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STRUCTURAL EIGENFREQUENCY OPTIMIZATION BASED ON LOCAL SUB-DOMAIN ”FREQUENCIES”

Pauli Pedersen*[1] and Niels L. Pedersen[2]

Department of Mechanical Engineering, Solid Mechanics
Technical University of Denmark
Nils Koppels Allé, Building 404, DK-2800 Kgs.Lyngby, Denmark
[1]pauli@mek.dtu.dk

[2]nlp@mek.dtu.dk

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Abstract. *The engineering approach of fully stressed design is a practical tool with a theoretical foundation. The analog approach to structural eigenfrequency optimization is presented here with its theoretical foundation. A numerical redesign procedure is proposed and illustrated with examples.For the ideal case, an optimality criterion is fulfilled if the design have the same sub-domain ”frequency” (local Rayleigh quotient). Sensitivity analysis shows an important relation between squared system eigenfrequency and squared local sub-domain frequency for a given eigenmode. Higher order eigenfrequencies may also be controlled in this manner.The presented examples are based on 2D finite element models with the use of subspace iteration for analysis and a recursive design procedure based on the derived optimality condition. The design that maximize a frequency depend on the total amount of available material and on a necessary interpolation as illustrated by different design cases.*

1 INTRODUCTION

In most vibrating structures or continua there are sub-domains of high stiffness with only small displacements and other sub-domains of large displacements with relatively small stiffness. An optimal design with extremal eigenfrequency should have more uniformity among sub-domains, which may be elements (or group of elements) in a finite element (FE) model. This qualitative description is quantified and analytical described in the present paper.

To obtain optimal design for eigenfrequency or more generally control of eigenfrequencies we need three steps: analysis, sensitivity analysis and a procedure for redesign. For analysis the numerical method of subspace iteration is chosen and will not be further commented. The sensitivity analysis is to the largest possible extend presented analytically and optimality criterion methods are the basis for redesign.

The essential point of eigenfrequency optimization is the sensitivity analysis. More than 50 years ago rates of change of eigenvalues were published in [1]. A review on eigenfrequency optimization is presented in [2] and the book [3] includes a chapter on sensitivity of discrete systems with sensitivities for eigenvalues as well as for eigenmodes. The present analysis put focus on the possibility for localized results with simple physical interpretation. Searching for literature with focus on localized results, we noted the proceedings [4]. From the here presented analysis follows that the difference between a local Rayleigh quotient (”local squared frequency”) and the system Rayleigh quotient (system squared eigenfrequency) gives the overall information.

In material design with non-dimensional density as design parameter, a linear stiffness interpolation is seldom acceptable and a non-linear interpolation function must be introduced. This strongly influences the sensitivity analysis. In the present paper an interpolation that is controlled by two parameters, the slope at zero and full density, is used. Interpolation are in topology optimization mostly used as a penalization tool to obtain ”black and white” or ”0 and 1” designs, i.e., designs with no intermediate densities, and tools in addition to penalization are often necessary. In the present research the goal is not to obtain black and white design and the necessary interpolation is interpreted from the physical point of view, where a linear density dependency is seldom realistic.

Optimal design may iteratively be obtained based on an optimality criterion, as described in the book [5] and in the review [6]. For redesign the engineering approach of fully stressed design is a practical tool with a theoretical foundation. The analog approach to structural eigenfrequency optimization is here presented with its theoretical foundation, with a numerical redesign procedure and with illustrative examples. Sensitivity analysis shows an important relation between squared system eigenfrequency and squared local sub-domain frequency (Rayleigh quotient) for a mode. Higher order eigenfrequencies may also be controlled similarly.

The layout of the paper is as follows: in Section 2 the results from eigenfrequency analysis by the subspace iteration method is defined and discussed on system level as well as on element level. The importance of energy relations, i.e., Rayleigh quotients is emphasized, and a simple sensitivity analysis is derived, with modifications due to stiffness interpolation.

Section 3 define a specific optimization problem and an optimality criterion is derived. Based on this Section 4 present three different solution procedures, from mathematical programming the sequential linear programming (SLP) approach and then two alternative recursive procedures based directly on optimality criterion and sensitivity analysis. The recursive procedures (close to fully stress iterations) are applied in Section 5, and design histories show the effectiveness. In general fast convergence and stable iteration histories are obtained with these heuristic methods, with linear interpolation as well as with non-linear interpolation.

2 ANALYSIS AND SENSITIVITY ANALYSIS

The analysis by finite element (FE) and the eigenfrequency analysis by the subspace method are traditional and only the obtained results are commented. Thus the present section concentrates on sensitivity analysis including the important influence from interpolation of stiffnesses as a function of material density.

2.1 Results from eigenvalue analysis

For a given continuum/structure, analysis by subspace iteration gives a series of modes, described then individually by the displacement eigenvector $\{D\}$ (orthogonal to other eigenvectors and normalized) so that the specific kinetic energy $T = 1$. With this normalization of the eigenvector the specific elastic energy U is numerically equal to the eigenvalue ω^2, i.e., for the numerical values $U = \omega^2 = \omega^2 T$. (The T and U are the time independent amplitudes). Contained in this system displacement vector $\{D\}$ are the components of the element (sub-domain) displacement vector $\{D_e\}$ (as it follows from finite element interpretation). The system specific elastic energy U (twice the strain energy) may be accumulated from element energies U_e

$$U = \{D\}^T[S]\{D\} = \sum U_e = \sum \{D_e\}^T[S_e]\{D_e\} \tag{1}$$

where $[S]$ is the system stiffness matrix (say of the order 100000) while the element stiffness matrix $[S_e]$ is of the order 10. Analogously the system specific kinetic energy T (not containing the factor $\omega^2/2$) may be accumulated from element energies T_e

$$T = \{D\}^T[M]\{D\} = \sum T_e = \sum \{D_e\}^T[M_e]\{D_e\} \tag{2}$$

where $[M]$ is the system mass matrix and $[M_e]$ is element mass matrix.

In the following sensitivity analysis we use the results of an analysis

$$\begin{gathered} \omega^2, U_e, T_e \ \text{ with } \ \{D\} \ \text{ normalized such that} \\ U = \sum U_e = \omega^2, \ \ T = \sum T_e = 1, \ \ \omega^2 = U/T \end{gathered} \tag{3}$$

where U/T is the system Rayleigh quotient.

2.2 Sensitivity analysis

The design parameters ρ_e are assumed to be local, positive non-dimensional quantities in the interval $0 < \rho_e \leq 1$. With later interpolation extensions we at first assume both the element stiffness matrix $[S_e]$ and the element mass matrix $[M_e]$ to be proportional to ρ_e, i.e.,

$$[S_e] = \rho_e[\mathcal{S}_e], \ \ [M_e] = \rho_e[\mathcal{M}_e] \tag{4}$$

with both $[\mathcal{S}_e]$ and $[\mathcal{M}_e]$ independent of design.

The gradient $d\omega^2/d\rho_e$ is determined at the element level

$$\frac{d\omega^2}{d\rho_e} = \frac{\partial\omega^2}{\partial\rho_e} + \frac{\partial\omega^2}{\partial\{D\}}\frac{d\{D\}}{d\rho_e} = \frac{\frac{\partial U}{\partial\rho_e}T - \frac{\partial T}{\partial\rho_e}U}{T^2} = \frac{1}{T}\left(\frac{\partial U_e}{\partial\rho_e} - \omega^2\frac{\partial T_e}{\partial\rho_e}\right) \tag{5}$$

because $\partial\omega^2/\partial\{D\} = 0$. Inserting the assumptions of linear dependency $\partial U_e/\partial\rho_e = U_e/\rho_e$ and $\partial T_e/\partial\rho_e = T_e/\rho_e$ gives the local result where the gradient is expressed by local energies

$$\frac{d\omega^2}{d\rho_e} = \frac{1}{T\rho_e}\left(U_e - \omega^2 T_e\right) = \frac{T_e}{T\rho_e}\left(\omega_e^2 - \omega^2\right) \tag{6}$$

The gradient is proportional to the difference between the local ratio of energies (local Rayleigh quotient or termed local squared frequency) ω_e^2 and the system squared eigenfrequency ω^2.

From expression (6) follows directly the sign of the gradient as all T_e, T, ρ_e are non-negative

$$\frac{d\omega^2}{d\rho_e} > 0 \text{ for } \omega_e^2 > \omega^2, \quad \frac{d\omega^2}{d\rho_e} < 0 \text{ for } \omega_e^2 < \omega^2, \quad \frac{d\omega^2}{d\rho_e} = 0 \text{ for } \omega_e^2 = \omega^2 \tag{7}$$

To increase the frequency of the continuum/structure we increase ρ_e for $\omega_e^2 > \omega^2$ and decrease ρ_e for $\omega_e^2 < \omega^2$. A design change may be limited by a volume constraint and by the fact that sensitivity analysis will change when changing the design. The solution to these problems is part of our recursive optimization.

In the present research we have for simplicity assumed a single (non multiple) eigenfrequency and tested that this remains the case throughout the optimization. For the extended analysis with multiple eigenfrequencies, see the paper [7] or in truss design see [8].

2.3 Modifications from stiffness interpolation

The assumed linear dependence of the element stiffness matrix on the design parameter ρ_e (4) may be questioned and an analysis like in [9] gives the involved modifications. We do not change the assumption of linear dependence for the element mass matrix, but modify (4) to

$$[S_e] = f(\rho_e)[\mathcal{S}_e], \ \ [M_e] = \rho_e[\mathcal{M}_e] \tag{8}$$

where $f(\rho_e)$ is an interpolation function, still with $[\mathcal{S}_e]$ independent of design parameter. From this follows

$$\frac{\partial U_e}{\partial\rho_e} = \frac{df/d\rho_e}{f(\rho_e)}U_e = \frac{f'(\rho_e)}{f(\rho_e)}U_e \tag{9}$$

which for linear interpolation $f(\rho_e) = \rho_e$ gives $\partial U_e/\partial\rho_e = U_e/\rho_e$ as earlier applied.

With the modification (8) the gradient of the squared eigenfrequency (6) is modified to

$$\frac{d\omega^2}{d\rho_e} = \frac{T_e}{T\rho_e}\left(\Lambda(\rho_e)\omega_e^2 - \omega^2\right) \text{ with the definition } \Lambda(\rho_e) = \frac{\rho_e f'(\rho_e)}{f(\rho_e)} \tag{10}$$

Appendix A describes in detail and explicitly the applied version of a two parameter interpolation. The conclusions in (7) is modified to

$$\frac{d\omega^2}{d\rho_e} > 0 \text{ for } \Lambda(\rho_e)\omega_e^2 > \omega^2, \ \frac{d\omega^2}{d\rho_e} < 0 \text{ for } \Lambda(\rho_e)\omega_e^2 < \omega^2, \ \frac{d\omega^2}{d\rho_e} = 0 \text{ for } \Lambda(\rho_e)\omega_e^2 = \omega^2 \tag{11}$$

In appendix A it is seen that $\Lambda(\rho_e) > 1$ for $\rho_e > 0$.

2.4 Remarks on sensitivities

The gradients (6) and (10) constitute the basis for an optimization procedure and different possibilities exist. The gradients are important for general control of eigenfrequencies and the simplicity gives a direct physical interpretation:

- The change in squared eigenfrequency ω^2 when a local density ρ_e is changed is determined directly by local quantities from ordinary analysis.
- The gradient is proportional to the local relative kinetic energy amplitude T_e/T.
- The gradient is inversely proportional to the actual density ρ_e.
- As stated in (6) and (10) the gradient is proportional to the difference between a weighted local squared frequency (local Rayleigh quotient) and system squared eigenfrequency, i.e., $(\Lambda(\rho_e)\omega_e^2 - \omega^2)$, with $\Lambda(\rho_e) = 1$ for linear stiffness interpolation.

3 AN OPTIMIZATION PROBLEM AND ITS OPTIMALITY CRITERION

We study the optimization problem to maximize an eigenvalue (assumed single and often the lowest one) ω^2 for a given total amount of material, specified by the volume $\mathcal{V}$. We assume this volume constraint to be active and state the problem with non-dimensional densities ρ_e as design variables

$$\begin{aligned}
&\text{Maximize } \ \omega^2 \quad \text{(objective)}\\
&\text{for all densities } \ 0 < \rho_{min} \leq \rho_e \leq \rho_{max} \leq 1 \quad \text{(size limits)}\\
&\text{and } \ g = \sum \rho_e \mathcal{V}_e - \mathcal{V} = 0 \quad \text{(active constraint)}
\end{aligned} \tag{12}$$

The optimality criterion with only a single, active constraint is proportionality between the gradients of the objective and the gradients of the constraint, i.e.,

$$\frac{d\omega^2}{d\rho_e} = z\frac{dg}{d\rho_e} = z\mathcal{V}_e \quad \Rightarrow \quad z_e = \frac{1}{\mathcal{V}_e}\frac{d\omega^2}{d\rho_e} = z \tag{13}$$

with the same value z for all elements (sub-domains) e having an active design parameter ρ_e where $\rho_{min} < \rho_e < \rho_{max}$. With linear stiffness interpolation the gradient $d\omega^2/d\rho_e$ is given in (6) and the optimality criterion is

$$z_e = \frac{T_e}{T}\frac{1}{\rho_e\mathcal{V}_e}(\omega_e^2 - \omega^2) = z \tag{14}$$

With stiffness interpolation function $f(\rho_e)$ the gradient $d\omega^2/d\rho_e$ is given in (10) and the optimality criterion is

$$z_e = \frac{T_e}{T}\frac{1}{\rho_e\mathcal{V}_e}(\Lambda(\rho_e)\omega_e^2 - \omega^2) = z \ \text{ with the definition } \ \Lambda(\rho_e) = \frac{\rho_e f'(\rho_e)}{f(\rho_e)} \tag{15}$$

The function $\Lambda(\rho_e)$ is graphically illustrated in appendix A.

Based on the optimality criterion (15), a number of models each with different specific parameter cases are designed and rather generally we for the active design parameter sub-domains of the optimal designs find rather constant values of $\Lambda(\rho_e)\omega_e^2$. The following discussion supports the understanding of this. The optimality criterion is a necessary condition related to the active design sub-domains. Assume for simplicity that all sub-domains are active, then we write (15) with a summation over the total domain

$$\sum \frac{T_e}{T}(\Lambda(\rho_e)\omega_e^2 - \omega^2) = \sum \rho_e \mathcal{V}_e z = \mathcal{V} z \tag{16}$$

and assuming same value for all sub-domains $\Lambda(\rho_e)\omega_e^2 = \Omega^2$ and with $\sum T_e = T$ this is a sufficient condition (but not necessary condition) for satisfying the optimality condition by

$$z = (\Omega^2 - \omega^2)/\mathcal{V} \tag{17}$$

4 POSSIBLE SOLUTION PROCEDURES

For the use of mathematical programming such as linear programming (LP) the problem (12) may also be formulated as a number of redesign problems, where the design variables are $(\Delta\rho_e)_i$ and the objective in redesign number i with unchanged volume $\Delta\mathcal{V}$ = 0 is $(\Delta\omega^2)_i$, i.e., with linear expansion

$$\begin{aligned} &\text{Maximize } \ (\Delta\omega^2)_i = \sum (\frac{d\omega^2}{d\rho_e})_i (\Delta\rho_e)_i \quad \text{(redesign objective)} \\ &\text{for } \ g = \sum (\Delta\rho_e)_i \mathcal{V}_e = 0 \quad \text{(no volume change)} \end{aligned} \tag{18}$$

This is a linear programming (LP) problem, that for a large number of unknowns may not be easily solved, but for small and medium size problems the global solution to this sub-problem is straight forward. The total non-linear design problem (12) is then solved by a sequence of these redesign problems, with the procedure termed sequential LP or SLP. With a limited number of unknowns the use of this procedure is illustrated in [10].

An alternative procedure to mathematical programming for solving (12) is based on the optimality criterion, say (15) with (14) as the more simple case for linear stiffness interpolation. A recursive procedure for optimization based on (15) is separated according to the sign of the gradients (sign of $(\Lambda(\rho_e)\omega_e^2 - \omega^2)$).

$$\begin{aligned} &\text{For positive gradients } \ (\Lambda(\rho_e)\omega_e^2 - \omega^2 > 0) : (\rho_e)_{new} = (\rho_e)_{current}(1 + 4.0 z_e/z_{max})^{0.8}\eta \\ &\text{For negative gradients } \ (\Lambda(\rho_e)\omega_e^2 - \omega^2 < 0) : (\rho_e)_{new} = (\rho_e)_{current}(1 - 0.8 z_e/z_{min})^{0.8}\eta \end{aligned} \tag{19}$$

where the values of $z_{min} < 0, z_{max} > 0$ are determined during the evaluation of the gradients. The specific values in (19) 4.0, 0.8, 0.8 are chosen from experience, acting as a kind of move-limits and influence the number of recursive redesigns (number of eigenvalue analysis). The iteratively (without FE analysis) determined volume correction factor η relate to the fact that the densities at the limits ρ_{min} or ρ_{max} are not known in advance. Factor η strictly keep the specified volume by inner iterations where the ρ_e at the size limits are localized. This is described in detail in [9]. The value 0.8 of the power also limits the change of ρ_e in one redesign, and such a power (also with a lower value) is often applied for similar recursive procedures.

An even more simple recursive procedure is based only on the sign of the gradients as in (7) or (11). For non-linear stiffness interpolation it is suggested to recursive redesign according to

$$(\rho_e)_{new} = (\rho_e)_{current}(\Lambda(\rho_e)\omega_e^2/\omega^2)^{0.8}\eta \tag{20}$$

again with the power 0.8 in order not to change ρ_e too much, and with η to strictly fulfill the volume constraint and satisfying size limits $\rho_{min} \leq \rho_e \leq \rho_{max}$. For different examples the design history, when applying recursive design procedures (19) or (20) are compared. Note, that (20) agrees with the optimality criterion (15) for the specific case of $z = 0$. However, in general the best designs are obtained when applying the total optimality criterion in (19).

5 EXAMPLES WITH RECURSIVE ITERATIONS

For three specific examples the lowest eigenfrequency is maximized, based on the recursive formula (19). Final design results from initial uniform designs are shown. Isotropic material parameters, total domain and boundary conditions are not changed. However, the solutions depend rather much on the total amount of material $\mathcal{V} = \rho_{mean} \sum \mathcal{V}_e$ and therefore results for ρ_{mean} = 0.15, 0.3 and 0.6 (corresponding to 15%, 30% and 60% material) are presented. Also the chosen interpolation function $f(\rho)$ that gives $\Lambda(\rho_e)$ in (20) is important for the solution. Therefore, both linear interpolation $f(\rho) = \rho$ and non-linear interpolation with specifically $\kappa_0 = 0.2$ and $\kappa_1 = 4$ (see the appendix) are studied. For each specific example this gives 3 $\times$ 2 = 6 cases.

5.1 A square disc example

The first example is a 2D model for a square disc supported at one side such that this side is only displaced in its plane with fixed middle point. In order to avoid designs with most material near the supported boundary, the side opposite to the supported side has a strip which is not changed during redesign and is given the initial value ρ_{mean}. Aluminum material described by E, ν, ρ_{mass} = 0.7 10^{11} Pa, 0.3, 2700 kg/m^3 is applied for this example, with cases of 15%, 30% and 60% material and two different stiffness interpolations. The total design domain has dimensions 1m $\times$ 1m with thickness 0.01m.

	15% material	30% material	60% material
Uniform, linear interpolation	1	1	1
Optimized, linear interpolation	2.08	1.99	1.47
Uniform, non-linear interpolation	0.50	0.56	0.69
Optimized, non-linear interpolation	1.21	1.32	1.38

Table 1: For the square disc example, the relative lowest eigenfrequencies for uniform designs and for optimized designs with different total amount of materials and linear as well as non-linear interpolation.

A design with uniform density and linear interpolation has an eigenfrequency that is independent of the total amount of material. This is valid for all three treated examples as it follows from Tables 1, 2 and 3. Table 1 shows the relative values of the lowest eigenfrequencies, normalized to 1 for linear interpolation of uniform design and therefore independent of the total amount of material. We note, that for linear interpolation and less material, more is obtained by optimization, expecting the optimized frequency in principle to keep on increasing for ρ_{mean} decreasing. With non-linear interpolation more is generally obtained by optimization. The relative values between non-linear interpolation and linear interpolation for uniform design, i.e., 0.50, 0.56 and 0.69 are the same for all three examples, as also seen in Tables 2 and 3.

5.2 A beam/bridge example

The second example is a model for half a simply supported beam/bridge structure, for which we assume symmetric eigenmode and model the boundary conditions accordingly. Steel material described by E, ν, ρ_{mass} = 2.1 10^{11} Pa, 0.3, 7850 kg/m^3 is applied for this example. The design domain for this model is 2.4 $\times$ 0.4m with thickness 0.01 m. Again we study total volume percent 15%, 30% and 60% and two different stiffness interpolations.

	15% material	30% material	60% material
Uniform, linear interpolation	1	1	1
Optimized, linear interpolation	1.52	1.45	1.29
Uniform, non-linear interpolation	0.50	0.56	0.69
Optimized, non-linear interpolation	1.33	1.37	1.28

Table 2: For the beam/bridge example, relative lowest eigenfrequencies for uniform designs and for optimized designs with different total amount of materials and linear as well as non-linear interpolation.

Table 2 shows the relative values of the lowest eigenfrequencies, normalized to 1 for linear interpolation of uniform design. We note, that for linear interpolation and less material more is obtained by optimization while with non-linear interpolation and less material the gain is not so clear.

5.3 A cantilever example

The third example is a model of a cantilever structure, for which we assume boundary conditions that is only vertical supported in the lowest node. To get a design with material also at the cantilever tip, a strip (non-uniform) at the upper side has fixed density, equal to ρ_{mean}. Steel material described by E, ν, ρ_{mass} = 2.1 10^{11} Pa, 0.3, 7850 kg/m^3 is applied for this example. The design domain is 14m long, 6m height at left edge and 4m height at right edge, with thickness 0.01 m. Again we study total volume percent 15%, 30% and 60% and two different stiffness interpolations.

	15% material	30% material	60% material
Uniform, linear interpolation	1	1	1
Optimized, linear interpolation	2.61	2.64	1.97
Uniform, non-linear interpolation	0.50	0.56	0.69
Optimized, non-linear interpolation	1.58	1.77	1.89

Table 3: For the cantilever example, relative lowest eigenfrequencies for uniform designs and for optimized designs with different total amount of materials and linear as well as non-linear interpolation.

Table 3 shows the relative values of the lowest eigenfrequencies, normalized to 1 for linear interpolation of uniform design. We note, that for linear interpolation most is obtained by optimization for less material while for non-linear interpolation the gain is opposite.

5.4 Summary of comments on the examples.

- All examples with parameter cases have been optimized also based on the more simple recursive procedure (20). Although not based on a derived optimality criterion, this heuristic simple recursive procedure is also effective and converges in less than 10 iterations with results close to the shown results, although including more dominating sub-domains with resulting ρ_{min}, that may result in a lower optimized frequency.

- All design optimizations are initiated with uniform material (density) design.

- The treated FE models are with 16384, 8192 and 8192 design parameters. Interactive optimization on a small computer is performed.

- Linear stiffness interpolation as well as two parameter non-linear stiffness interpolation is applied. With the density formulation, application to axisymmetric models and 3D models can directly be done.
- The study includes different total amount of total material (volume), corresponding 15%, 30% and 60% of the total prescribed design domain, and optimized design as well as obtained eigenfrequency depend strongly on this parameter.

6 CONCLUSIONS

The finite element analysis, the subspace eigenvalue analysis, the sensitivity analysis, the optimality criterion and the recursive iterations are not restricted to the 2D problems as studied here, but are directly applicable to other models such as for axisymmetric problems and for 3D problems.

For uniform density and linear interpolation, the eigenfrequency is independent of the total amount of material. For zero material the eigenfrequency is zero or at least not defined. The three treated examples give no clear information about the optimal total amount of material, and a parametric study is therefore suggested. It should be based on a realistic, necessary interpolation.

The obtained sensitivities with local, direct physical interpretation is found valuable independent of the actual use, and as such constitutes basic vibrational knowledge.

Although simple recursive iteration based mainly on the sign of the sensitivities are possible, it is found more reliable to base the iterations on the total optimality criterion.

Linear mass interpolation is assumed realistic, but non-linear stiffness interpolation as a function of density may be a physical reality. Such an interpolation then strongly influence the optimized design as well as the resulting eigenfrequency.

A The interpolation function NLPI with boundary slope parameters

In research on optimal material design, one parameter functions such as SIMP and RAMP are mostly applied. For the present research we applied a two parameter function $f = f(\rho)$ that as independent parameters has the boundary slopes: κ_0 at $\rho = 0$ and κ_1 at $\rho = 1$. It is derived based on a Bézier curve, see [11], and are here given in explicit form, assuming

$$0 \leq \kappa_0 < 1, \quad \kappa_1 > 1, \quad \text{and} \quad \kappa_0 + \kappa_1 > 2 \tag{21}$$

The explicit form of this Non Linear Penalization or Interpolation (NLPI) function may be given as

$$f(\rho) = r_4(1-t)t + t^2 \;\Rightarrow \tag{22}$$

$$f'(\rho) = \frac{-\kappa_1 t + \kappa_0(1 - t + \kappa_1(2t-1))}{1 + \kappa_1(t-1) + t(\kappa_0 - 2)} \;\text{ with }\; t = r_1 - \sqrt{r_2 + r_3\rho} \tag{23}$$

$$\begin{aligned} r_1 &= (\kappa_1 - 1)/(\kappa_0 + \kappa_1 - 2), \quad r_2 = r_1^2 \\ r_3 &= (\kappa_0 - \kappa_1)/(\kappa_0 + \kappa_1 - 2), \;\; r_4 = 2\kappa_0(1-\kappa_1)/(\kappa_0 - \kappa_1) \end{aligned} \tag{24}$$

The important factor $\Lambda = \Lambda(\rho)$ from the optimality criterion (15) is

$$\Lambda(\rho) = \rho f'(\rho)/f(\rho) \tag{25}$$

as derived in the sensitivity analysis.

For specific values of the parameters κ_0, κ_1 Figures 1, 2 and 3 shows function plots of $f(\rho), f'(\rho)$ and $\Lambda(\rho)$. We note that all the shown functions are monotonically increasing and the functions $f(\rho), f'(\rho)$ always has a positive curvature. In the interval $(0 \leq \rho \leq 1)$ the factor Λ in the optimality criterion increases from 1 to κ_1. This function may change curvature but is always monotonically increasing. The example in Figure 2 middle with κ_0 = 0.2 and κ_1 = 4 is used for the presented optimizations with non-linear interpolation.

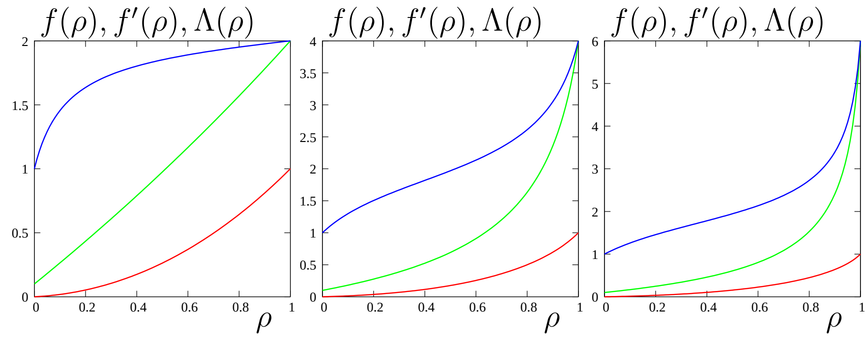

Figure 1: Function plots for low interpolation start κ_0 = 0.1. Red curves for $f(\rho)$, green for $f'(\rho)$ and blue for $\Lambda(\rho) = \rho f'(\rho)/f(\rho)$. Left frame for κ_1 = 2, mid frame for κ_1 = 4, and right frame for κ_1 = 6.

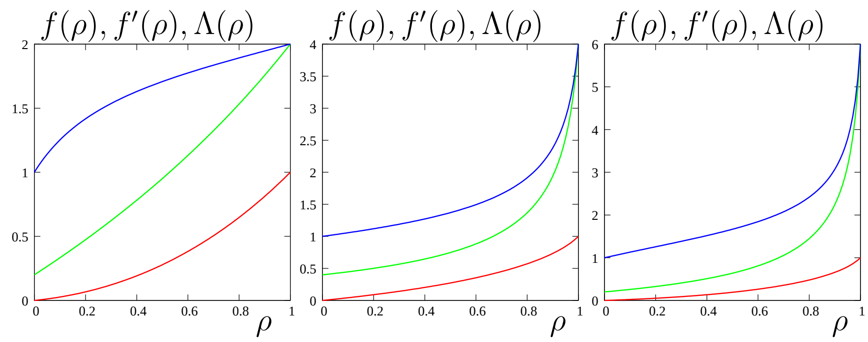

Figure 2: Function plots for increased interpolation start κ_0 = 0.2. Red curves for $f(\rho)$, green for $f'(\rho)$ and blue for $\Lambda(\rho) = \rho f'(\rho)/f(\rho)$. Left frame for κ_1 = 2, mid frame for κ_1 = 4, and right frame for κ_1 = 6.

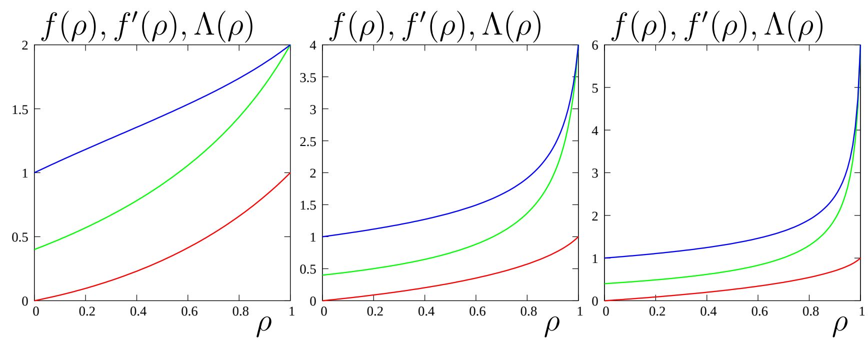

Figure 3: Function plots for high interpolation start κ_0 = 0.4. Red curves for $f(\rho)$, green for $f'(\rho)$ and blue for $\Lambda(\rho) = \rho f'(\rho)/f(\rho)$. Left frame for κ_1 = 2, mid frame for κ_1 = 4, and right frame for κ_1 = 6.

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