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ANALYSIS OF TIME-PERIODIC SYSTEMS, WHEN CORRESPONDING EQUATIONS DO NOT CONTAIN A SMALL PARAMETER EXPLICITLY

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Abstract. *There are many approaches to study equations with time-periodic coefficients, in which a small parameter may be assigned, in particular the classical asymptotic methods. The present paper is concerned with the analysis of the applicability of these approaches for solving nonlinear equations, which don't contain a small parameter explicitly. A new modification of the method of direct separation of motions (MDSM) [1,2], which may be employed to study such equations, is proposed. As an example a classical problem about the stability of a pendulum with vibrating suspension axis is considered in an unconventional case, when the frequency of external loading and the natural frequency of the pendulum in the absence of this loading are of the same order. As the result it is shown that in the considered range of parameters not only the effective "stiffness" of the system changes due to the external loading, but also its effective "mass". It is noted that application of the classical asymptotic methods in this case leads to erroneous results.*

A correlation between the proposed modification of the MDSM and Ritz's method of harmonic balance, Van der Pol's method of slowly varying amplitudes, the classical asymptotic methods and other approaches is discussed.

1 INTRODUCTION

Widely used approaches for the analysis of equations, in which a small parameter may be assigned, are the classical asymptotic methods, in particular the method of averaging [3,4], the multiple scales method (MSM) [5] and other approaches [6,7]. Application of these methods for solving equations, which do not contain a small parameter explicitly, is rather cumbersome and may lead to erroneous results.

The present paper is concerned with the analysis of the applicability of the method of direct separation of motions (MDSM) [1,2] for solving such equations. A new modification of the method, which may be employed for their studying, is introduced. A classical problem about the stability of a pendulum with vibrating suspension axis is considered as an illustrative example. The unconventional case, when the frequency of external loading and the natural frequency of the pendulum in the absence of this loading are of the same order, is studied. Vibration intensity is assumed to be relatively low. Pendulum's motion near its upper position is studied. In the considered case the corresponding linearized equation, i.e. the Mathieu equation, doesn't contain a small parameter explicitly. The aim is to obtain solutions of this equation in the stability domain. It may be noted that previously such solutions were not determined analytically (see, e.g. [8-12]).

It is shown that application of the classical asymptotic methods, particularly of the MSM, for solving the considered equation in the studied range of parameters leads to erroneous results. Thereby, the applicability range of the MDSM turns out to be broader than the one of these methods. A correlation between the MDSM and Ritz's method of harmonic balance [13,14], Van der Pol's method of slowly varying amplitudes [14], the classical asymptotic methods, and other approaches [15] is revealed. The present study may be considered as a development and continuation of the ideas proposed in the monograph [16] for the analysis of equations which don't contain a "natural" small parameter: An assumption regarding the type of the solution sought is employed in order to introduce a small parameter.

2 THE MODIFIED MDSM

Consider a classical problem about the stability of a pendulum with vibrating suspension axis in the simplest formulation, i.e. in the case of small deviations from the upper position. In this case motion of the pendulum is described by the following equation:

$$I\ddot{\varphi} - ml(g + G\Omega^2 \cos\Omega t)\varphi = 0 \tag{1}$$

Here φ is small angle of pendulum deviation from the upper position, I, m and l are the moment of inertia, the mass and the distance from the pendulum center of gravity to the axis of suspension, G and Ω are the amplitude and the frequency of vertical oscillations of the suspension axis, g is the acceleration of gravity, dot designates the time derivative.

Introducing two dimensionless parameters $\delta = \dfrac{mlg}{I\Omega^2}$, $\chi = \dfrac{G\Omega^2}{g}$ and dimensionless time $t_0 = \Omega t$, rewrite equation (1) in the form:

$$\frac{d^2\varphi}{dt_0^2} - \delta(1 + \chi\cos t_0)\varphi = 0 \tag{2}$$

The case $O(\delta) = O(\chi) = 1$ is considered: the frequency of external loading and the natural frequency of the pendulum in the absence of this loading are of the same order; the amplitude of acceleration of vertical oscillations of pendulum's suspension axis is of the same order as g. In this case equation (2) doesn't contain a small parameter explicitly.

For studying equation (2) the vibrational mechanics approach [1,2] is employed. The MDSM is applied in the form, which differs from the conventional one [1,2]; the solutions are sought in the form:

$$\varphi = \alpha(t_1) + \psi(t_1, t_0) \tag{3}$$

where $t_1 = \varepsilon t_0$, $\varepsilon << 1$ is small parameter, α is "slow", and ψ is "fast", 2π - periodic in dimensionless time t_0 variable, with average zero:

$$\langle \psi(t_1, t_0) \rangle = 0$$

Here $\langle ... \rangle$ designates averaging in the period 2π on time t_0, i.e. for function $h(t_1, t_0)$ we have $\langle h(t_1, t_0) \rangle = \frac{1}{2\pi}\int_0^{2\pi} h(t_1, t_0) dt_0$.

As is seen, a hypothesis regarding the type of the solution sought is used in order to introduce a small parameter ε. In fact, searching solution of the initial equation in the form (3) we assume that system in the considered range of parameters performs oscillations with slowly varying characteristics. If this hypothesis is not correct, then the trivial solution or solution which doesn't meet the sense of the problem will be obtained. Otherwise the parameters of oscillations of the considered type will be determined.

While solving equation (2) consider variables t_1 and t_0 as independent, so that $\frac{d^2}{dt_0^2} = \frac{\partial^2}{\partial t_0^2} + 2\varepsilon \frac{\partial^2}{\partial t_1 \partial t_0} + \varepsilon^2 \frac{\partial^2}{\partial t_1^2}$. By averaging equation (2) on time t_0 we obtain the following equation of pendulum's "slow" motion

$$\varepsilon^2 \frac{d^2\alpha}{dt_1^2} - \delta(\alpha + \chi\langle \psi \cos t_0 \rangle) = 0, \tag{4}$$

Equation of pendulum's "fast" motions may be obtained by subtracting equation (4) from equation (2)

$$\frac{\partial^2\psi}{\partial t_0^2} + 2\varepsilon \frac{\partial^2\psi}{\partial t_1 \partial t_0} + \varepsilon^2 \frac{\partial^2\psi}{\partial t_1^2} - \delta\psi = \delta\chi\left((\alpha + \psi)\cos t_0 - \langle \psi \cos t_0 \rangle\right) \tag{5}$$

2.1 Solution of the fast motions equation

In conventional cases of the MDSM application [1] corresponding equation of fast motions is solved only approximately, because the equation of slow motion is the one of the primary interest. In particular, while solving fast motions equation all involved slow variables are considered as constants ("frozen"), and terms $2\varepsilon \frac{\partial^2\psi}{\partial t_1 \partial t_0}$ and $\varepsilon^2 \frac{\partial^2\psi}{\partial t_1^2}$ are neglected, because they are small in comparison with $\frac{\partial^2\psi}{\partial t_0^2}$. In the present case this simplification has to be abandoned. Indeed, terms of order of ε^2 are retained in equation (4) of pendulum's slow motion, and $\delta \sim 1$, $\chi \sim 1$, so solution of fast motions equation should be found with the accuracy of order of ε^2. Thus terms of this order, particularly $2\varepsilon \frac{\partial^2 \psi}{\partial t_1 \partial t_0}$ and $\varepsilon^2 \frac{\partial^2 \psi}{\partial t_1^2}$, should be retained in equation (5).

Taking into account that $\psi(t_1, t_0)$ is a time t_0 periodic function, the solution of fast motions equation (5) is sought in the form of series

$$\psi = B_{11}(t_1)\cos t_0 + B_{12}(t_1)\sin t_0 + B_{21}(t_1)\cos 2t_0 + B_{22}(t_1)\sin 2t_0 + ... \tag{6}$$

Amplitudes $B_{11}(t_1)$, $B_{12}(t_1)$,… are determined by multiplying equation (5) sequentially by $\cos t_0, \sin t_0$,… and averaging on time t_0. Consequently, the following system of equations is derived

$$\begin{aligned}
&-B_{11} + 2\varepsilon \frac{dB_{12}}{dt_1} + \varepsilon^2 \frac{d^2 B_{11}}{dt_1^2} - \delta B_{11} = \delta\chi\left(\alpha + \frac{B_{21}}{2}\right)\\
&-B_{12} - 2\varepsilon \frac{dB_{11}}{dt_1} + \varepsilon^2 \frac{d^2 B_{12}}{dt_1^2} - \delta B_{12} = \delta\chi \frac{B_{22}}{2}\\
&...\\
&-n^2 B_{n1} + 2n\varepsilon \frac{dB_{n2}}{dt_1} + \varepsilon^2 \frac{d^2 B_{n1}}{dt_1^2} - \delta B_{n1} = \delta\chi \frac{B_{n-1,1}}{2}\\
&-n^2 B_{n2} - 2n\varepsilon \frac{dB_{n1}}{dt_1} + \varepsilon^2 \frac{d^2 B_{n2}}{dt_1^2} - \delta B_{n2} = \delta\chi \frac{B_{n-1,2}}{2}
\end{aligned} \tag{7}$$

Amplitudes B_{11}, B_{12}, B_{21}, B_{22},… are “slowly varying” functions. The following relation should hold true for them

$$\left| \frac{dB_{nm}}{dt_0} \frac{1}{B_{nm0}} \right| << 1 \tag{8}$$

here B_{nm0} is characteristic amplitude of function B_{nm}. In essence this relation reflects the requirement for amplitude B_{nm} to vary slowly in comparison with the fast variable ψ. The fulfillment of this requirement is a necessary condition of the MDSM applicability.

The general solution of the homogeneous system of equations which corresponds to (7) has the form:

$$\begin{aligned}
&B_{11} = C_{11}\exp(\lambda_1(\delta,\chi)t_0) + C_{12}\exp(\lambda_2(\delta,\chi)t_0) + C_{13}\exp(\lambda_3(\delta,\chi)t_0) + C_{14}\exp(\lambda_4(\delta,\chi)t_0) + ...\\
&B_{12} = C_{21}\exp(\lambda_1(\delta,\chi)t_0) + C_{22}\exp(\lambda_2(\delta,\chi)t_0) + ..., B_{21} = ..., B_{22} = ...
\end{aligned} \tag{9}$$

Here C_{nm} are arbitrary constants, $\lambda_1(\delta,\chi)$, $\lambda_2(\delta,\chi)$, $\lambda_3(\delta,\chi)$, $\lambda_4(\delta,\chi)$,… are the corresponding eigenvalues. To determine a particular solution of the initial system (7) the classical procedure of expansion in the small parameter ε may be employed. As the result we obtain:

$$B_{11}(t_1) = -F(\delta,\chi)\alpha(t_1) + \varepsilon^2 F_2(\delta,\chi)\frac{d^2\alpha}{dt_1^2} + O(\varepsilon^3),\ B_{12}(t_1) = \varepsilon F_1(\delta,\chi)\frac{d\alpha}{dt_1} + O(\varepsilon^3), \text{ etc.} \tag{10}$$

Here $F(\delta,\chi)$, $F_1(\delta,\chi)$, $F_2(\delta,\chi)$ are functions of parameters δ and χ which depend on the number of retained harmonics in series (6). Thus the general solution of equations (7) is obtained in the form of sum of (9) and (10).

However, modules of eigenvalues $\lambda_1(\delta,\chi)$, $\lambda_2(\delta,\chi)$, $\lambda_3(\delta,\chi)$, $\lambda_4(\delta,\chi)$,… exceed unity for all positive values of parameters δ and χ. Therefore, constants C_{nm} should be equal to zero to satisfy condition (8) which reflects the requirement for amplitudes B_{11}, B_{12}, B_{21}, B_{22},… to be slowly varying functions. Thus, the sought solution of the considered system of equations (7) takes the form (10).

Certain terms in the solution of the fast motions equation were discarded due to previous assumptions regarding its character. Other approaches, particularly the asymptotic methods [3-7], e.g. the MSM, imply similar operations. In many cases of the MSM application, in particular, all non-periodic terms in the solution are discarded due to previously made assumptions. So the MDSM, the MSM, and other similar approaches may be called “non-rigorous”: additional a posteriori verification of the results obtained by their means is required. For the case under study such verification is conducted in Section 4.

2.2 Equation of pendulum’s slow motion

So we have determined expressions for amplitudes $B_{11}(t_1)$, $B_{12}(t_1)$,…, which are correct with the accuracy of order of ε^2. Employing solution of fast motions equation (10) and equality $t_1 = \varepsilon t_0$, the equation of slow motion is composed in the form

$$(1-\delta\frac{\chi}{2}F_2(\delta,\chi))\frac{d^2\alpha}{dt_0^2}-\delta(1-\frac{\chi}{2}F(\delta,\chi))\alpha=0 \qquad (11)$$

As it was noted above, functions $F(\delta,\chi)$, $F_2(\delta,\chi)$ depend on the number of retained harmonics in series (6). While solving equation of fast motions terms of order of ε^2 were taken into account, so number n of the harmonic, which may be discarded, is determined by the relation

$$\frac{1}{4}\frac{\delta\chi}{\delta+n^2}\frac{\delta\chi}{\delta+(n-1)^2}\sim\varepsilon^3 \qquad (12)$$

From equation (11) it follows that not only the effective stiffness of the system changes due to the external loading, but also its effective mass. This fact is especially remarkable. As is shown in the classical papers (see, e.g. [17,18]), in the studied system only the effective stiffness changes under the action of vibration. The distinction is due to consideration of different ranges of the parameters. In [17,18] the case $\delta\sim\varepsilon^2<<1$ is studied, when the frequency of external loading is much higher than the natural frequency of the pendulum in the absence of this loading, whereas in the present paper we consider the case $\delta\sim 1$.

The effective natural frequency of the pendulum becomes equal to

$$\lambda=\sqrt{\frac{mlg}{I}\frac{\frac{G\Omega^2}{2g}F\left(\frac{mlg}{I\Omega^2},\frac{G\Omega^2}{g}\right)-1}{1-\frac{mlG}{2I}F_2\left(\frac{mlg}{I\Omega^2},\frac{G\Omega^2}{g}\right)}} \qquad (13)$$

From obtained equation (11) it follows that the upper position of the pendulum becomes stable due to action of vibration if the following condition holds true:

$$\frac{\chi F(\delta,\chi)/2-1}{1-\delta\chi F_2(\delta,\chi)/2}>0 \tag{14}$$

We note that this condition is in good agreement with the classical Ince-Strutt diagram [8-10].

3 ON THE VALIDITY OF THE RESULTS OBTAINED BY THE PROPOSED MODIFICATION OF THE MDSM

In order to substantiate the proposed modification of the MDSM a correlation between this method and other approaches should be revealed. Consider the classical method of harmonic balance in Ritz interpretation [13,14]. Application of this method also implies a hypothesis regarding the type of the solution sought. It is assumed that this solution is a time periodic function, i.e. that system in the considered range of parameters performs stationary oscillations. If this hypothesis is not correct, then the trivial solution or solution which doesn't meet the sense of the problem is obtained. Otherwise the parameters of oscillations of the considered type are determined. It should be noted that Ritz's method was validated in many classical papers (see, e.g. [13,14]).

So, the main difference between the MDSM and the classical method of harmonic balance lies in the hypothesis regarding the type of the solution sought. The MDSM may be interpreted as a method, by means of which not only stationary oscillations, but also oscillations with slowly varying characteristics may be determined. It should be noted that in this sense the MDSM is close to the classical method of slowly varying amplitudes [14], which was proposed by Van der Pol for solving the equation named after him. However, Van der Pol's method implies only the first harmonic in the solution to be taken into account. Another approach, correlation with which should be mentioned, is the projection method [15]. This approach also implies the representation of the solution as a series of orthogonal functions with coefficients which vary slowly in comparison with these functions.

In accordance with [1,2] the MDSM requires an additional a posteriori analysis of the obtained results. It should be assayed whether the characteristics of the defined oscillations vary indeed slowly in comparison with these oscillations, or not. For the considered problem taking into account that all amplitudes $B_{11}(t_1)$, $B_{12}(t_1)$,… depend on variable $\alpha(t_1)$ this verification may be reduced to the assessment of the fulfillment of the following relation

$$\frac{d\alpha}{dt_0}\frac{1}{\alpha_A}<<\frac{1}{\psi_A}\frac{d\psi}{dt_0} \tag{15}$$

where α_A and ψ_A are characteristic amplitudes of variables $\alpha(t_1)$ and $\psi(t_1,t_0)$. Taking into account equation (11) relation (15) may be rewritten as

$$\sqrt{\left|\delta\frac{\frac{\chi}{2}F(\delta,\chi)-1}{1-\delta\frac{\chi}{2}F_2(\delta,\chi)}\right|}<<1 \tag{16}$$

Thereby, the obtained equation of pendulum's slow motion (11) and the corresponding expressions for amplitudes $B_{11}(t_1)$, $B_{12}(t_1)$,… are correct if condition (16) holds true. Taking into account equality (13) this condition may be written in the traditional form $\lambda<<\Omega$ [1,2].

It should be noted that application of the classical asymptotic methods, e.g. the MSM, also implies a posteriori analysis of the results obtained (see, e.g. [5,9]). It is required, in particular to assess a time interval, in which the defined solution is correct.

Taking into account that small parameter ε is present in decomposition (3), a correlation between the proposed modification of the MDSM and the classical asymptotic methods may be revealed. As in the case of these methods, application of the modified MDSM requires an accurate account of orders of the parameters in the considered equation. This account is necessary to identify terms, which may be neglected while solving corresponding equations of fast and slow motions. In the considered case, when $\delta \sim 1$ and $\chi \sim 1$, solution of fast motions equation (5) was found with the accuracy of order of ε^2 to compose correct equation of slow motion (4). It should be noted that in the classical case $\delta \sim \varepsilon^2$ and $\chi \sim \frac{1}{\varepsilon}$ [17,18] this solution may be determined with the accuracy of order of ε^0.

At the same time employment of the MDSM doesn't require the presence of a small parameter in the initial equation. This distinction broadens significantly its applicability range in comparison with the classical asymptotic methods. In particular, in the considered case it is impossible to assign a small parameter in equation (2), since $O(\delta) = O(\chi) = 1$ and $\delta \not\approx 1$. So, application of the MSM and other classical asymptotic methods in this case leads to erroneous results.

4 COMPARISON WITH THE RESULTS OF NUMERICAL EXPERIMENTS

In accordance with the results obtained in Section 2 in the considered range of parameters not only the effective stiffness of the system changes due to the external loading, but also its effective mass. A series of numerical experiments was conducted to verify this conclusion. Initial equation (2) was integrated directly by means of the Wolfram Mathematica 7; corresponding results were compared with the derived analytical solution.

Consider the case $\delta = 0.4$ as an illustrative example. For such δ and $\chi \sim 1$ condition (12) fulfils for $n = 4$, so only three harmonics may be taken into account in solution (6) of pendulum's fast motions equation. Condition (14) of pendulum's upper position stabilization in this case reduces to $\chi > 2.587$. The dependence of pendulum's deflection on time t_0 at $\delta = 0.4$, $\chi = 2.59$ is shown in Figure 1 (a) for initial conditions $\varphi(0) = 0.01$, $\dot{\varphi}(0) = 0$. Solid line is the numerical solution of the initial equation (2), dotted line is the solution $\alpha(t_0)$ of the obtained equation of pendulum's slow motion (11), and dashed line is the analytical solution, i.e. the sum of $\alpha(t_0)$ and the solution of pendulum's fast motions equation.

As another illustrative example consider the case $\delta = 1.4$. For such δ condition (12) fulfils only for $n = 6$, so five harmonics were taken into account while solving equation (5) of pendulum's fast motions. Condition (14) of pendulum's upper position stabilization in this case reduces to $\chi > 1.73944$. The dependence of pendulum's deviation angle on time t_0 at $\delta = 1.4$, $\chi = 1.7395$ is shown in Figure 1 (b) for initial conditions $\varphi(0) = 0.01$, $\dot{\varphi}(0) = 0$.

As is seen from Figure 1, obtained analytical solution is in good agreement with the results of numerical experiments. In particular, the conclusion that not only the effective stiffness of the system changes due to the external loading, but also its effective mass is confirmed.

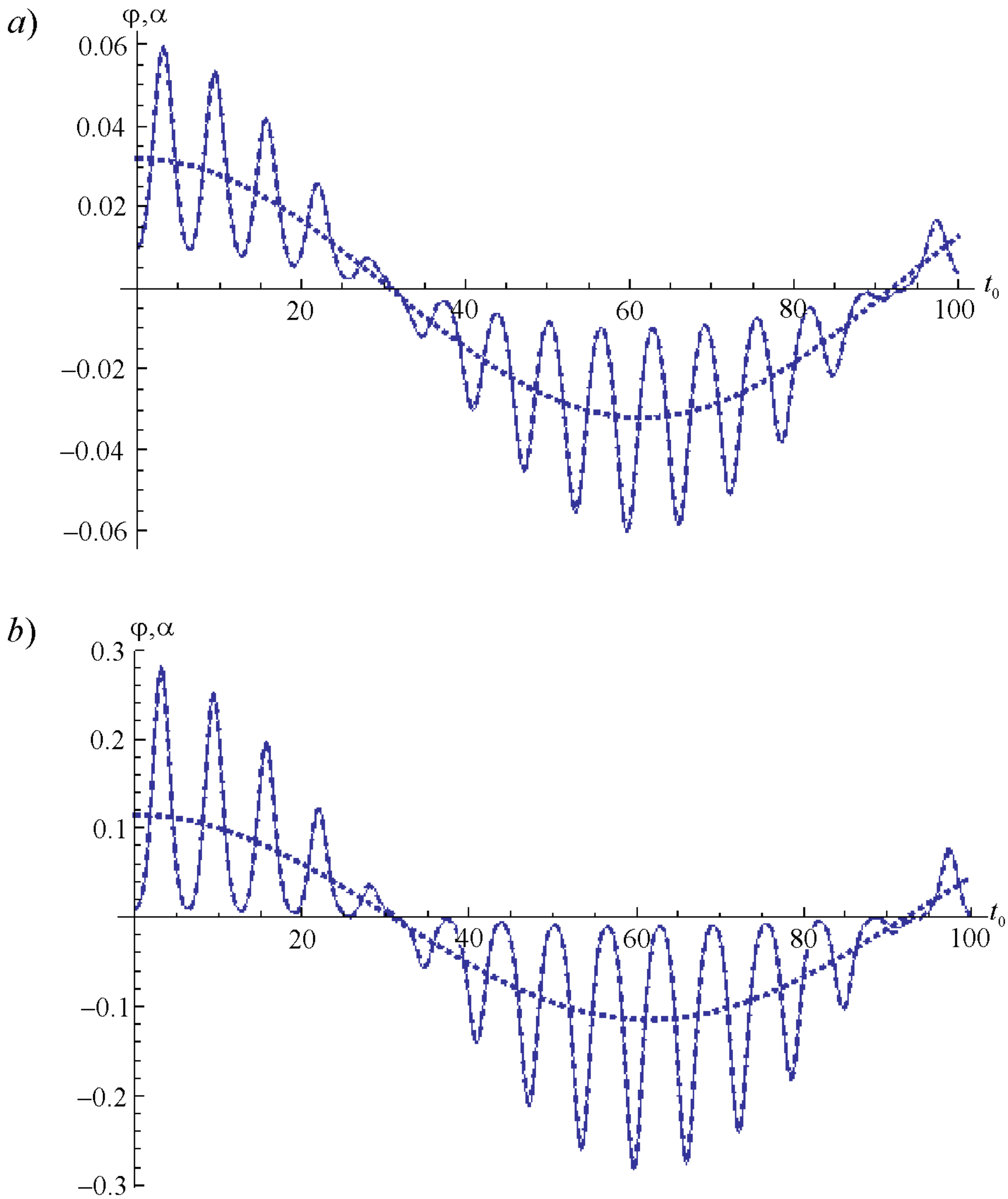

Figure 1: The dependence of pendulum's deflection φ on time t_0 at a) $\delta = 0.4$, $\chi = 2.59$, b) $\delta = 1.4$, $\chi = 1.7395$ and initial conditions $\varphi(0) = 0.01$, $\dot{\varphi}(0) = 0$. Solid line is the numerical solution of the initial equation (2), dotted line is the solution of the obtained equation of pendulum's slow motion (11), and dashed line is the analytical solution of the initial equation (2).

5 CONCLUSIONS

A modification of the MDSM applicable for solving equations which don't contain a small parameter explicitly is proposed in the paper. As an illustrative example a classical problem about the stability of a pendulum with vibrating suspension axis is considered at unconventional values of parameters. Case, when frequency of external loading and the natural frequency of the pendulum in the absence of this loading are of the same order, is studied. Vibration intensity is assumed to be relatively low. As the result, it is revealed that in the considered range of parameters not only the effective stiffness of the system changes due to the external loading, but also its effective mass. This fact is especially remarkable, since in the classical case, when the frequency of external loading is much higher than the natural frequency of the pendulum, only the effective stiffness of the system changes.

The validity of the results obtained by the proposed modification of the MDSM is confirmed. A correlation between the MDSM and Ritz's method of harmonic balance, Van der Pol's method of slowly varying amplitudes, the classical asymptotic methods and other approaches is revealed. It is shown that application of the asymptotic methods for solving the considered equation in the studied range of parameters leads to erroneous results. So, the applicability range of the MDSM turns out to be broader than the one of these methods.

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