11th International Conference on Vibration Problems
Z. Dimitrovová *et.al.* (eds.)
Lisbon, Portugal, 9–12 September 2013

# A FINITE NONLINEAR DYNAMICS FORMULATION BASED ON MINIMAL SET PARAMETERIZATIONS OF THE FINITE ROTATIONS

**S. Lopez***

Dipartimento di Modellistica per l'Ingegneria, Università della Calabria, 87030 Rende, Italy
salvatore.lopez@unical.it

**Keywords:** Computational nonlinear dynamics, three-dimensional finite rotations.

**Abstract.** *The treatment of the finite rotations is a crucial topic in geometrically nonlinear elastodynamics because complex manipulations are required to obtain conservative descriptions and well-posed transformation matrices. Then computationally efficient definitions of the parameters for finite rotation representations are required.*

*In this work an updated Lagrangian formulation for three-dimensional beam elements in the hypotheses of large rotations and small strains is described. In the formulation slopes are used, instead of rotation parameters, to compute the nonlinear representations of the strain measures. The presented approach is based on that used in [6] and [7] for the statical and dynamical analysis of beams. In the cited formulations, the nonlinear rigid motion is recovered by referring to three unit and mutually orthogonal vectors. All nine components of such vectors in the global inertial frame of reference are assumed as unknown. As demonstrated, the rotational degree of freedom of the element is reduced to only three by six well-posed constraint conditions.*

*Instead, the use of the internal constraint equations is replaced by the intrinsic definition of the related rotation parameters by obtaining a three parameter description here. As demonstrated, such a reduction from nine to three unknown components is well posed under widely applicable hypotheses. The linear tranformations connecting the spin vector and the time derivative of the parametrization vector are also defined. Then the resulting formulation can utilize an additive approach where only linear vector operations are involved.*

*In relation to the Newmark approximations for the incremental velocity and acceleration vectors, approaches based on the trigonometrical rotational vector and the algebraic Rodrigues-Cayley parameterizations for rotation matrix definition are also employed (see for example Felippa [2]). In particular, once the admissible variations for the evaluation of the coefficients in the internal and inertial force vectors and related tangent stiffness and mass matrices are performed, the implementation effort and arithmetical operations required by the used formulations are compared.*

## 1 INTRODUCTION

Geometric nonlinearity in elastodynamics is an important field in structural analysis and in the last thirty years there has been extensive research into time integration algorithms. In particular, considerable work has been devoted to developing models for three-dimensional elastic frame structures for small strains and in the presence of large rotations. In this context, here we present an extension of the corotational approach described in [8] to the time-stepping nonlinear analysis of three-dimensional rotations models.

The treatments of the finite rotations, typically based on the rotation vector of the Euler theorem to describe finite rotations, have an economical definition of the rotated local reference system because only three parameters are used. Such a minimal set approach for the parameterization of the rotations suffers from the singularities in the transformation matrices for several angles. However, in the use of updated Lagrangian procedures, only the composition of rotations is strictly required because within a single time increment rotations it does not exceed the singularity angle. Statical and dynamical formulations based on finite rotation updates in incremental-iterative procedures can be found for example in Ibrahimbegović and Al Mikdad [3] and Pimenta et al. [9].

The singularities for arbitrarily large rotations, however, are not just inherent in the corotational approach, but in all methods based on a minimal set of parameters. The choice of the three parameters, therefore, is usually made by considering the characteristics of a specific application, without, however, avoiding singularities. In the quaternion descriptions, free singularity parameterizations can be obtained by adding a further parameter, and by subjecting the related Lagrange multiplier to a constraint equation. In respect to the minimal description, more nonlinear equations have to be solved while computationally expensive evaluations of the coefficients in the force vector and in the tangent stiffness matrix persist.

More recently, formulations where the nonlinear rigid motion is recovered by referring to three unit and mutually orthogonal vectors attached to the elements have been presented. All nine components of such vectors in the global inertial frame of reference are assumed as unknown and the rotational degree of freedom of the element is reduced to only three by six constraint conditions. In the work of Betsch and Steinmann [1] orthonormality of the directors is enforced by using six scalar products. In particular, three unit length and three orthogonality conditions are imposed on the directors by using the scalar products among them and reciprocally, respectively. In [6] and [7], instead, three scalar products and one cross product are used to define the constraint conditions. In particular, one orthogonality and two unit length conditions are imposed by scalar products while the complete definition of a director is obtained by referring to a cross product. Hence, the formulation proves to be well posed for any finite rotations.

The minimal set approach for the parameterization of the rotations used here, in the following denoted by director parameterization, as said is based on that described in [6] and [7] for the statical and dynamical geometrically nonlinear analysis of beams. Here, the use of the internal constraint equations is replaced by the intrinsic definition of the related rotation parameters by obtaining a three parameter description. We demonstrate in [8] that such a reduction from nine to three unknown components is well posed under widely applicable hypotheses. The linear tranformations connecting the spin vector and the time derivative of the rotation parameter vector are also defined. Then the resulting formulation can utilize an additive approach where only linear vector operations are involved.

As regards beam element modeling, here we use a small strain - finite displacement formulation of a two-node finite element based on the Timoshenko beam theory. The actual configuration of the element is rigidly translated and rotated, and deformed according to selected linear modes. Rigid and deformation modes are referred to the nodes at the boundaries of the element with six unknowns per node. The nonlinear motion is recovered by referring to three unit and mutually orthogonal vectors attached to the nodes. As stated, three of the nine components of such vectors in the global inertial frame of reference are assumed as unknown. Afterward, the deformative modes are summed up in the strain tensor definition.

By using a Newmark approximation for the incremental velocity and acceleration vectors, approaches based on the trigonometrical rotational vector and the algebraic Rodrigues-Cayley parameterizations for rotation matrix definition are also employed in tha paper. In particular, once the admissible variations for the evaluations of the coefficients in the internal and inertial force vectors and related tangent stiffness and mass matrices are performed, the implementation effort and arithmetical operations required by the used formulations are compared.

The paper is set out in the following way. In Section 2 we describe the treatment of rotations by the cited director, rotational vector and Rodrigues-Cayley parameterizations. In Section 3 we define the kinematics of the beam element and the connected variational formulation of the motion. Section 4 contains the description of the solution algorithm and the related numerical examples. Final conclusions are drawn in Section 5.

## 2 PARAMETRIZATION AND UPDATE TREATMENT OF ROTATIONS

In the following, we denote with Latin indices $i$ and $j$ the values $[1, ..., 3]$ while $\delta_{ij}$ is the Kronecker delta. Let $\mathbf{g}_i = \{g_{ij}\}$ and $\hat{\mathbf{g}}_i = \{\hat{g}_{ij}\}$ be, respectively, the actual and the initial configuration of three unit mutually orthogonal vectors in the inertial reference basis $\mathbf{k}_i = \{k_{ij}\} = \{\delta_{ij}\}$. Matrix $\hat{\mathbf{G}}$ links $\hat{\mathbf{g}}_i$ and $\mathbf{k}_i$ vectors by $\hat{\mathbf{g}}_i = \hat{\mathbf{G}}\mathbf{k}_i$ while $\mathbf{G}$ maps $\hat{\mathbf{g}}_i$ into $\mathbf{g}_i$ vectors by $\mathbf{g}_i = \mathbf{G}\hat{\mathbf{g}}_i$. Now let $\bar{\mathbf{g}}_i = \{\bar{g}_{ij}\}$ be an intermediate configuration of the $\mathbf{g}_i$ vectors and $\bar{\mathbf{e}}_i = \{\bar{e}_{ij}\}$ the related representation in the $\hat{\mathbf{g}}_i$ basis. Analogously, $\mathbf{e}_i = \{e_{ij}\}$ is the counterpart of $\mathbf{g}_i$ in the $\bar{\mathbf{g}}_i$ reference. So the right-application of the rotation operator $\mathbf{E} = [\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3]$ to the actual rotation $\bar{\mathbf{E}} = [\bar{\mathbf{e}}_1|\bar{\mathbf{e}}_2|\bar{\mathbf{e}}_3]$ describes the tranformation $\mathbf{g}_i = \bar{\mathbf{E}}\mathbf{E}\hat{\mathbf{g}}_i$.

In the approach described in the works [6] and [7] the $g_{ij}$ components of the $\mathbf{g}_i$ vectors are assumed as unknown parameters. The nine $g_{ij}$ unknown components are subject to six constraint conditions so that the related direction cosine matrix $\mathbf{G} = [\mathbf{g}_1|\mathbf{g}_2|\mathbf{g}_3]$ is orthogonal, i.e. $\mathbf{G}\mathbf{G}^T = \mathbf{G}^T\mathbf{G} = \mathbf{I}$, with $\mathbf{I}$ as identity matrix. Here we abandon the unbounded validity for finite rotations of such an approach in order to reduce the description to a minimal set of parameters. In effect, under widely applicable hypotheses on the $\mathbf{e}_i$ incremental vectors from the updated $\bar{\mathbf{e}}_i$ ones, by the cited conditions $\mathbf{e}_2 \cdot \mathbf{e}_3 = 0$, $\mathbf{e}_2 \cdot \mathbf{e}_2 - 1 = 0$, $\mathbf{e}_3 \cdot \mathbf{e}_3 - 1 = 0$ and $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$, an explicit definition of the $e_{ij}$ components as a function of the $\mathbf{d}^T = \{d_1, d_2, d_3\} = \{e_{32}, e_{31}, e_{21}\}$ director parameters is given. In particular, let $\theta_{ij}$ and $\pi/2 + \theta_{ij}$ be the angles between $\mathbf{e}_i$ and $\bar{\mathbf{e}}_j$ vectors for $i = j$ and $i \neq j$, respectively. So $e_{ij} = cos\theta_{ij}$ for $i = j$ while $e_{ij} = sin\theta_{ij}$ for $i \neq j$. The $\theta_{ij} \in (-\theta^*, +\theta^*)$ with $\theta^* = \arcsin(\sqrt{2}/2) = 45^o$ is the range of validity for the director based parametrization. As at present the formulation is based on the updated Lagrangian procedure, such a value is not a really restricting bound.

To obtain the updated treatment of rotations we refer to the $\mathbf{g}_{i(k)} = \mathbf{G}_{(k)}\hat{\mathbf{g}}_i = \bar{\mathbf{E}}_{(k)}\mathbf{E}_{(k)}\hat{\mathbf{g}}_i$ expression for the actual configuration of the $\mathbf{g}_i$ orthonormal triad at the $k$-th integration step. Vectors $\mathbf{e}_{i(k)}$ defining $\mathbf{E}_{(k)}$, then, represent the incremental rotation from the $\bar{\mathbf{e}}_{i(k)}$ previously computed configuration. The subsequent $k+1$ step, afterward, refers to the

$$\bar{\mathbf{E}}_{(k+1)}=\bar{\mathbf{E}}_{(k)}\mathbf{E}^*_{(k)}, \quad \bar{\mathbf{E}}_{(k+1)}=[\bar{\mathbf{e}}_{1(k+1)} \mid \bar{\mathbf{e}}_{2(k+1)} \mid \bar{\mathbf{e}}_{3(k+1)}] \quad \mathbf{E}^*_{(k)}=[\mathbf{e}^*_{1(k)} \mid \mathbf{e}^*_{2(k)} \mid \mathbf{e}^*_{3(k)}] \tag{1}$$

updated configuration with the $\mathbf{e}^*_{i(k)}$ established configuration of $\mathbf{e}_i$. The process is initialized by $\bar{\mathbf{E}}_{(0)} = \mathbf{I}$. As we can see, simple products are used recursively to compose as many successive rotations as necessary. In particular, the $\bar{\mathbf{E}}$ matrix takes into account the previously computed rotations of the $\hat{\mathbf{g}}_i$ in the $\bar{\mathbf{g}}_i$ frame while $\mathbf{e}_i$ vectors map $\bar{\mathbf{g}}_i$ in the actual $\mathbf{g}_i$ frame. In fact, we note that the updated values of director parameters $\bar{d}_1$, $\bar{d}_2$ and $\bar{d}_3$ are directly the $\bar{E}_{23}$, $\bar{E}_{13}$ and $\bar{E}_{12}$ coefficients of the (1) updated rotational matrix. Finally, in the $k$-th step, vectors $\mathbf{e}_{i(k)}$ of $\mathbf{E}(\mathbf{d})$ are completely defined as a function of the $d_{1(k)}$, $d_{2(k)}$ and $d_{3(k)}$ unknown parameters by the recursive evaluations

$$\begin{gathered} h=1/(1-d_2^2), \quad c_1=\sqrt{1-d_3^2-d_2^2}, \quad c_3=\sqrt{1-d_2^2-d_1^2}, \quad e_{11}=c_1, \quad e_{33}=c_3, \\ e_{22}=-(d_3d_2d_1-c_1c_3)h, \;\; e_{23}=-(d_3d_2c_3+c_1d_1)h, \;\; e_{12}=-(d_2d_1c_1+d_3c_3)h, \;\; e_{13}=-(d_2c_1c_3-d_3d_1)h. \end{gathered} \tag{2}$$

In the treatment of rotations based on the rotation vector $\boldsymbol{\psi} = \varphi\boldsymbol{\phi}$, $\boldsymbol{\phi}^T\boldsymbol{\phi} = 1$ of the Euler theorem to describe finite rotations, the representation of rotation operators is: $\mathbf{E}(\boldsymbol{\psi}) = \mathbf{I} + \sin\varphi/\varphi\boldsymbol{\psi}_\times + (1-\cos\varphi)/\varphi^2\boldsymbol{\psi}_\times\boldsymbol{\psi}_\times$, where $\boldsymbol{\psi}_\times$ denotes the skew symmetric tensor obtained by the related components of the axial vector $\boldsymbol{\psi}$. The $\boldsymbol{\psi} = \mathbf{axial}(\boldsymbol{\psi}_\times)$ is then the converse operation of $\boldsymbol{\psi}_\times = \mathbf{Skew}(\boldsymbol{\psi})$ that extracts the $\boldsymbol{\psi}$ vector from the skew symmetric tensor $\boldsymbol{\psi}_\times$. In the use of the $\mathbf{G} = \bar{\mathbf{E}}\mathbf{E}$ composition, let $\boldsymbol{\psi}_G = G^{-1}(\mathbf{G})$ be the inverse problem defined as the operation of obtaining the $\boldsymbol{\psi}_G$ rotation vector based on the knowledge of the $\mathbf{G}$ rotation matrix. We stress that in such a parametrization the inverse problem can be solved only by case sensitive procedures, as the no ill-conditioning Spurrier algorithm. A parameterization where the solution of the inverse problem is not case sensitive is used in [9]. Such a representation of the rotation operator refers to the Rodrigues-Cayley rotation matrix definition and is given by $\mathbf{E}(\boldsymbol{r}) = \mathbf{I} + 4/(4+r^2)(\mathbf{r}_\times + 1/2\mathbf{r}_\times\mathbf{r}_\times)$. In contrast to the rotational vector, the actual parameterization vector is here directly computed from the updated configuration by using $\boldsymbol{r}_G = 4/(4-\boldsymbol{r}^{*T}\bar{\boldsymbol{r}})(\boldsymbol{r}^* + \bar{\boldsymbol{r}} + 1/2\boldsymbol{r}^* \times \bar{\boldsymbol{r}})$.

Note that expressions of the incremental rotation given in the director approach are computationally slightly less expensive than those given in the rotational vector and Rodrigues-Cayley ones. In effect, in the director approach the three components $e_{21}$, $e_{31}$ and $e_{32}$ of $\mathbf{E}$ are assumed directly as unknown parameters. An inverse problem to be solved or case statements, furthermore, are not present in the coding of the director based formulation.

## 3 KINEMATICS AND VARIATIONAL FORMULATION OF THE MOTION OF THE BEAM ELEMENT

Let $\xi$ be the referential coordinate along the beam element centerline $-h/2 \le \xi \le +h/2$. In the following, we denote with $n$ and $m$ respectively the nodes in $\xi = -h/2$ and $\xi = +h/2$, while $o$ refers to $\xi = 0$. Along the beam centerline we define the displacement vector $\mathbf{u}(\xi) = \{u_i(\xi)\}$ and the three orthonormal vectors $\mathbf{g}_1(\xi) = \{g_{1i}(\xi)\}$, $\mathbf{g}_2(\xi) = \{g_{2i}(\xi)\}$ and $\mathbf{g}_3(\xi) = \{g_{3i}(\xi)\}$ in the global inertial frame of reference $\mathbf{k}_i$. Director vectors $\mathbf{g}_2$ and $\mathbf{g}_3$ are along the principal axes of inertia of the cross-section $A$. In the beam element, global displacement vector $\mathbf{u}(\xi)$ is composed of rigid and deformation components. In particular, we refer to the $\bar{\mathbf{u}}(\xi) = \{\bar{u}_i(\xi)\}$ rigid displacements defined in the inertial frame of reference while the deformation $\tilde{\mathbf{u}}(\xi) = \{\tilde{u}_i(\xi)\}$ displacements and $\tilde{\boldsymbol{\alpha}}(\xi) = \{\tilde{\alpha}_i(\xi)\}$ rotations are defined in the local rigidly rotated frame of reference. The deformation kinematics is assumed by the linear interpolations: $\tilde{u}_1(\xi) = \varepsilon\xi$, $\tilde{u}_2(\xi) = \gamma_2\xi$, $\tilde{u}_3(\xi) = \gamma_3\xi$, $\tilde{\alpha}_1(\xi) = \theta\xi$, $\tilde{\alpha}_2(\xi) = \chi_3\xi$ and $\tilde{\alpha}_3(\xi) = \chi_2\xi$ for local displacements and rotations.

Based on the above definitions and by referring to the $\overset{o}{\mathbf{g}}_i$ columns of $\overset{o}{\mathbf{G}}$, local rotation vector

$\tilde{\boldsymbol{\alpha}}(\xi)$ and director components $\mathrm{g}_i(\xi)$ of $\mathbf{G}(\xi)$ are now linked by the vectorial operation

$$\mathbf{G}(\xi)\hat{\mathbf{G}} = \overset{o}{\mathbf{G}}\hat{\mathbf{G}}(\mathbf{I} + \tilde{\boldsymbol{\alpha}}_\times(\xi)), \tag{3}$$

where $\tilde{\boldsymbol{\alpha}}$ is the material incremental rotation. We note that the first order accuracy of the (3) representation leads to local evaluations consistent with the small strains hypotheses. By evaluating the (3) relation for $\xi = -h/2$ and $\xi = h/2$, respectively in the $n$ and $m$ nodes, we have $\overset{o}{\mathbf{g}}_i = (\overset{n}{\mathbf{g}}_i + \overset{m}{\mathbf{g}}_i)/2$ and, by using orthonormality of the directors, $\theta = (\overset{n}{\mathbf{g}}_3 \cdot \overset{m}{\mathbf{g}}_2 - \overset{n}{\mathbf{g}}_2 \cdot \overset{m}{\mathbf{g}}_3)/2h$, $\chi_2 = -(\overset{n}{\mathbf{g}}_2 \cdot \overset{m}{\mathbf{g}}_1 - \overset{n}{\mathbf{g}}_1 \cdot \overset{m}{\mathbf{g}}_2)/2h$, $\chi_3 = (\overset{n}{\mathbf{g}}_3 \cdot \overset{m}{\mathbf{g}}_1 - \overset{n}{\mathbf{g}}_1 \cdot \overset{m}{\mathbf{g}}_3)/2h$. Analogously, being

$$\mathbf{u}(\xi) = \overset{o}{\mathbf{u}} + \xi(\overset{o}{\mathbf{g}}_1 - \hat{\mathbf{g}}_1) + \xi(\varepsilon\overset{o}{\mathbf{g}}_1 + \gamma_2\overset{o}{\mathbf{g}}_2 + \gamma_3\overset{o}{\mathbf{g}}_3) \tag{4}$$

the motion of the $\xi$ point, we deduce that $\overset{o}{\mathbf{u}} = (\overset{n}{\mathbf{u}} + \overset{m}{\mathbf{u}})/2$ is the central point displacement and $\varepsilon = [\overset{o}{\mathbf{g}}_1 \cdot (\overset{m}{\mathbf{u}} - \overset{n}{\mathbf{u}}) - h + h\overset{o}{g}_{11}]/h$, $\gamma_2 = [\overset{o}{\mathbf{g}}_2 \cdot (\overset{m}{\mathbf{u}} - \overset{n}{\mathbf{u}}) + h\overset{o}{g}_{21}]/h$, $\gamma_3 = [\overset{o}{\mathbf{g}}_3 \cdot (\overset{m}{\mathbf{u}} - \overset{n}{\mathbf{u}}) + h\overset{o}{g}_{31}]/h$ are the expressions of the axial and shear deformations as a function of nodal displacement and director components.

For the evaluation of the energetic quantities of the beam element, now we consider the referential coordinates $(\xi_i)$ in the element, with $\xi_1 = \xi$ and where $\xi_2$ and $\xi_3$ are along the principal axes of inertia of the cross-section. In the following, if not specified, quantities refer to the central point of the element. We denote with $\mathbf{u}_P(\xi_i)$ the displacement of the generic point $P$ in the element represented in the global reference frame. Then, by defining $\tilde{\mathbf{u}}_P^T(\xi_i) = \xi_1\{\varepsilon + \chi_2\xi_2 - \chi_3\xi_3, \gamma_2 - \theta\xi_3, \gamma_3 + \theta\xi_2\}$, we can refer to the expression

$$\mathbf{u}_P(\xi_i) = \mathbf{u} + \sum_i \xi_i(\mathbf{g}_i - \hat{\mathbf{g}}_i) + \sum_i \tilde{u}_{Pi}(\xi_i)\mathbf{g}_i = \mathbf{u} + \sum_i \xi_i(\mathbf{G} - \mathbf{I})\hat{\mathbf{g}}_i + \mathbf{G}\hat{\mathbf{G}}\tilde{\mathbf{u}}_P(\xi_i), \tag{5}$$

where rigid and deformation components of the motion are recognizable. To summarize the derivation of the variational formulation and related consistent linearization we firstly denote respectively by $\boldsymbol{\varepsilon}^T = \{\varepsilon, \gamma_2, \gamma_3\}$ and $\boldsymbol{\theta}^T = \{\theta, \chi_2, \chi_3\}$ the deformation and the curvature vector. By defining the $dv = (\overset{m}{v} - \overset{n}{v})/h_1$ discrete counterpart of the derivative of $v$ with respect to the reference coordinate $\xi_1$, from the (5) expressions we have

$$\boldsymbol{\varepsilon} = \hat{\mathbf{G}}^T\mathbf{G}^T(d\mathbf{u} + \hat{\mathbf{G}}\mathbf{k}_1) - \mathbf{k}_1 = \hat{\mathbf{G}}^T\mathbf{G}^T d\mathbf{x} - \mathbf{k}_1, \tag{6}$$

where $\mathbf{x}$ is the position vector of the centroid point. Furthermore, from the (3) vectorial operation we can write $\overset{m}{\mathbf{G}}\hat{\mathbf{G}} = \mathbf{G}\hat{\mathbf{G}}(\mathbf{I} + \overset{m}{\tilde{\boldsymbol{\alpha}}}_\times)$ and $\overset{n}{\mathbf{G}}\hat{\mathbf{G}} = \mathbf{G}\hat{\mathbf{G}}(\mathbf{I} + \overset{n}{\tilde{\boldsymbol{\alpha}}}_\times)$. Then, relation $d\mathbf{G}\hat{\mathbf{G}} = \mathbf{G}\hat{\mathbf{G}}d\tilde{\boldsymbol{\alpha}}_\times$ leads to the following expression

$$\boldsymbol{\theta}_\times = d\tilde{\boldsymbol{\alpha}}_\times = \hat{\mathbf{G}}^T\mathbf{G}^T d\mathbf{G}\hat{\mathbf{G}} = \hat{\mathbf{G}}^T\mathbf{G}^T(d\bar{\mathbf{E}}\mathbf{E} + \bar{\mathbf{E}}d\mathbf{E})\hat{\mathbf{G}} = \hat{\mathbf{G}}^T(\mathbf{E}^T d\bar{\boldsymbol{\alpha}}_\times\mathbf{E} + d\boldsymbol{\alpha}_\times)\hat{\mathbf{G}}, \tag{7}$$

where definitions $d\boldsymbol{\alpha}_\times = \mathbf{E}^T d\mathbf{E}$ and $d\bar{\boldsymbol{\alpha}}_\times = \bar{\mathbf{E}}^T d\bar{\mathbf{E}}$ were used. As we can see, the classical definitions of material strain and curvature are obtained in (6) and (7), respectively.

Because rotation operator belongs to the Lie group of proper orthogonal linear transformations $SO(3) = \{\mathbf{E} : \mathcal{R}^3 \to \mathcal{R}^3 \mid \mathbf{E}^T\mathbf{E} = \mathbf{I}, \det\mathbf{E} = +1\}$, admissible variations have to be performed by $\delta\mathbf{E} = \mathbf{E}\delta\boldsymbol{\alpha}_\times$. Then, by (6) and (7) we have

$$\delta\boldsymbol{\varepsilon} = \hat{\mathbf{G}}^T(\delta\boldsymbol{\alpha}_\times^T\mathbf{G}^T d\mathbf{x} + \mathbf{G}^T d\delta\mathbf{u}) \tag{8}$$

and

$$\delta\boldsymbol{\theta}_\times = \hat{\mathbf{G}}^T(d\delta\boldsymbol{\alpha}_\times + d\boldsymbol{\alpha}_\times\delta\boldsymbol{\alpha}_\times - \delta\boldsymbol{\alpha}_\times d\boldsymbol{\alpha}_\times + \mathbf{E}^T d\bar{\boldsymbol{\alpha}}_\times\mathbf{E}\delta\boldsymbol{\alpha}_\times - \delta\boldsymbol{\alpha}_\times\mathbf{E}^T d\bar{\boldsymbol{\alpha}}_\times\mathbf{E})\hat{\mathbf{G}}. \tag{9}$$

respectively. By extracting the axial vectors by the skew symmetric matrices in (7) and (9) we obtain

$$\boldsymbol{\theta} = \hat{\mathbf{G}}^T(\mathbf{E}^T d\bar{\boldsymbol{\alpha}} + d\boldsymbol{\alpha}) \quad \text{and} \quad \delta\boldsymbol{\theta} = \hat{\mathbf{G}}^T[d\delta\boldsymbol{\alpha} + (d\boldsymbol{\alpha}_\times + \mathbf{E}^T d\bar{\boldsymbol{\alpha}}_\times \mathbf{E})\delta\boldsymbol{\alpha}], \tag{10}$$

respectively. In the previous calculations, classical vectorial relations have been used. Finally, if $\mathbf{n} = \mathbf{C}_\varepsilon \boldsymbol{\varepsilon}$, $\mathbf{m} = \mathbf{C}_\theta \boldsymbol{\theta}$, with the material and cross-sectional properties $\mathbf{C}_\varepsilon$=Diag$\{EA, GA, GA\}$ and $\mathbf{C}_\theta$=Diag$\{GJ_t, EJ_2, EJ_3\}$, denote the internal stress resultants energy conjugated to the $\delta\boldsymbol{\varepsilon}$, $\delta\boldsymbol{\theta}$ virtual strain measures, the expression for the internal virtual work is written as: $\delta U = h[\mathbf{n}^T\delta\boldsymbol{\varepsilon} + \mathbf{m}^T\delta\boldsymbol{\theta}]$.

Variation of the kinetic energy $T$ is evaluated by referring to the following expressions:

$$\begin{aligned} \delta\mathbf{u}_P(\xi_i) &= \delta\mathbf{u} + \textstyle\sum_i \xi_i \delta\mathbf{G}\hat{\mathbf{g}}_i + \delta\mathbf{G}\hat{\mathbf{G}}\tilde{\mathbf{u}}_P(\xi_i) + \mathbf{G}\hat{\mathbf{G}}\delta\tilde{\mathbf{u}}_P(\xi_i), \\ \dot{\mathbf{u}}_P(\xi_i) &= \dot{\mathbf{u}} + \textstyle\sum_i \xi_i \dot{\mathbf{G}}\hat{\mathbf{g}}_i + \dot{\mathbf{G}}\hat{\mathbf{G}}\tilde{\mathbf{u}}_P(\xi_i) + \mathbf{G}\hat{\mathbf{G}}\dot{\tilde{\mathbf{u}}}_P(\xi_i), \\ \ddot{\mathbf{u}}_P(\xi_i) &= \ddot{\mathbf{u}} + \textstyle\sum_i \xi_i \ddot{\mathbf{G}}\hat{\mathbf{g}}_i + \ddot{\mathbf{G}}\hat{\mathbf{G}}\tilde{\mathbf{u}}_P(\xi_i) + 2\dot{\mathbf{G}}\hat{\mathbf{G}}\dot{\tilde{\mathbf{u}}}_P(\xi_i) + \mathbf{G}\hat{\mathbf{G}}\ddot{\tilde{\mathbf{u}}}_P(\xi_i), \end{aligned} \tag{11}$$

obtained by variation and time differentiation of displacement vector $\mathbf{u}_P$ in (5). In (11) the evaluations $\dot{\mathbf{G}} = \mathbf{G}\dot{\boldsymbol{\alpha}}_\times$ and $\ddot{\mathbf{G}} = \mathbf{G}(\ddot{\boldsymbol{\alpha}}_\times + \dot{\boldsymbol{\alpha}}_\times\dot{\boldsymbol{\alpha}}_\times)$ result, where the time derivatives $\dot{\boldsymbol{\alpha}}$ and $\ddot{\boldsymbol{\alpha}}$ are respectively the material angular velocity and acceleration vectors. Furthemore, time differentation of the (6) and (10) expressions provides the strain rate vectors as follows

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}} &= \hat{\mathbf{G}}^T(\dot{\boldsymbol{\alpha}}_\times^T\mathbf{G}^T d\mathbf{x} + \mathbf{G}^T d\dot{\mathbf{u}}), \quad \ddot{\boldsymbol{\varepsilon}} = \hat{\mathbf{G}}^T[(\ddot{\boldsymbol{\alpha}}_\times + \dot{\boldsymbol{\alpha}}_\times\dot{\boldsymbol{\alpha}}_\times)^T\mathbf{G}^T d\mathbf{x} + 2\dot{\boldsymbol{\alpha}}_\times^T\mathbf{G}^T d\dot{\mathbf{u}} + \mathbf{G}^T d\ddot{\mathbf{u}}], \\ \dot{\boldsymbol{\theta}} &= \hat{\mathbf{G}}^T[d\dot{\boldsymbol{\alpha}} + (d\boldsymbol{\alpha}_\times + \mathbf{E}^T d\bar{\boldsymbol{\alpha}}_\times\mathbf{E})\dot{\boldsymbol{\alpha}}], \\ \ddot{\boldsymbol{\theta}} &= \hat{\mathbf{G}}^T[d\ddot{\boldsymbol{\alpha}} + (d\dot{\boldsymbol{\alpha}}_\times + \dot{\boldsymbol{\alpha}}_\times^T\mathbf{E}^T d\bar{\boldsymbol{\alpha}}_\times\mathbf{E} + \mathbf{E}^T d\bar{\boldsymbol{\alpha}}_\times\mathbf{E}\dot{\boldsymbol{\alpha}}_\times)\dot{\boldsymbol{\alpha}} + (d\boldsymbol{\alpha}_\times + \mathbf{E}^T d\bar{\boldsymbol{\alpha}}_\times\mathbf{E})\ddot{\boldsymbol{\alpha}}]. \end{aligned} \tag{12}$$

The integration over the section area $A$ of $\ddot{\mathbf{u}}_P^T(\xi_i)\delta\mathbf{u}_P(\xi_i)$ in (11), followed by a further integration over the beam centerline, leads to

$$\begin{aligned} \delta T = \rho h[A\ddot{\mathbf{u}}^T\delta\mathbf{u} + \textstyle\sum_i J_i^r\hat{\mathbf{g}}_i^T(\ddot{\boldsymbol{\alpha}}_\times + \dot{\boldsymbol{\alpha}}_\times\dot{\boldsymbol{\alpha}}_\times)^T\delta\boldsymbol{\alpha}_\times\hat{\mathbf{g}}_i] + \tfrac{1}{12}\rho h^3[A\ddot{\boldsymbol{\varepsilon}}^T\delta\boldsymbol{\varepsilon} + \ddot{\boldsymbol{\theta}}^T\mathbf{J}^d\delta\boldsymbol{\theta}] \\ + \tfrac{1}{12}\rho h^3 A[\hat{\mathbf{g}}_1^T(\ddot{\boldsymbol{\alpha}}_\times + \dot{\boldsymbol{\alpha}}_\times\dot{\boldsymbol{\alpha}}_\times)^T\hat{\mathbf{G}}\delta\boldsymbol{\varepsilon} + \ddot{\boldsymbol{\varepsilon}}^T\hat{\mathbf{G}}^T\delta\boldsymbol{\alpha}_\times\hat{\mathbf{g}}_1], \end{aligned} \tag{13}$$

where we have introduced the geometrical coefficient matrices $\mathbf{J}^r$=Diag$\{Ah^2/12, J_2, J_3\}$ and $\mathbf{J}^d$=Diag$\{J_1, J_2, J_3\}$, while $\rho$ is the mass density. In (13) the rigid, deformation and mixed kinetic terms can be recognized. In particular we note how the terms $J_i^r(\ddot{\boldsymbol{\alpha}}_\times + \dot{\boldsymbol{\alpha}}_\times\dot{\boldsymbol{\alpha}}_\times)\hat{\mathbf{g}}_i$ reproduce the rigid inertial components of the angular momentum of the element.

We note that the definitions hitherto given use $\boldsymbol{\alpha}$ as primary variable to describe rotations. These definitions are consistent when a multiplicative representation is used in relation to the $\boldsymbol{\alpha}$ incremental rotation. The (1) update procedure must be performed at every Newton iteration within the time step $k$ of the solution process. When the linear tranformation $\delta\boldsymbol{\alpha} = \mathbf{T}(\mathbf{p})\delta\mathbf{p}$ between $\delta\boldsymbol{\alpha}$ angular and $\delta\boldsymbol{p}$ parameterization vector variations consistent with the $\delta\mathbf{E} = \mathbf{E}\delta\boldsymbol{\alpha}_\times$ relation is given, the above definitions become admissible with the additive representation: $\boldsymbol{p}_{(k+1)} = \bar{\boldsymbol{p}}_{(k)} + \boldsymbol{p}_{(k)}$, $\bar{\mathbf{E}}_{(k+1)} = \mathbf{E}(\boldsymbol{p}_{(k+1)})$, where $\boldsymbol{p}_{(k)}$ is the incremental parameterization vector, as $\boldsymbol{d}_{(k)}$, $\boldsymbol{\psi}_{(k)}$ or $\boldsymbol{r}_{(k)}$. Essentially, by the linear tranformation the configuration space is simplified to a linear vector space. The $\boldsymbol{p}$ parameterization vector is used as primary variable while, by denoting with $\delta W$ the virtual work of external non-conservative loads, $\int_{t_1}^{t_2}(\delta T - \delta U + \delta W)dt = 0$ is the calssical form of the Hamilton's principle to find the $\mathbf{f}(\mathbf{u}, \boldsymbol{p})^T\{d\delta\mathbf{u}, \delta\boldsymbol{p}, d\delta\boldsymbol{p}\} = 0$ equilibrium equations. We note that the tangent stiffness matrix related to the multiplicative representation is unsymmetric although inexpensive calculations are required to compute its coefficients.

In contrast, in the additive representation the linearization provides symmetric but complex tangent stiffness matrices. In effect, seconde derivative of the $\mathbf{T}(\boldsymbol{p})$ operator is involved in the linerization process which would be complicated to calculate. However, as said the multiplicative process requires frequently matrices products to the composition of rotations.

Here an additive approach is used and the linear transformation $\delta\boldsymbol{\alpha} = \mathbf{T}(\mathbf{d})\delta\mathbf{d}$ is established by imposing $d_\epsilon \mathbf{E}(\mathbf{d} + \epsilon\delta\mathbf{d}) = \mathbf{E}(\mathbf{d})\delta\boldsymbol{\alpha}_\times$, where we use the symbol $d_\epsilon$ to denote operator $d/d\epsilon\ |_{\epsilon=0}$. Since in the director parameterization the $E_{23}$, $E_{13}$ and $E_{12}$ coefficients of $\mathbf{E}$ compose directly the rotation vector $\mathbf{d}$, we can write: $d_\epsilon[\mathbf{E}(\mathbf{d} + \epsilon\delta\mathbf{d})]_{23} = d_\epsilon(e_{32} + \epsilon\delta e_{32}) = \delta d_1 = \delta\alpha_2 e_{12} - \delta\alpha_1 e_{22}$, $d_\epsilon[\mathbf{E}(\mathbf{d} + \epsilon\delta\mathbf{d})]_{13} = d_\epsilon(e_{31} + \epsilon\delta e_{31}) = \delta d_2 = \delta\alpha_2 e_{11} - \delta\alpha_1 e_{21}$, $d_\epsilon[\mathbf{E}(\mathbf{d} + \epsilon\delta\mathbf{d})]_{12} = d_\epsilon(e_{21} + \epsilon\delta e_{21}) = \delta d_3 = -\delta\alpha_3 e_{11} + \delta\alpha_1 e_{31}$. Then, we obtain

$$\delta\mathbf{d}=\begin{bmatrix}-e_{22} & e_{12} & 0\\ -e_{21} & e_{11} & 0\\ e_{31} & 0 & -e_{11}\end{bmatrix}\delta\boldsymbol{\alpha}=\mathbf{T}(\mathbf{d})^{-1}\delta\boldsymbol{\alpha},\ \ \delta\boldsymbol{\alpha}=\frac{1}{e_{33}}\begin{bmatrix}-e_{11} & e_{12} & 0\\ -e_{21} & e_{22} & 0\\ -e_{31} & e_{12}e_{31}/e_{11} & e_{33}/e_{11}\end{bmatrix}\delta\mathbf{d}=\mathbf{T}(\mathbf{d})\delta\mathbf{d}. \quad (14)$$

In the inverse relation of (14) the $e_{33} = e_{22}e_{11} - e_{21}e_{12}$ relation was used. From (14) it is clearly seen that $\mathbf{T}(\mathbf{d})^{-1}$ and $\mathbf{T}(\mathbf{d})$ operators are well defined in the assigned range of validity of the parameterization and that $\mathbf{T}(\mathbf{d})$ reduces to $\mathbf{I}$ as $\mathbf{d}$ goes to zero. We note that analogous relations $d\boldsymbol{\alpha} = \mathbf{T}(\mathbf{d})d\mathbf{d}$ and $\dot{\boldsymbol{\alpha}} = \mathbf{T}(\mathbf{d})\dot{\mathbf{d}}$ are established for the space and time differentiation, respectively. Linear transformations related to the $\boldsymbol{\psi}$ and $\mathbf{r}$ parameters, instead, are given by the expressions: $\mathbf{T}(\boldsymbol{\psi}) = \mathbf{I} + (1 - \cos\varphi)/\varphi^2\boldsymbol{\psi}_\times + (\varphi - \sin\varphi)/\varphi^3\boldsymbol{\psi}_\times\boldsymbol{\psi}_\times$ and $\mathbf{T}(\mathbf{r}) = \mathbf{I}{+}4/(4 + r^2)(\mathbf{I}{+}1/2\mathbf{r}_\times)$, respectively.

External work $W$, finally, is defined by referring to the (5) expressions of the displacement vector. Note that as the kinematics of the element being modelled as a three dimensional body, only external forces must be assigned. In particular, spatially fixed moments are modelled as forces following the motion of points of the beam element.

## 4 NONLINEAR DYNAMICAL ANALYSIS

We refer to dynamical systems with $\mathbf{f}(\mathbf{q}(t), \dot{\mathbf{q}}(t), \ddot{\mathbf{q}}(t)) = \mathbf{0}$ motion equation, where $\mathbf{q}$ is the vector of the unknown components of the element. Unknown vector $\mathbf{q}$ is composed of the incremental $u_i$ displacements and $p_i$ rotation parameters at the nodes of the element. For the time integration of the related semidiscrete initial value problem we refer to the constant time step $\Delta t = t_{k+1} - t_k$. By assuming the state variables $\bar{\mathbf{q}}_{(k)}$, $\dot{\bar{\mathbf{q}}}_{(k)}$, $\ddot{\bar{\mathbf{q}}}_{(k)}$, as known at the time $t_k$ and making the external forces $\mathbf{g}(t)$ for all $t$, the time integration is restricted to the subsequent solution of the state variables at the end of each step $\mathbf{q}_{(k)}$, $\dot{\mathbf{q}}_{(k)}$, $\ddot{\mathbf{q}}_{(k)}$.

In order to realize the step by step integration, the set of variables is reduced to the incremental unknowns $\mathbf{q}_{(k)}$ only by the average acceleration Newmark scheme. In particular, for the displacements we have the usually expressions $\dot{\mathbf{u}}_{(k)} = 2/\Delta t\mathbf{u}_{(k)} - \dot{\bar{\mathbf{u}}}_{(k)}$, $\ddot{\mathbf{u}}_{(k)} = 4/\Delta t^2\mathbf{u}_{(k)} - 4/\Delta t\dot{\bar{\mathbf{u}}}_{(k)} - \ddot{\bar{\mathbf{u}}}_{(k)}$, while for the rotations we use the expressions

$$\dot{\boldsymbol{\alpha}}_{(k)} = \frac{2}{\Delta t}\mathbf{T}(\mathbf{p}_{(k)})\mathbf{p}_{(k)} - \dot{\bar{\boldsymbol{\alpha}}}_{(k)}, \quad \ddot{\boldsymbol{\alpha}}_{(k)} = \frac{4}{\Delta t^2}\mathbf{T}(\mathbf{p}_{(k)})\mathbf{p}_{(k)} - \frac{4}{\Delta t}\dot{\bar{\boldsymbol{\alpha}}}_{(k)} - \ddot{\bar{\boldsymbol{\alpha}}}_{(k)}. \quad (15)$$

The proposed Newmarks integration (15), then, uses angular velocities and accelerations instead of $\dot{\mathbf{p}}$ and $\ddot{\mathbf{p}}$ time derivatives of rotation parameters. Note that in the $\Delta t$ time step, being $\mathbf{p}$ the incremental parametrization vector, the term $\mathbf{T}(\mathbf{p}_{(k)})\mathbf{p}$ is the consistent incremental rotation.

By inserting the Newmarks integration relations in the motion equation, we arrive at the nonlinear equation $\mathbf{f}(\mathbf{q}_{(k)}) = 0$. This represents the nonlinear system of algebraic equations defined at the $t_k$ time with the $\mathbf{q}_{(k)}$ unknown vector. The velocities and accelerations at the end of the time step can then be obtained by the given Newmarks relations, while displacements and parameters are updated by $\bar{\mathbf{u}}_{(k+1)} = \bar{\mathbf{u}}_{(k)} + \mathbf{u}^{(*)}_{(k)}$ and $\bar{\mathbf{p}}_{(k+1)} = \bar{\mathbf{p}}_{(k)} + \mathbf{p}^{(*)}_{(k)}$, respectively. The Newton iterative method was used to compute the $n$-th approximation $\mathbf{q}^{(n)}_{(k)}$ of the solution point $\mathbf{q}^{(*)}_{(k)}$.

Two numerical examples compare the application of the (DIR) proposed formulation with (ROT) rotational vector and (RC) Rodrigues-Cayley based parameterization for the time integration of the motion equations. Stable behaviour of the time integration method, in the absence of numerical dissipation, is verified by referring to the conservation of the total energy condition $U_{(k)} + T_{(k)} = W_{(k)}$. We use the trapezoidal rule $\Delta W = 1/2(\mathbf{u}_{(k)} - \bar{\mathbf{u}}_{(k)})^T(\mathbf{g}_{(k)} + \bar{\mathbf{g}}_{(k)})$ to calculate the increment of the external forces work. Let $steps$ be the number of time steps effected by the integration process to analyze the behaviour for $t = 0...T$. In the following we refer to the mean value $N_m = \sum_{k=1}^{steps} Nw_{(k)}/steps$ of the $Nw_{(k)}$ Newton iterations in the $k$-th step. The ($t_s$) CPU time (s) spent in the whole analysis is also recorded.

When compared to reference results, similar equilibrium states computed by the described treatments of finite rotations and beam element model are obtained. Additionally, the differences between the computed equilibrium paths obtained by DIR, ROT and RC like parameterizations are negligible, unless the integration process becomes unstable ($div$). In effect, whereas the treatment of the rotations is different, the beam finite element used is the same. Finally, at present, we are primarily concerned with comparing the number of arithmetical operations and simplicity in programming rather than discretization error.

### 4.1 Toss rule in space

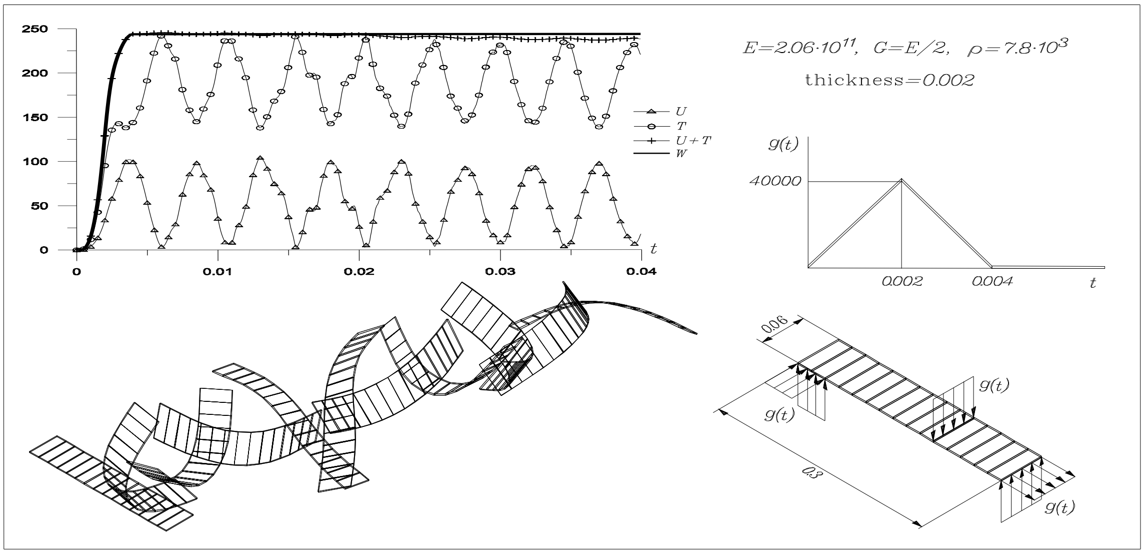


Figure 1: Toss rule in space: problem definition, motion of the toss rule, energies vs time.

This example refers to the three-dimensional movement of a toss rule (see Kuhl and Ramm [5] for a solution to such a dynamical problem). The rule, as described in Fig. 1 and with zero initial displacements and velocities, is discretized by 15 and 45 described finite elements. Deformed configurations after each $0.004$ time increments computed within $t = 0...0.04$ and

| | 15 elements | | | | | | 45 elements | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta t$ | 0.00001 | | 0.00005 | | 0.0001 | | 0.00001 | | 0.00005 | | 0.0001 | |
| | $t_s$ | $N_m$ | $t_s$ | $N_m$ | $t_s$ | $N_m$ | $t_s$ | $N_m$ | $t_s$ | $N_m$ | $t_s$ | $N_m$ |
| DIR | 13.95 | 3.000 | 3.50 | 4.013 | 2.64 | 5.463 | 35.66 | 3.000 | 9.34 | 4.033 | 5.07 | 5.781 |
| ROT | 14.89 | 3.000 | 3.82 | 4.121 | *div* | | 38.22 | 3.000 | 10.57 | 4.079 | *div* | |
| RC | 12.91 | 3.000 | 3.45 | 4.058 | *div* | | 33.24 | 3.000 | 8.87 | 4.042 | *div* | |

Table 1: Toss rule in space: computational characteristics of the considered formulations.

conservation of the total energy condition by the DIR representation with $\Delta t = 0.00005$ and 15 elements are also shown in Fig. 1. Tab. 1 shows the behaviour of the different schemes with respect to several $\Delta t$ time integration steps.

### 4.2 Right-angle cantilever

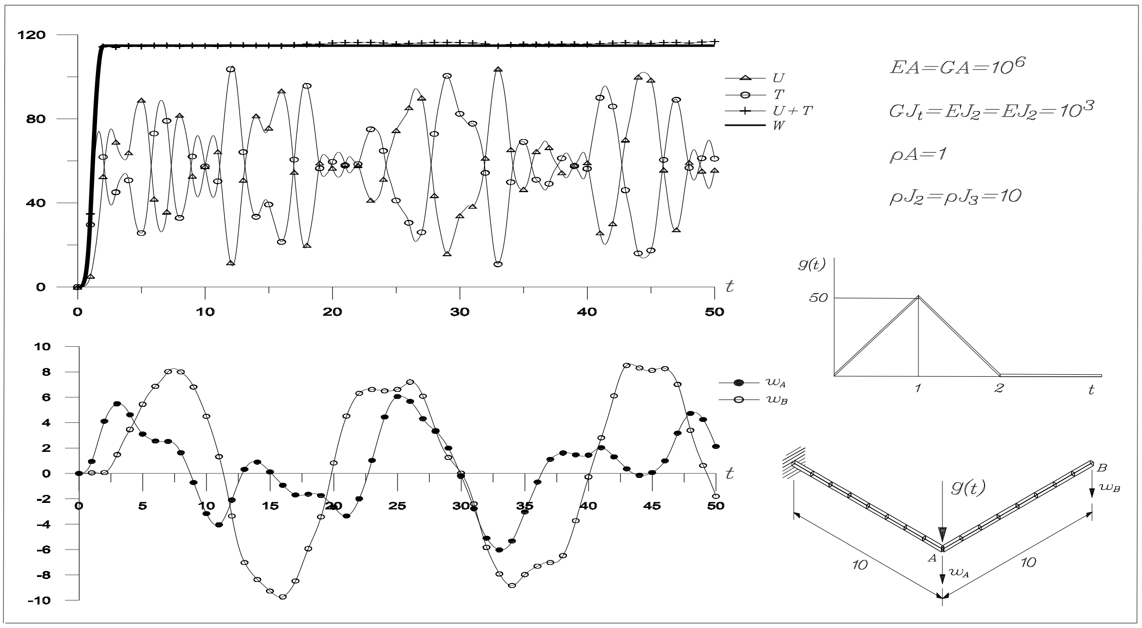


Figure 2: Right-angle cantilever: problem definition, out-of-plane displacements, energies vs time.

For this test we refer to the reference results obtained in Jelenic and Crisfield [4]. An L shape cantilever beam (see Fig. 2) is set into motion by applying an out-of-plane concentrated load at its elbow. In the time interval of interest, until $T = 50$, the cantilever is undergoing free vibrations of finite amplitude under combined bending and torsion. The model uses 16 or 32 linear beam elements for the entire structure. The computed response for the elbow and the tip out-of-plane displacements and behaviour of the energies for $\Delta t = 0.05$ and 16 elements are also given in Fig. 2. Also here, Tab. 2 shows the behaviour of the schemes with respect to several $\Delta t$ time steps.

| | 16 elements | | | | | | 32 elements | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta t$ | 0.01 | | 0.05 | | 0.1 | | 0.01 | | 0.05 | | 0.1 | |
| | $t_s$ | $N_m$ | $t_s$ | $N_m$ | $t_s$ | $N_m$ | $t_s$ | $N_m$ | $t_s$ | $N_m$ | $t_s$ | $N_m$ |
| DIR | 17.53 | 3.000 | 3.94 | 3.555 | 2.26 | 4.218 | 31.26 | 3.000 | 7.41 | 3.644 | 4.25 | 4.246 |
| ROT | 18.87 | 3.000 | 4.23 | 3.613 | 2.42 | 4.255 | 33.34 | 3.000 | 8.01 | 3.682 | 4.61 | 4.307 |
| RC | 16.56 | 3.000 | 3.60 | 3.596 | 2.13 | 4.234 | 29.28 | 3.000 | 6.92 | 3.647 | 3.97 | 4.271 |

Table 2: Right-angle cantilever: computational characteristics of the considered formulations.

## 5 CONCLUSIONS

Based on the classical implicit one-step time integration scheme with Newmark approximations, a technique to analyse the dynamical behaviour of large three-dimensional rotations beam frames has been presented. By utilizing vectorial operations, the approach uses an updated Lagrangian description of rotations. The presented parameterization leads to computationally efficient expressions in the equations of the resultant nonlinear system. In the numerical tests a similar number of mean value of Newton iterations to complete the analysis was used for the compared parameterizations. By the numerical tests we have that fewer arithmetical operations in respect to the ROT formulation and less implementation effort with respect to the ROT and RC ones are required. Overall, the proposed formulation shows simplicity of the analysis while computational effectiveness and algorithmic reliability are retained.

## REFERENCES


[1] P. Betsch, P. Steinmann, Constrained dynamics of geometrically exact beams. *Comput Mech* **31**, 49–59, 2003.

[2] C.A. Felippa, *A systematic approach to the element-independent corotational dynamics of finite elements*. Technical Report CU-CAS-95-06 Center for Aerospace, Structures Dept of Aerospace Engineering, Univ. Colorado, Boulder, 2000.

[3] A. Ibrahimbegović, M. Al Mikdad, Finite rotations in dynamics of beams and implicit time-stepping schemes. *Int J Numer Methods Eng*, **41**, 781–814, 1998.

[4] G. Jelenic, M.A. Crisfield, Interpolation of rotational variables in ninlinear dynamics of 3D beams. *Int J Numer Methods Eng*, **43**, 1193–1222, 1998.

[5] D. Kuhl, E. Ramm, Generalized energy-momentum method for non-linear adaptive shell dynamics, *Comp Meth Appl Mech Eng*, **178**, 343–366, 1999.

[6] S. Lopez, A three-dimensional beam element undergoing finite rotations based on slopes and distance measures. *Int J Comput Methods*, **9**, 1250054, 2012.

[7] S. Lopez, Relaxed representations and improving stability in time-stepping analysis of three-dimensional structural nonlinear dynamics. *Nonlinear Dyn*, **69**, 705–720, 2012.

[8] S. Lopez, Three-dimensional finite rotations treatment based on a minimal set parameterization and vector space operations in beam elements. *Comput Mech*, in press.

[9] P.M. Pimenta, E.M.B. Campello, P. Wriggers, An exact conserving algorithm for nonlinear dynamics with rotational DOFs and general hyperelasticity. Part 1: Rods. *Comput Mech*, **42**, 715–732, 2008.