion related to the variation of the functional $J(\hat{\mathbf{F}}^{(k)}) = \frac{1}{2} \left\| \mathbf{X} - \mathbf{H}\hat{\mathbf{F}}^{(k)} \right\|_2^2 + \frac{\lambda^{(k)}}{2} \left\| \mathbf{W}^{(k-1)} \mathbf{L}\hat{\mathbf{F}}^{(k)} \right\|_2^2$ between two successive iterations is used. The algorithm is automatically stopped when a prescribed tolerance defined by the user is reached. Here, the tolerance is set to 10^{-8} .

4 NUMERICAL VALIDATION

In the present numerical validation, one seeks to identify a point force of unit amplitude acting on a thin simply supported steel plate with dimensions $0.21 \times 0.27 \times 0.007$ m³. The coordinates of the point force, measured from the lower left corner of the plate, are $(x, y) = (0.17$ m, 0.12 m). Moreover, to simulate the vibration field \mathbf{X} , a finite element model of the plate made up with 616 (22×28) quadrilateral shell elements has been used. It is worth to mention that an additive Gaussian white noise has been added to the data to simulate the measured vibration field as shown in Figure 1.

Finally, a FE model of the structure with free boundary conditions is used to compute the transfer functions matrix \mathbf{H} .

Figure 2a presents the excitation field $\hat{\mathbf{F}}$ from the standard Tikhonov regularization. This figure clearly shows that the location of the point force is properly estimated while the amplitude

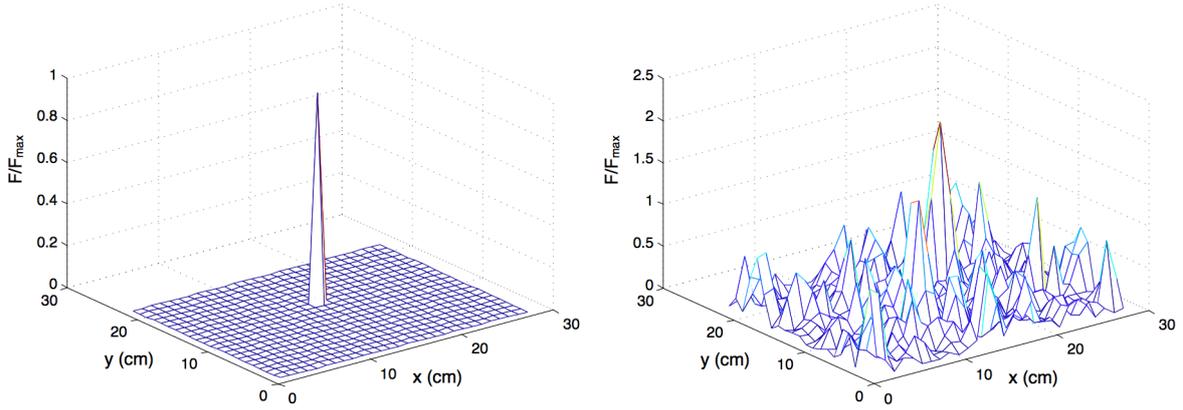
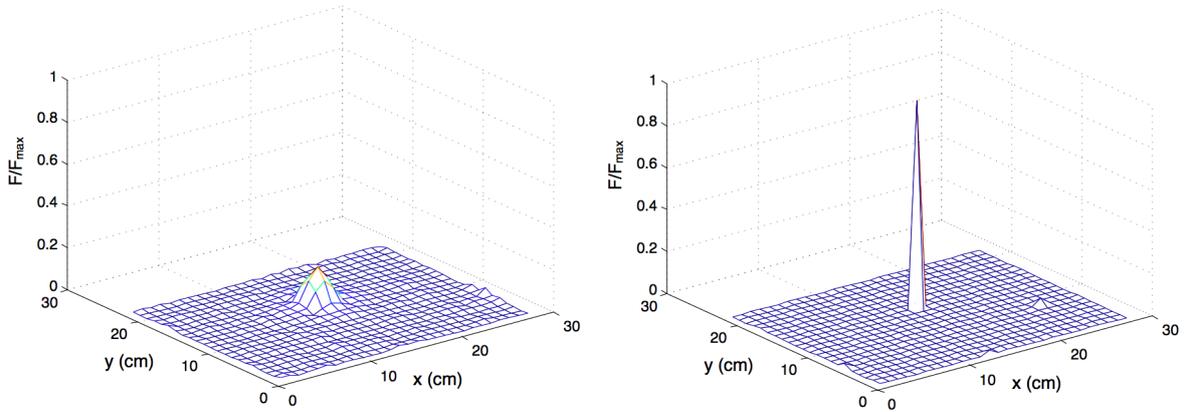


Figure 1: (a) reference solution; (b) direct identification with noisy data

Figure 2: Identification: (a) standard Tikhonov regularization; (b) proposed method with $q = 0.5$ and $L = \text{Id}$

is greatly underestimated. This contrasted result can be explained by the fact that the global a priori used in the standard Tikhonov regularization gives priority to smooth solutions. To improve the quality of the identification, the prior knowledge on the nature of sources acting on the source has to be exploited. Indeed, as presented in Figure 1a, one knows that the excitation field to identify is very sparse. Consequently, the structure can be characterized with a sparsity-promoting regularization term ($q < 2$). Figure 2b presents the excitation field identified from the proposed approach using $q = 0.5$. The comparison of this result with the reference solution clearly shows that the proposed methodology allows identifying properly not only the location and the amplitude of the point force (0.998 instead of 1).

5 EXPERIMENTAL VALIDATION

In the experimental validation, one seeks to identify a point force of known amplitude acting on a thin simply supported steel plate with dimensions $0.35 \times 0.29 \times 0.005 \text{ m}^3$. The coordinates of the point force, measured from the lower left corner of the plate, are $(x, y) = (0.241 \text{ m}, 0.192 \text{ m})$. The measurement of the vibration field is made on a 1015 (35×29) quadrilateral mesh centered on the plate. To compute the transfer functions matrix \mathbf{H} , a FE model of the structure with free boundary conditions is used.

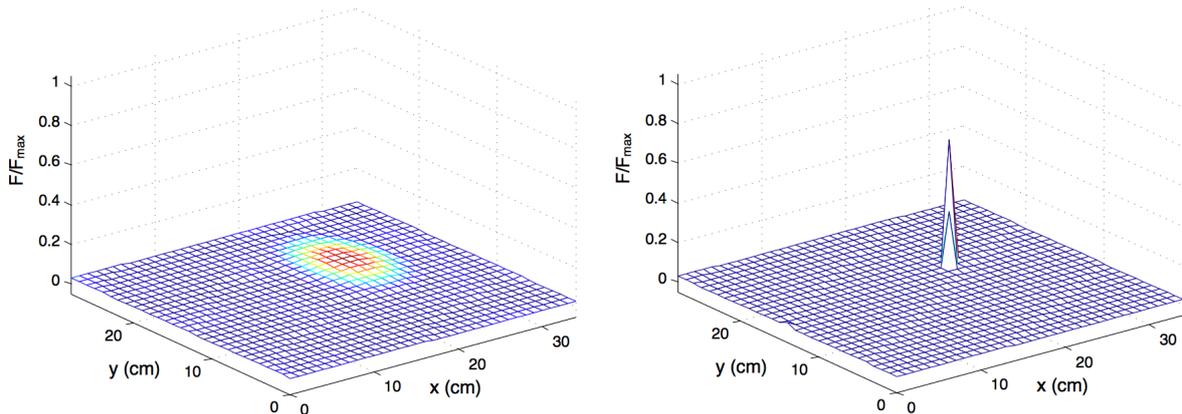


Figure 3: (a) standard Tikhonov regularization (b) proposed method with $q = 0.5$

Figure 3a presents the excitation field identified with a standard Tikhonov regularization. The location and amplitude of the acting point is barely visible. Figure 3b presents the excitation field identified from the proposed approach with $q = 0.5$ for a sparse excitation. The comparison of this result with the Tikhonov solution clearly shows that the proposed methodology allows identifying properly the location. One get a normalized force of 0.6 for coordinates (0.245 m, 0.195 m) and a second force of 0.3 for coordinates (0.235 m, 0.185 m). The identified value of the amplitude of the two points force is due to the fact that the excitation point is not just above a measurement point. A refined mesh in this domain could enhance the quality of the identified solution.

6 CONCLUSION

In the present study, the motivation was to propose a Bayesian formulation of the structural source identification problem able to fully exploit information available a priori on the nature and the location of the sources. To this end, the nature of the sources and of the noise are characterized with different distributions. A classic Gaussian law is taken for the noise, and a generalized Gaussian law is used for the sources. This formulation is very flexible, as the tuning parameters of these laws can be precisely adjusted to mathematically transcribe the a priori knowledge of the experimenter on the studied system. Formally, the solution of the problem is defined as the maximum a posteriori estimate, but practically, one seeks the solution of the dual minimization problem, which is solved from a Generalized Iteratively Reweighted Least-Squares algorithm. Numerical and experimental results show that exploiting a priori information substantially improves the quality of the source identification.

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