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FREE VIBRATION OF A CRACKED TIMOSHENKO BEAM USING THE DYNAMIC STIFFNESS METHOD

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Abstract. *The dynamic stiffness theory of a cracked Timoshenko beam is developed to investigate its free vibration characteristics. The cracked beam is idealised by two intact beam elements connected by a cracked element of infinitesimally small length. The intact beam elements are modelled using the Timoshenko beam theory which customarily accounts for the effects of shear deformation and rotary inertia. For the cracked element, first the flexibility matrix and then the subsequent stiffness matrix are established by using fracture mechanics. The dynamic stiffness matrix of the whole cracked beam is assembled by connecting the dynamic stiffness matrices of the two intact elements at two ends with the cracked element in the middle. The formulation leads to a non-linear eigenvalue problem and the Wittrick-Williams algorithm is applied to compute the natural frequencies and mode shapes. Numerical results are examined and illustrated for a cracked beam with cantilever boundary conditions. The effects of crack location and crack depth on the natural frequencies and mode shapes are highlighted. Some results are compared with those in the published literature to confirm the validity and accuracy of the proposed method. The theory developed can be extended to frame works containing single or multi crack elements.*

1 INTRODUCTION

The investigation of the free vibration analysis of cracked beams appears to have been predominantly based on the finite element method, see for example, [1-2]. This is not unusual because the finite element method is unquestionably a universal tool in structural analysis. However, the dynamic stiffness method (DSM) for free vibration of a cracked Bernoulli-Euler beam has recently been developed [3] as a powerful, but highly viable alternative. Following this recent work, development of the DSM to solve the free vibration problem of a cracked Timoshenko beam is timely and pertinent and indeed, a significant step forward, particularly when the slenderness ratio of the beam is small. It is well recognised that the DSM [4] gives exact results and as a consequence, it has much better model accuracy than the finite element and other approximate methods. For instance, a single structural element can be used in the DSM to compute any number of its natural frequencies of a structure without any loss of accuracy. The DSM can also be applied to complex structures and yet computational accuracy will not be unduly compromised. Clearly this uncompromisingly high accuracy is impossible in the finite element and other approximate methods. The main reason for this striking difference is that the shape function used in the DSM is exact unlike the finite element and other approximate methods in which it is generally assumed as a polynomial or interpolation function.

The purpose of this paper is to develop the dynamic stiffness matrix of a cracked Timoshenko beam to investigate its free vibration characteristics. The following steps are involved to achieve this objective. The first step is to develop the governing differential equations of motion for the two intact uniform beams which are modelled using the Timoshenko beam theory and thus accounting for the effects of shear deformation and rotary inertia. The governing differential equations of motion and natural boundary conditions are obtained by applying Hamilton's principle. In the second step the equations are solved for axial and flexural displacements as well as for bending rotation when the oscillatory motion is harmonic. The expressions for shear force and bending moment obtained from the natural boundary conditions as a result of the Hamiltonian formulation are utilised in the next step. The procedure for generating the governing differential equations of motion and natural boundary conditions of the beam benefitted from the application of symbolic computation [5]. The dynamic stiffness matrix of the intact beam is derived by relating the amplitudes of loads to those of the responses at the two ends of the beam. This is followed by the step to derive the compliance properties of the cracked element using facture mechanics theory. The final step is to assemble the overall dynamic stiffness matrix by connecting the two intact beam elements with the cracked element. Once the dynamic stiffness matrix of the cracked beam is derived, the Wittrick-Williams algorithm [6] is applied as a solution technique to compute its natural frequencies and mode shapes. The results are computed for various boundary conditions, crack length and crack location. For illustrative purpose, the results for the demanding case of cantilever boundary conditions are provided in the paper. Finally, the principal findings are summarised.

2 THEORY

Figure 1 shows the coordinate system (XOY) of a rectangular beam of width B, thickness h, and length L. The cracked beam system is modelled by two intact beam elements (I and II) of lengths L_1 and L_2 respectively, connected by a cracked element (III) having crack depth a at a distance L_1 from the origin. The beam is allowed to deflect in the XY plane undergoing axial displacement (u), bending displacement (w), and bending rotation (θ). The axial, bending and shear rigidities of the beam are EA, EI, and kAG respectively. The mass per unit length of the beam is m, the density of material is ρ, the cross-section area is A, the second moment of area is I and the Young's modulus is E.

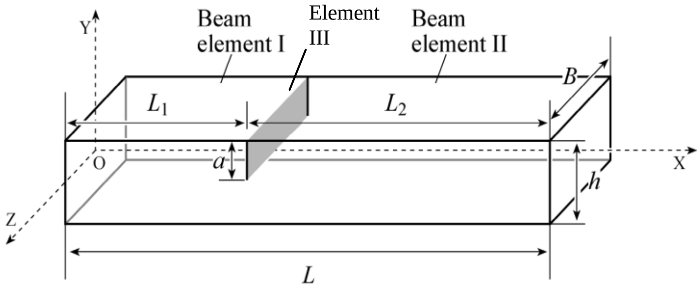

Figure 1: Coordinate system and notation for a cracked Timoshenko beam

2.1 Intact Timoshenko Beam

Timoshenko's beam theory is used to investigate the two intact beam elements (I and II). Note that the Timoshenko theory does not affect the axial vibration. The kinetic (T) and potential (V) energies of the beam can be obtained routinely [7] as follows:

$$T = \frac{1}{2}\int_0^L \left\{\rho A\dot{u}^2 + \rho A\dot{w}^2 + \rho I\dot{\theta}^2\right\}dx\,, \quad V = \frac{1}{2}\int_0^L \left\{EAu'^2 + EI\theta'^2 + kGA(w'-\theta)^2\right\}dx \tag{1}$$

where a prime and an over dot denote differentiation with respect to space and time respectively.

Applying Hamilton's principle, the following governing differential equations of motions in free vibration are obtained:

$$EAu'' - \rho A\ddot{u} = 0\,, \quad kAG(w'' - \theta') - \rho A\ddot{w} = 0\,, \quad EI\theta'' - \rho I\ddot{\theta} + kAG(w' - \theta) = 0 \tag{2}$$

The expressions for axial force (P), shear force (S) and bending moment (M) are obtained through Hamiltonian formulation [5] as follows:

$$P(x) = -EAu'\,, \quad S(x) = -kAG(w' - \theta), \quad M(x) = -EI\theta' \tag{3}$$

Introducing the non-dimensional length,

$$\xi = \frac{x}{L} \tag{4}$$

and assuming harmonic oscillation so that,

$$u(x,t) = U(x)e^{i\omega t}\,, \quad w(x,t) = W(x)e^{i\omega t}\,, \quad \theta(x,t) = \Theta(x)e^{i\omega t} \tag{5}$$

where ω is the angular frequency, $U(x)$, $W(x)$ and $\Theta(x)$ are amplitudes of u, w and θ in free vibration.

Eqs. (2) with the help of Eqs. (4) and (5) are written as ordinary differential equations to give

$$(D^2 + \mu^2)U(\xi) = 0\,, \quad (D^2 + bs)W(\xi) - LD\Theta(\xi) = 0\,, \quad \frac{1}{L}DW(\xi) + [sD^2 + (brs - 1)]\Theta(\xi) = 0 \tag{6}$$

where

$$\mu^2 = \frac{\rho A}{EA} L^2 \omega^2, \quad b = \frac{\rho A}{EI} L^4 \omega^2, \quad s = \frac{EI}{kAGL^2}, \; r = \frac{I}{AL^2} \;, \quad D = \frac{d}{d\xi} \tag{7}$$

Noting that the axial motion is uncoupled, the last two Eqs. (6) in $W(\xi)$ and $\Theta(\xi)$ can be combined to give

$$(D^4 + 2a_1 D^2 + a_2)H = 0 \tag{8}$$

where

$$2a_1 = b(r+s), \quad a_2 = b(brs-1) \tag{9}$$

Now the flexural or bending motion is dealt with first and the relatively simple case of axial motion is dealt with later, see Eq. (16) below. The solution of Eq. (8) can be obtained in terms of two sets of constants A_j and B_j (j = 3 to 6) as:

$$W(\xi) = A_3 \cosh \alpha\xi + A_4 \sinh \alpha\xi + A_5 \cos \beta\xi + A_6 \sin \beta\xi$$

$$\Theta(\xi) = B_3 \cosh \alpha\xi + B_4 \sinh \alpha\xi + B_5 \cos \beta\xi + B_6 \sin \beta\xi \tag{10}$$

where α and β are the roots of the auxiliary equation of Eq. (8) as

$$\alpha^2 = -a_1 + \sqrt{a_1^2 - a_2}\,, \quad \beta^2 = a_1 + \sqrt{a_1^2 - a_2} \tag{11}$$

Two sets of constants A_j and B_j can be related to each other to give:

$$B_3 = (c_1 / L)A_4, \;\; B_4 = (c_1 / L)A_3, \;\; B_5 = (c_2 / L)A_6, \;\; B_4 = -(c_2 / L)A_5 \tag{12}$$

where

$$c_1 = \frac{\alpha^2 + bs}{\alpha}, \quad c_2 = \frac{-\beta^2 + bs}{\beta} \tag{13}$$

The shear force and bending moment are expressed in the non-dimensional form as

$$S(\xi) = -kAG\left[\frac{1}{L}W' - \Theta\right], \qquad M(\xi) = -\frac{EI}{L}\theta' \tag{14}$$

The rotation, shear force and bending moment are rewritten in terms of the constants A_j as

$$\Theta(\xi) = \frac{1}{L}\{c_1 A_3 \sinh\alpha\xi + c_1 A_4 \cosh\alpha\xi + c_2 A_5 \sin\beta\xi - c_2 A_6 \cos\beta\xi\}$$

$$S(\xi) = \frac{EIb}{L^3}\{\frac{1}{\alpha} A_3 \sinh \alpha\xi + \frac{1}{\alpha} A_4 \cosh \alpha\xi + \frac{1}{\beta} A_5 \sin \beta\xi - \frac{1}{\beta} A_6 \cos \beta\xi\}$$

$$M(\xi) = -\frac{EI}{L^2}\{c_1\alpha A_3 \cosh \alpha\xi + c_1\alpha A_4 \sinh \alpha\xi + c_2\beta A_5 \cos \beta\xi + c_2\beta A_6 \sin \beta\xi\} \tag{15}$$

The governing differential equation of motion of the beam in axial vibration is the first of Eqs. (6), i.e.

$$(D^2 + \mu^2)U(\xi) = 0 \tag{16}$$

For harmonic oscillation, the solution of Eq. (16) is

$$U(\xi) = A_1 \cos\mu\xi + A_2 \sin\mu\xi \tag{17}$$

where A_1 and A_2 are two constants.

The axial force $P(\xi)$ with the help of Eq. (3) is expressed in the non-dimensional form as

$$P(\xi) = -\frac{EA}{L}\frac{dU}{d\xi} = \frac{EA}{L}\mu(A_1 \sin\mu\xi - A_2 \cos\mu\xi) \tag{18}$$

The sign convention for axial force, shear force and bending moment when applying the boundary conditions is shown in Figure 2. The boundary conditions for displacements and forces shown in Figure 3 are given as:

$$\text{At } \xi = 0 \ (x = 0):\ U = U_1,\ W = W_1,\ \Theta = \Theta_1,\ P = -P_1,\ S = S_1,\ M = M_1$$

$$\text{At } \xi = 1 \ (x = L):\ U = U_2,\ W = W_2,\ \Theta = \Theta_2,\ P = P_2,\ S = -S_2,\ M = -M_2 \tag{19}$$

The dynamic stiffness matrix of the beam relating the amplitudes of forces $\mathbf{F}$ to those of the displacements $\boldsymbol{\delta}$ is

$$\mathbf{F} = \mathbf{K}\,\boldsymbol{\delta} \tag{20}$$

where

$$\mathbf{F} = \begin{bmatrix} P_1 & S_1 & M_1 & P_2 & S_2 & M_2 \end{bmatrix}^{\mathrm{T}},\ \boldsymbol{\delta} = \begin{bmatrix} U_1 & W_1 & \Theta_1 & U_2 & W_2 & \Theta_2 \end{bmatrix}^{\mathrm{T}} \tag{21}$$

$\mathbf{K}$ is the 6×6 frequency dependent dynamic stiffness matrix as

$$\mathbf{K} = \begin{bmatrix} k_{11} & 0 & 0 & k_{14} & 0 & 0 \\ 0 & k_{22} & k_{23} & 0 & k_{25} & k_{26} \\ 0 & k_{23} & k_{33} & 0 & -k_{26} & k_{36} \\ k_{14} & 0 & 0 & k_{11} & 0 & 0 \\ 0 & k_{25} & -k_{26} & 0 & k_{22} & -k_{23} \\ 0 & k_{26} & k_{36} & 0 & -k_{23} & k_{33} \end{bmatrix} \tag{22}$$

where

$$\Phi_1 = -c_1 c_2(\alpha^2 + \beta^2)(c_1 \sinh\alpha\cos\beta - c_2 \cosh\alpha\sin\beta),$$

$$\Phi_2 = c_1 c_2\left[(c_1\alpha + c_2\beta)(1 - \cosh\alpha\cos\beta) + (c_1\beta - c_2\alpha)\sinh\alpha\sin\beta\right],$$

$$\Phi_3 = c_1 c_2(\alpha^2 + \beta^2)(c_1 \sinh\alpha - c_2 \sin\beta),$$

$$\Phi_4 = -c_1 c_2(\alpha^2 + \beta^2)(\cosh\alpha - \cos\beta),$$

$$\Phi_5 = (\alpha^2 + \beta^2)(c_1 \cosh\alpha\sin\beta + c_2 \sinh\alpha\cos\beta),$$

$$\Phi_6 = -(\alpha^2 + \beta^2)(c_1 \sin\beta + c_2 \sinh\alpha),$$

$$\Delta = 2c_1 c_2(\cosh\alpha\cos\beta - 1) + (c_1^2 - c_2^2)\sinh\alpha\sin\beta,$$

$$k_{11} = (EA/L)\mu\cot\mu, \quad k_{14} = -(EA/L)\mu\,\mathrm{cosec}\,\mu, \quad k_{22} = (EI/L^3)(\Phi_1/\Delta),$$

$$k_{23} = (EI/L^2)(\Phi_2/\Delta), \quad k_{33} = (EI/L)(\Phi_5/\Delta), \quad k_{25} = (EI/L^3)(\Phi_3/\Delta),$$

$$k_{26} = (EI / L^2)(\Phi_4 / \Delta)\,, \quad k_{36} = (EI / L)(\Phi_6 / \Delta) \tag{23}$$

Eq. (22) can be rewritten in sub-matrices form for the beam elements I and II respectively as:

$$\mathbf{K}^{\mathrm{I}} = \begin{bmatrix} \mathbf{K}^{\mathrm{I}}_{12} & \mathbf{K}^{\mathrm{I}}_{12} \\ \mathbf{K}^{\mathrm{I}}_{21} & \mathbf{K}^{\mathrm{I}}_{22} \end{bmatrix}, \ \mathbf{K}^{\mathrm{II}} = \begin{bmatrix} \mathbf{K}^{\mathrm{II}}_{11} & \mathbf{K}^{\mathrm{II}}_{12} \\ \mathbf{K}^{\mathrm{II}}_{21} & \mathbf{K}^{\mathrm{II}}_{22} \end{bmatrix} \tag{24}$$

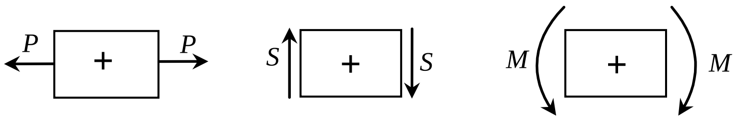

Figure 2: Sign convention for positive axial force *P*, shear force *S* and bending moment *M*.

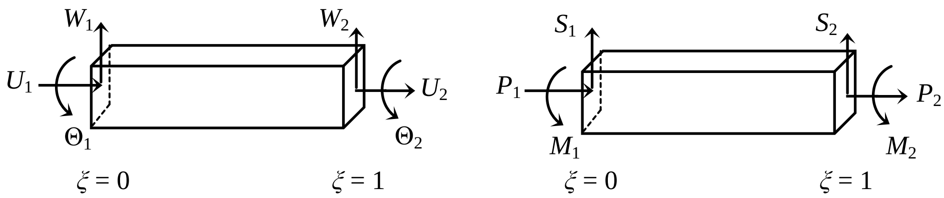

Figure 3: Boundary conditions for displacements and forces.

2.2 Cracked Element

The explicit 3×3 flexibility matrix $\mathbf{C}$ of a cracked element for both rectangular and circular cross-section beams has been investigated in the literature by many, e.g. see [2] in terms of the cross sectional dimensions and the crack depth *a* through the thickness *h*. The best fitted formulas for flexibility coefficients can be found in Ref [2]. The details are not repeated here. The flexibility matrix **C** for a crack element is as follows [2]:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{22} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \tag{25}$$

The dynamic stiffness matrix $\mathbf{K}^{\mathrm{III}}$ of the cracked element (III) can be constructed by taking the inverse of **C** as:

$$\mathbf{K}^{\mathrm{III}} = \begin{bmatrix} \mathbf{C}^{-1} & -\mathbf{C}^{-1} \\ -\mathbf{C}^{-1} & \mathbf{C}^{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{K}^{\mathrm{III}}_{11} & \mathbf{K}^{\mathrm{III}}_{12} \\ \mathbf{K}^{\mathrm{III}}_{21} & \mathbf{K}^{\mathrm{III}}_{22} \end{bmatrix} \tag{26}$$

The 6×6 matrix $\mathbf{K}^{\mathrm{III}}$ represents the force displacement relationship at the left and right hand ends for the cracked element (III).

2.3 The Cracked Beam System

The overall frequency dependent dynamic stiffness matrix **K**(ω) can be obtained by assembling the element stiffness matrices of two intact Timoshenko beams (I and II) and the cracked element (III) in the usual way. In matrix notation, the overall dynamic stiffness matrix **K**(ω) can be obtained as:

$$\mathbf{K}(\omega)=\begin{bmatrix}\mathbf{K}_{11}^{\mathrm{I}} & \mathbf{K}_{12}^{\mathrm{I}} & 0 & 0\\ \mathbf{K}_{21}^{\mathrm{I}} & \mathbf{K}_{22}^{\mathrm{I}}+\mathbf{K}_{11}^{\mathrm{III}} & \mathbf{K}_{12}^{\mathrm{III}} & 0\\ 0 & \mathbf{K}_{21}^{\mathrm{III}} & \mathbf{K}_{22}^{\mathrm{III}}+\mathbf{K}_{11}^{\mathrm{II}} & \mathbf{K}_{12}^{\mathrm{II}}\\ 0 & 0 & \mathbf{K}_{21}^{\mathrm{II}} & \mathbf{K}_{22}^{\mathrm{II}}\end{bmatrix} \tag{27}$$

The assembled stiffness matrix will now be used to compute the natural frequencies and mode shapes of the cracked Timoshenko beam. An effective way is to apply the Wittrick-Williams algorithm which is generally used as a solution technique in solving transcendental eigenvalue problems as in the present case employing the DSM. Appropriate boundary conditions can be applied by deleting the particular rows and columns of $\mathbf{K}(\omega)$, corresponding to zero displacements when computing the natural frequencies and mode shapes of individual cases such as cantilever, simply-supported and clamped-clamped cracked beams. A non-uniform cracked beam can be analysed for its free vibration characteristics by assembling it by many uniform cracked beams.

3 NUMERICAL RESULTS AND DISCUSSIONS

Numerical results for natural frequencies and mode shapes of a cracked Timoshenko beam can be obtained for different boundary conditions. For illustrative purposes, the results are presented here only for the cantilever boundary conditions. The cracked beam model is made of mild steel and the data are taken from the literature [1] with the following material properties and beam dimensions: E=216GPa, ρ=7850kgm^{-3}, ν=0.28, L=0.2m, B=0.025m, h=0.0078m. The natural frequencies and mode shapes of the cracked Timoshenko beam have been examined for various crack location and depth ratios.

Table 1 shows the first four natural frequencies for the cracked and intact Timoshenko beams together with the results from Ref [1]. Excellent agreement is achieved between the current investigation using the DSM and the one using the finite element method in [1] as can be seen. The fundamental natural frequency ratios, defined as the ratio between natural frequencies of the cracked beam and the intact beam, are plotted against the crack location with respect to different crack depth ratio and the results are shown in Figure 4. The natural frequencies of the cracked beam are lower than the corresponding intact ones as expected. The difference naturally increases with the crack depth. It is clearly shown that a crack nearer to the built-in end of the beam has a much greater effect than the one located nearer to the free end because the maximum bending moment occurs at the built-in end and the bending moment gradually reduces towards the tip and finally becomes exactly zero at the tip. The natural frequencies are almost unchanged when the crack is far away from the fixed end. Figure 5 shows the fundamental natural frequency ratio against the crack depth ratio for two crack location ratios set to 0.4 and 0.6, respectively.

The next set of results was obtained to illustrate the mode shapes of the cracked beam when compared with the intact Timoshenko beam for the cantilever boundary conditions. Figure 6 shows the first three natural frequencies and mode shapes when the ratio of crack location and crack depth are both set to 0.4. The results for the intact beam are shown by dashed lines whereas the solid lines indicate results for the cracked beam. The natural frequencies of the intact beam are shown within parenthesis.

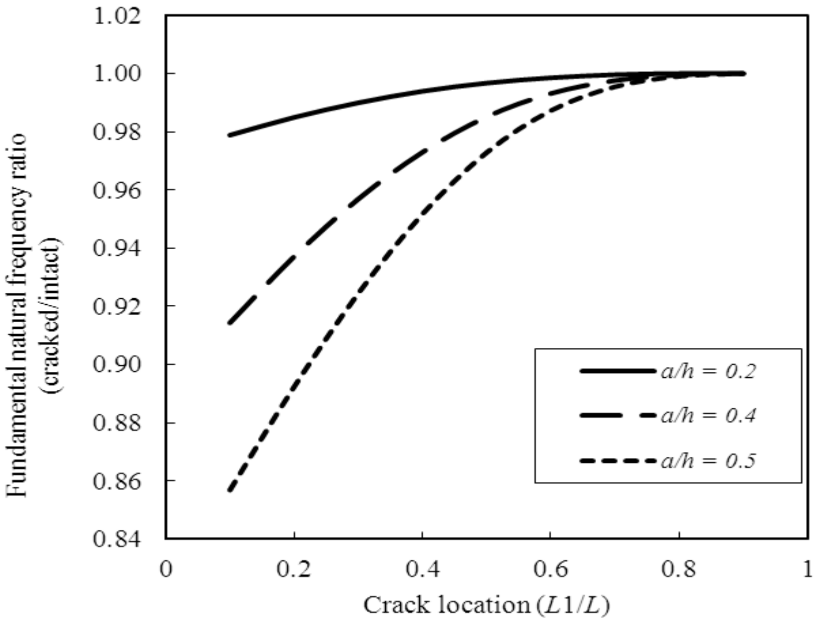

Figure 4: Fundamental natural frequency ratio against the crack locations for different crack depth ratios.

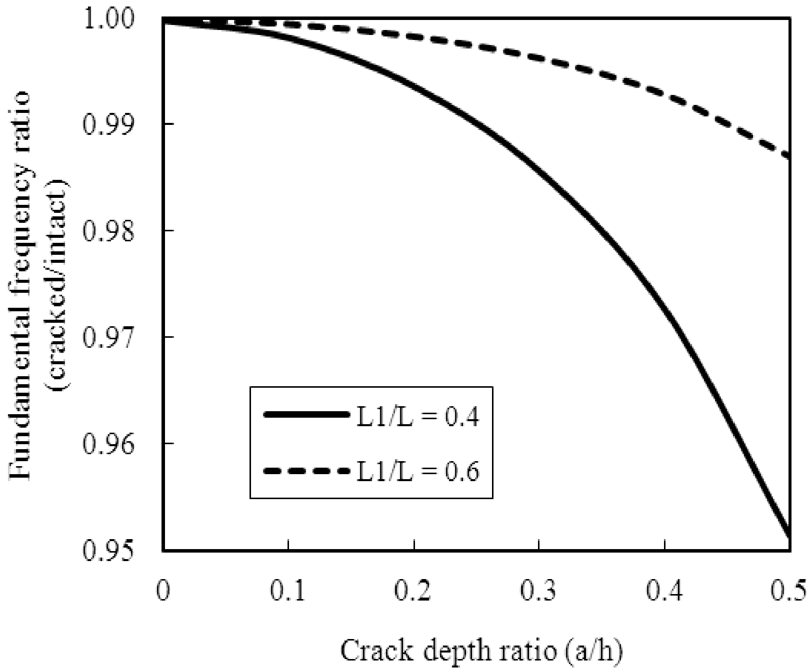

Figure 5: Fundamental natural frequency ratios against the crack depth ratio (a/h) for the crack location ratio (L_1/L = 0.4, 0.6).

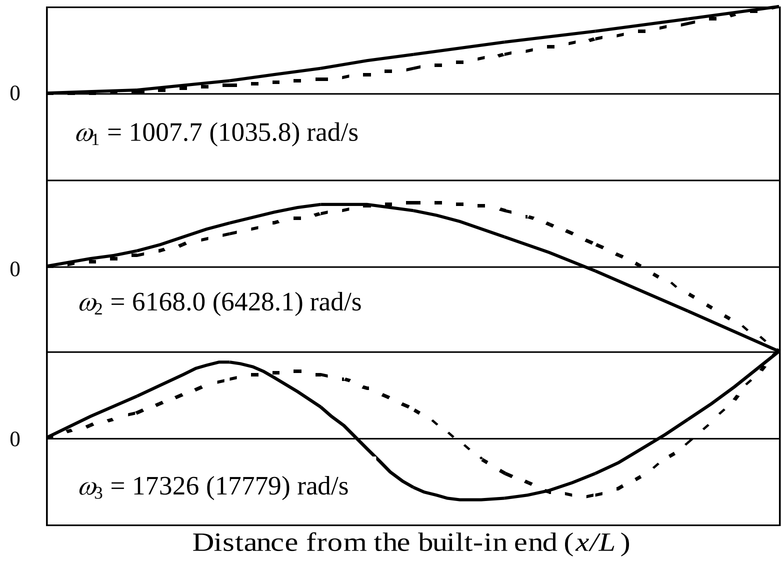

Figure 6: Mode shapes of a cantilever craked beam (solid lines) and the intact beam (dashed lines) for the crack location ratio L_1/L = 0.4 and the crack depth ratio a/h = 0.4.

L_1/L	ω_i (rad/s)	a/h 0.2		0.4		0.6	
		Current	Ref [1]	Current	Ref [1]	Current	Ref [1]
0.2	1	1019.916	1020.137	970.815	966.9525	852.480	842.2205
	2	6420.571	6457.396	6424.429	6454.483	6417.650	6448.175
	3	17651.16	17872.91	17433.77	17596.57	16876.49	16944.56
	4	33674.48	34553.13	32647.85	33100.42	30488.91	29796.26
0.4	1	1029.176	1030.095	1007.670	1006.856	947.811	942.7322
	2	6358.443	6389.394	6167.999	6174.539	5702.459	5689.841
	3	17608.81	17844.86	17326.41	17499.83	16611.97	16792.25
	4	33986.26	34866.97	33767.44	34420.09	33017.86	32971.51
0.6	1	1034.025	1035.284	1028.455	1029.262	1011.412	1010.864
	2	6339.555	6365.914	6064.305	6071.655	5400.865	5371.803
	3	17600.06	17807.94	17197.25	17359.27	16401.77	16478.82
	4	34026.17	34895.50	33713.21	34572.37	33073.52	33710.43
0.8	1	1035.504	1036.884	1035.067	1036.414	1033.700	1034.943
	2	6414.198	6440.057	6353.616	6375.921	6163.768	6174.710
	3	17595.15	17758.61	16951.02	17077.99	15255.65	15286.83
	4	33674.59	34393.87	32081.06	32639.52	29281.01	29529.79
Intact Beam	1	1035.812	1037.0189				
	2	6428.052	6458.3438				
	3	17778.74	17960.564				
	4	34279.67	34995.429				

Table 1: Variation of the first four natural frequencies (rad/s) of a cracked Timoshenko beam with crack location ratios (L_1/L) and crack depth ratio (a/h) for cantilever boundary condition.

4 CONCLUSIONS

The free vibration analysis of a cracked Timoshenko beam has been carried out using the dynamic stiffness method. The cracked beam is idealised by two intact Timoshenko beam elements and a cracked element. The dynamic stiffness matrix is formulated through an assembly procedure combining cracked and intact elements. The formulation resulted in a nonlinear eigenvalue problem which was solved by applying the Wittrick-Williams algorithm. Numerical results are presented for cantilever boundary conditions to serve as an illustrative example. The effects of crack location and crack depth on the free vibration behaviour are discussed and representative mode shapes are presented. The accuracy of results using the dynamic stiffness method is an important attribute of this research and the results provide a benchmark standard against the finite element and other approximate methods. The work carried out is expected to pave the way for further research on complex structural systems containing crack elements.

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