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NONLINEAR DYNAMICS OF THICK LEAF SPRING USING TIMOSHENKO BEAM FUNCTIONS

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Abstract. *Leaf springs are widely used in automobile suspension, vibrating feeder beds and in many other industrial applications. Conventional design consists of multiple leaves having gradually reducing slenderness ratio and analysis is based on Euler-Bernoulli beam theory, which is valid for small deflection only. However, the shear deformation effect of the smaller length leaves become significant compared to their bending deformation. In addition the deflection characteristics of leaf springs are highly nonlinear and modelling aims to find the optimized shapes of the leaf spring. To resolve these practical and theoretical issues the entire leaf spring can be modelled as a single thick cantilever beam. The proposed thick leaf spring would be easy to manufacture and maintain, and in addition it would sustain the required elastic deformation characteristics of an existing leaf spring. The present study undertakes a thick taper cantilever beam with geometric nonlinearity and reports the dynamic beam characteristic curves in normalized load-deflection plane. In the present study, in addition to the transverse deflection and cross-sectional rotation of a thick leaf spring, the effect of boundary restraints has also been taken into consideration. For better characterization of such beams geometric non-linear analysis is carried out and moreover updated Lagrangian analysis considers the changed geometry of beam as well. The problem is formulated by employing an energy based variational method.*

1 INTRODUCTION

A host of research papers are available in the domain of leaf springs due to its wide application in industries. Sugiyama et al. [1] reported development of nonlinear elastic leaf spring model for multi body vehicle systems. Roy and Saha [2] reported nonlinear analysis of leaf springs of Functionally Graded Materials. The initial research work was based on thin beam theory but many research work is found for thick beam modelling as well. The shear deformation and rotary inertia effects are included in the Timoshenko beam theory which calls for two basic reference quantities, rather than the flexural deflection alone as in classical beam theory. Huang [3] presented an early work on the determination of Timoshenko beam functions for a uniform beam with simple end conditions. Abbas [4] has solved the problem of an elastically supported Timoshenko beam by a finite element method. Saha et al. [5] have used the Timoshenko beam functions for the free vibration analysis of elastically restrained plates. Wattanasakulpong et al. [6] employed third order shear deformation theory to formulate a governing equation for predicting free vibration of layered functionally graded beams. Geometric nonlinear large amplitude forced vibration of beams was studied by Ribeiro [7] using shooting, Newton and p-version hierarchical finite element methods. Ding et al. [8] presented analytical and finite element analysis of the nonlinear cantilever response as a combined result of large deflection and non-vertical loading and investigated a general study of cantilever beam nonlinearity under a variety of loading conditions.

The review work reveals that much research has been carried out on the static and dynamic behaviour of Timoshenko beams and similarly various studies on geometric nonlinear behaviour have been carried out using Euler's thin beam theory. In the present paper, a combination of the two is intended to visualize the physical dynamic behaviour of leaf springs more accurately. It is also observed that various mathematical methods have been used in solving the beam problem in question and they have not been classified here to maintain brevity. In the present paper, problem is formulated by employing an energy based variational method and solution is sought by applying an improvised updated Lagrangian technique.

2 MATHEMATICAL FORMULATION

Figure 1 shows a leaf spring and its equivalent theoretical modelling with notations of significant dimensions and coordinate system for the present analysis. The equivalent beam is analyzed as a Timoshenko beam, which calls for two independent field variables w and ψ (or θ). All the notations used in this paper are indicated separately in Table 1 and hence they are not repeated in the body of the paper. As shown in the right hand side of Figure 2, Timoshenko beam theory refers to first order shear deformation, and although it is not the depiction of actual beam deformation it is rather an improvement over Euler beam theory. The actual rotation of the normal to the neutral axis of the beam is given by $\theta = (dw/dx) - \psi$ and hence the axial stretching strain and the shear strain induced in the beam are expressed as, $\varepsilon_{xx} = -z\dfrac{\partial \theta}{\partial x}$ and $\varepsilon_{xz} = \dfrac{1}{2}\left(-\theta + \dfrac{\partial w}{\partial x}\right)$. The shear stress induced in the beam is $\tau_{xz} = \kappa G \psi$, expressed through shear strain distribution co-efficient κ, which for a solid rectangular section is taken as $\{10(1+\nu)/(12+11\nu)\}$. Using these strain displacement relations and following generalized Hooke's law, the strain energy of the Timoshenko beam (U_b) is determined.

$$U_b = \int_0^L \int_{-h/2}^{h/2} \left\{ \frac{1}{2}(\tau_{xx}\varepsilon_{xx}) + (\tau_{xz}\varepsilon_{xz}) \right\} bdzdx = \frac{1}{2}\int_0^L \left\{ EI\left(\frac{d\theta}{dx}\right)^2 + \kappa GA\left(\frac{dw}{dx} - \theta\right)^2 \right\} dx. \quad (1)$$

Symbol	Description
A, b, h	c/s. area of beam, beam width and beam thickness at location x
b_0, b_1	beam width at root (ξ = 0) and tip (ξ = 1) respectively
$\{d_i\}$	vector of unknown co-efficients
E, ν, ρ	elasticity modulus, Poisson's ratio and density, respectively
f, g	set of functions for approximating w and θ
h_0, h_1	beam thickness at root (x = 0) and tip (x = L) respectively
I, I_0	area moment of inertia at general location and at root, respectively
[K], [M]	stiffness and mass matrix respectively
K_r, K_t	dimensional spring stiffness, rotational and translational, respectively
l, L	current horizontal length and overall length of the beam
P, p	concentrated and distributed load respectively
q_1, q_2	slenderness parameter and inertia parameter, respectively
S_r, S_t	normalized spring stiffness, rotational and translational, respectively
T, U, V	kinetic energy, strain energy and work energy functionals, respectively
U_b, U_s	strain energy stored in the beam and the springs respectively
w	beam deflection
α, β	parameters used in displacement functions
δ, δ^*	variational operator, normalized beam deflection, h/L
λ	load parameter, PL^2/EI_0, pL^3/EI_0 or $(P+pL)L^2/EI_0$
$\{\phi_i\}$	load vector
κ	Shear strain distribution correction factor
ψ, θ	shear deformation and bending slope of the beam, respectively
σ, ε	stress and strain
ω, Ω	normalized and dimensional frequency parameter, respectively
i, j	free subscript
0, 1	subscript used to indicate root (ξ = 0) and tip (ξ = 1) respectively
()'	d( )/dξ, etc.

Table 1: Nomenclatures

The conventional clamped free boundary conditions of an Euler beam fails to capture the actual rotation of the beam normal θ. In the first part of Figure 2, an elastically restrained Timoshenko beam is shown, which is used to model the boundary conditions of the leaf spring. The benefit of elastically restrained modelling is twofold: first it is able to capture any classical boundary condition by setting appropriate values to the spring constants and secondly it is possible to avoid shear locking phenomenon, which is manifested for clamped and free boundaries. The boundary conditions of the present elastically restrained modeling is given by,

$$\begin{aligned} &\theta' - S_{r0}\theta = 0 \text{ and } (w' - \theta) + S_{t0} q_1 w = 0 \text{ at } \xi = 0 \\ &\theta' + S_{r1}\theta = 0 \text{ and } (w' - \theta) - S_{t1} q_1 w = 0 \text{ at } \xi = 1 \end{aligned} . \tag{2}$$

Equation (2) is expressed in normalized coordinate ξ = x/L, and the normalized rotational and translational spring stiffness are expressed as, $S_t = K_t \mathrm{L}^3/EI$ and $S_r = K_r \mathrm{L}/EI$ respectively. To simulate clamped free boundary condition, values of non-dimensional translational and rotational restraints are taken as S_{r0}=10^7, S_{t0}=10^7, S_{r1}=10^{-7} and S_{t1}=10^{-7}, as suggested by Bapat et al. [9].

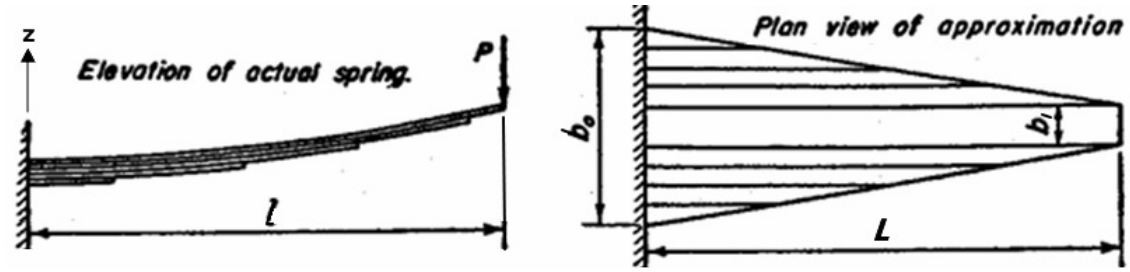

Figure 1: A leaf spring and its theoretical modelling.

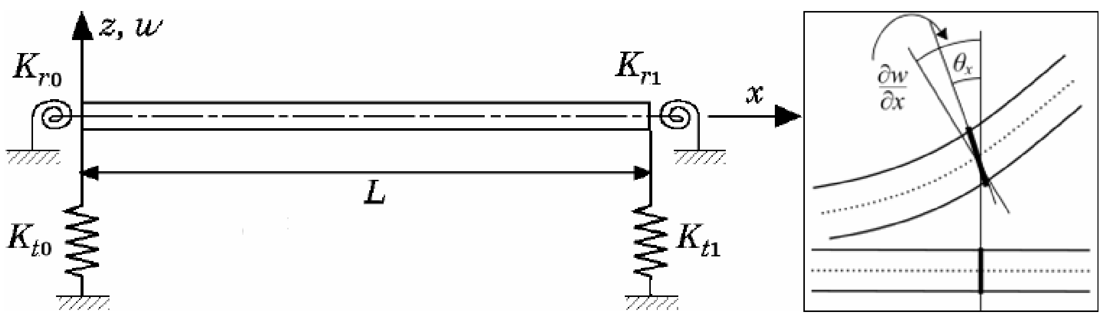

Figure 2: An elastically restrained Timoshenko beam and actual rotation of it's normal.

Determination of the two independent displacement fields for w and θ, in connection with the vibration of the leaf spring under consideration, requires solution of the following two equations.

$$\begin{aligned} & w^{\mathrm{IV}} + \omega^2(\mathrm{q}_1+\mathrm{q}_2)w^{\mathrm{II}} - \omega^2(1-\omega^2\mathrm{q}_1\mathrm{q}_2)w = 0, \\ & \theta^{\mathrm{IV}} + \omega^2(\mathrm{q}_1+\mathrm{q}_2)\theta^{\mathrm{II}} - \omega^2(1-\omega^2\mathrm{q}_1\mathrm{q}_2)\theta = 0 \,. \end{aligned} \tag{3}$$

The parameters used in Eq. (3) are $\mathrm{q}_1 = \mathrm{EI}/\kappa\mathrm{GAL}^2$, $\mathrm{q}_2 = I/\mathrm{AL}^2$ and $\omega^2 = \Omega^2\rho\mathrm{AL}^4/\mathrm{EI}$. Analytical solution of w and θ, as reported in [5] is used to determine the set of functions, which spans the domain and satisfies the boundary conditions. Some relevant equations are reproduced below for ready reference. The approximate analytical solutions, as obtained in this stage will be useful in the subsequent formulation based on energy functionals.

$$\begin{aligned} & \mathrm{w}(\xi) = \mathrm{C}_1 \cosh \omega\alpha\xi + \mathrm{C}_2 \sinh \omega\alpha\xi + \mathrm{C}_3 \cos \omega\alpha\xi + \mathrm{C}_4 \sin \omega\alpha\xi \,, \\ & \theta(\xi) = \overline{\mathrm{C}}_1 \cosh \omega\beta\xi + \overline{\mathrm{C}}_2 \sinh \omega\beta\xi + \overline{\mathrm{C}}_3 \cos \omega\beta\xi + \overline{\mathrm{C}}_4 \sin \omega\beta\xi \,. \end{aligned} \tag{4}$$

The parameters α and β of Eq. (4) are given by,

$$\alpha = \frac{1}{\sqrt{2}}\left\{\sqrt{(\mathrm{q}_1-\mathrm{q}_2)^2 + 4/\omega^2} - (\mathrm{q}_1+\mathrm{q}_2)\right\}^{1/2}, \; \beta = \frac{1}{\sqrt{2}}\left\{\sqrt{(\mathrm{q}_1-\mathrm{q}_2)^2 + 4/\omega^2} + (\mathrm{q}_1+\mathrm{q}_2)\right\}^{1/2}.$$

The governing set of equations for the dynamic system is derived following Hamilton's principle, which states that,

$$\delta\left(\int_{\tau 1}^{\tau 2}(T-U-V)dt\right)=0\,. \tag{5}$$

The kinetic energy (T) of the system and the potential energy (V) is given below,

$$T=\frac{1}{2}\rho L\int_0^1\left(\dot{w}^2+\dot{\theta}^2\right)A d\xi \text{ and } V=-Pw\,|_{x=L}-\int_0^L(pw)dx\,. \tag{6}$$

The potential energy comes from the work done by the external uniform (p) and concentrated (P) load at the tip of the beam. The strain energy of the beam system (U) consists of two parts, energy stored in beam (U_b) and energy stored in the springs (U_s). The expression of strain energy functional U_b has already been indicated in Eq. (1) and the expression for U_s is given below.

$$U_s=\frac{1}{2}K_{t0}\left(w\big|_{\xi=0}\right)^2+\frac{1}{2}K_{t1}\left(w\big|_{\xi=1}\right)^2+\frac{1}{2}K_{r0}\left(\theta\big|_{\xi=0}\right)^2+\frac{1}{2}K_{r1}\left(\theta\big|_{\xi=1}\right)^2. \tag{7}$$

The energy functionals (T, U and V) can be determined from the assumed transverse (w) and rotational (θ) dynamic displacement fields, which are defined at the neutral axis of the beam. Substitution of the complete energy functional expressions in Hamilton's equation, the governing equation of the system is obtained.

$$\int_{\tau 1}^{\tau 2}\left\{\int_0^L\left[\begin{array}{l}\left\{-\dfrac{\partial}{\partial t}\left(\rho I\dot{\theta}\right)+\dfrac{\partial}{\partial x}\left(EI\dfrac{\partial\theta}{\partial x}\right)+\kappa GA\left(\dfrac{\partial w}{\partial x}-\theta\right)\right\}\delta\theta+\\ \left\{-\dfrac{\partial}{\partial t}\left(\rho A\dot{w}\right)+\dfrac{\partial}{\partial x}\left[\kappa GA\left(\dfrac{\partial w}{\partial x}-\theta\right)\right]+p\right\}\delta w\end{array}\right]dx\right\}dt=0\,. \tag{8}$$

It should be mentioned that Eq. (8) is shown for uniformly distributed loading and neglecting the contributions of the spring energies. The displacements of the above equation are dynamic and assumed to be separable in space and time and they can be approximately represented by finite linear combinations of orthogonal admissible functions and a set of unknown coefficients (d_i), as follows.

$$w(\xi,\tau)\cong\sum d_i f_i\,\mathrm{e}^{\mathrm{i}\omega\tau},\ i=1,2,\ldots,nw \text{ and } \theta(\xi,\tau)\cong\sum d_{j+nw}g_j\,\mathrm{e}^{\mathrm{i}\omega\tau},\ j=1,2,\ldots,n\theta\,. \tag{9}$$

Here, ω represents response frequency of the system under the specified loading. The spatial functions f, g describe the displacements due to plate bending and rotation, respectively. The necessary set of functions f_i and g_j required for approximating w and θ are selected from Eq. (4). Substituting Eq. (9) in Eq. (8), gives the set of system governing equations in the following matrix form

$$-\omega^2[M]\{d\}+[K]\{d\}=\{0\} \tag{10}$$

where $[K]$, $[M]$ and $\{0\}$ are stiffness matrix, mass matrix and null vector respectively. Element wise details of the matrices are not furnished here to maintain brevity. The set of governing equations given in (10) are the equations of a standard eigen-value problem, but cannot be solved readily as the stiffness matrix $[K]$ is a function of the loading parameters.

Before addressing to the solution of Eq. (10), we need to look into another issue. The mathematical formulation as presented in Eq. (10) is applicable for small deflection only $\left(\delta^*\cong h/L\right)$, beyond which strain-displacement relations become non-linear. In the present paper, the effect of geometric nonlinearity is implemented iteratively, in which the total load on the beam is imposed incrementally. In each load step, a correction on projected beam length is carried out such that the length of the deflection curve remains constant to the original straight length of the beam. This is shown through a schematic representation of the bending curve, in Figure 3. The elemental beam length shown in figure is given by

$$ds = \left(1 + \left(dw/dx\right)^2\right)^{1/2} dx \ . \tag{11}$$

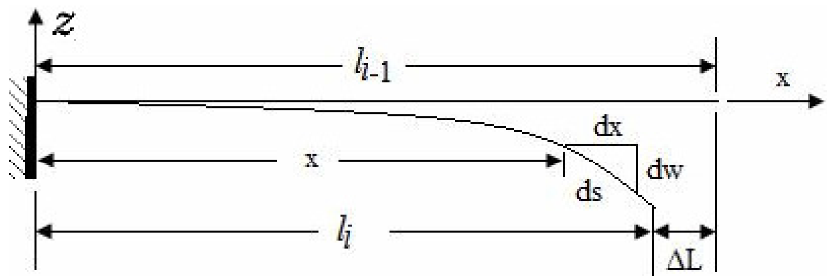

Figure 3: Representation of bending curve for an intermediate load step

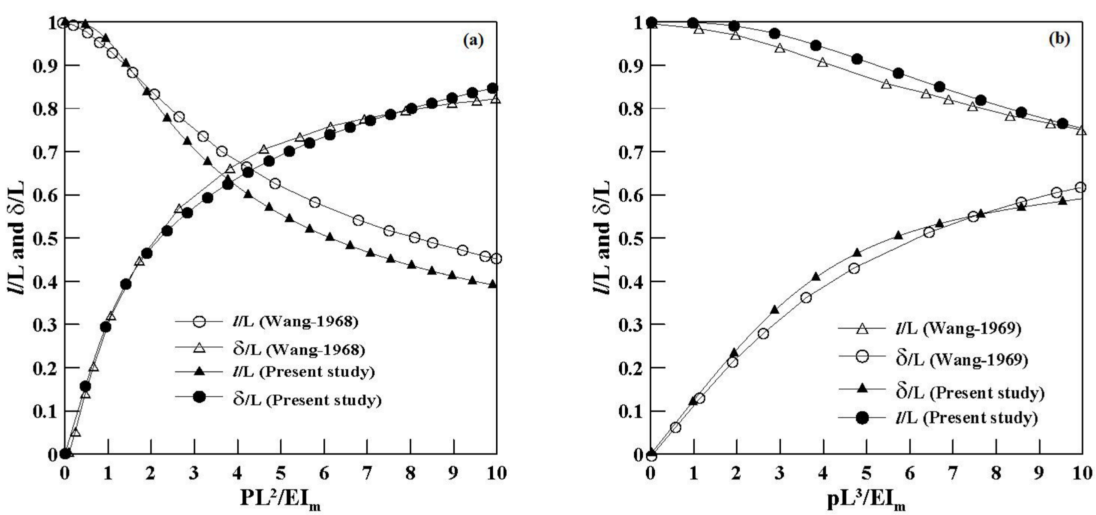

Figure 4: Validation study on δ/L and l/L for concentrated and uniform loading

Equation (11) is integrated to obtain the stretching of beam length

$$\Delta = \int_0^L \left[\left\{1 + \left(dw/dx\right)^2\right\}^{1/2} - 1\right] dx \, . \tag{12}$$

Δ has horizontal (ΔL) and vertical $\left(w\,|_{x=L}\right)$ components which are determined to obtain the effective beam length for the next incremental load step, i.e., $\left(l_i = l_{i-1} - \Delta L\right)$. It should be noted that, in each load step, analysis for deflection profile is carried out following Timoshenko beam theory. It is further assumed that direction of load remains unchanged during the process of beam deflection, which is in-line with the present formulation. The phenomenon of follower loading requires a different formulation altogether. The detail procedure of the proposed geometry updation technique is reported in [2] and hence is not repeated here.

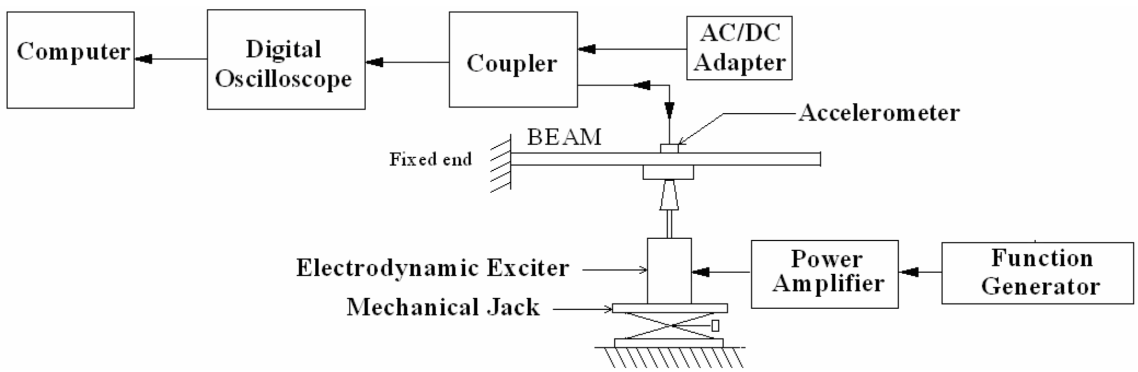

Figure 5: Schematic diagram of experimental setup

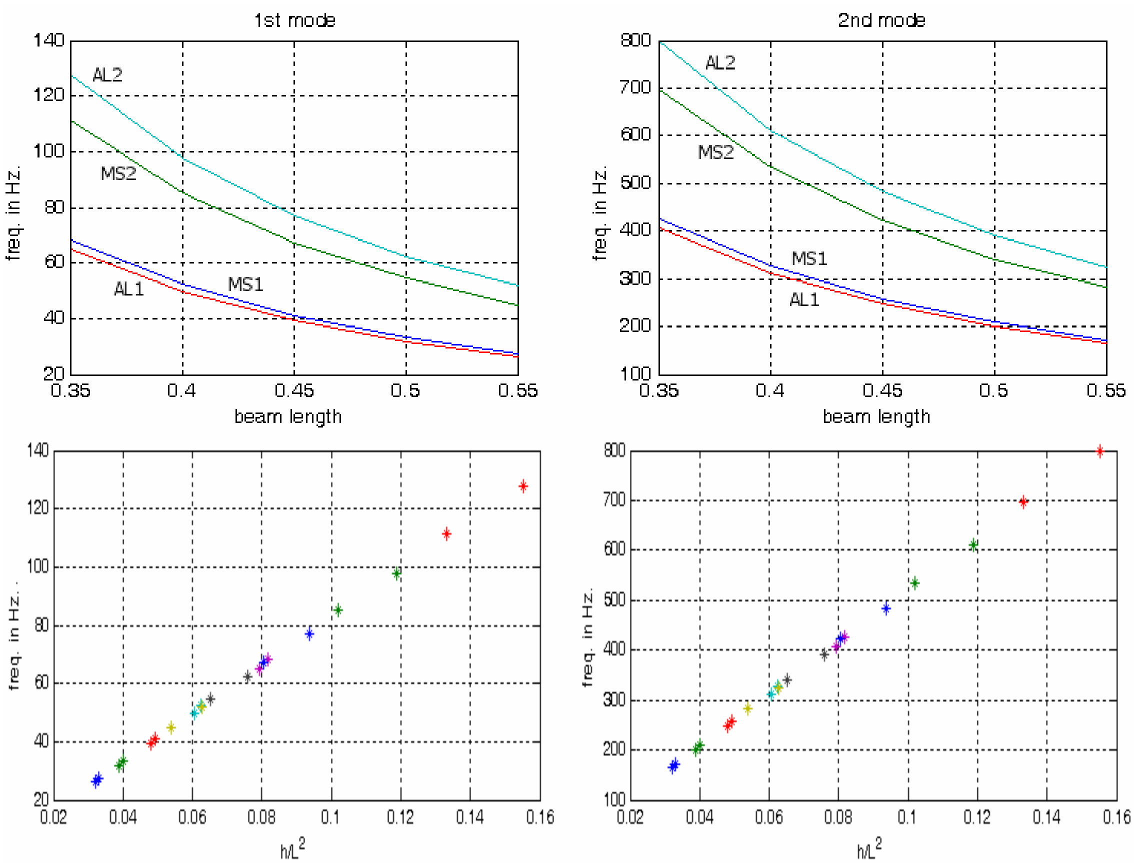

Figure 6: experimental results

Determination of ΔL requires slope of the beam deflection curve at free end and it is obtained from the slope of the current deflection profile. The process is continued incrementally till the value of normalized load parameter (λ) becomes 10. The tolerance value of the error limit for the numerical iteration scheme is chosen as 0.50% and the relaxation parameter is taken as 0.50.

In the present study vibration analysis in loaded condition is carried out in an indirect way. The basic assumption of the present method is that the geometric nonlinearity is accounted for by updation of the beam geometry. The dynamic system governing equation, Eq. (10), thus requires determination of stiffness matrix at each load step and finally the required result is obtained. The present method is validated by comparing the results with Wang [10, 11] for a static analysis of large deflection of a uniform beam and is shown in Figure 4. For validation study values of material properties assumed are: E = 210 GPa, ρ = 7800 kg/m^3 and υ = 0.30. The number of functions for each of the beam deflection and rotation displacements is taken as 11.

3 RESULTS AND DISCUSSION

The present analysis is undertaken to analyze the experimental results of uniform section beams, which are carried out for different beam geometries. The results are presented as non-dimensional frequency response curves with different system parameter values. The details of the specimens used for experiments are shown in Table 2 and a schematic diagram of experimental set up is shown in Figure 5.

Specimen Number	Width (b) mm	Thickness (h) mm	Specimen Id.	Material
1	40	10	AL1	Aluminium
2	32.4	16.3	AL2	Aluminium
3	51	9.62	MS1	Mild Steel
4	32	9.7	MS2	Mild Steel

Table 2: Details of specimens

Each of the specimens are tested for five different lengths, 350 mm to 550 mm, at an interval of 50 mm and corresponding vibration frequencies are noted for the first two modes of vibration. The experimental results are shown graphically in Figure 6, which is almost self explanatory. The two figures in the top row present results for individual specimens whereas, in the bottom row, first and second mode vibration frequencies are plotted for 20 different h/L^2 values. The comparison of experimental results with the theoretical ones is not presented in this paper. However, it is observed that theoretical vibration frequency values are marginally higher than the experimental ones which may be attributed to system damping and limitation in clamping arrangement in the experimental set-up.

4 CONCLUSIONS

- The present study undertakes a thick non-uniform cantilever beam with geometric nonlinearity and reports the dynamic beam characteristics.
- The problem is formulated by employing an energy based variational method and solution is sought by applying an improvised updated Lagrangian technique.
- Theoretical vibration frequency values are marginally higher than the experimental ones.

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