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# NUMERICAL INVESTIGATION OF CABLE PARAMETRIC VIBRATIONS

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**Abstract.** *It is well known that parametrically induced vibrations can be a considerable problem of cable lightweight structures. The scope of this paper is to investigate adequacy of applying different types of FEM elements for dynamic instability problem investigation of cable structures. We considered parametric vibrations of inclined cable structures induced by the vertical harmonic support excitation and investigated two types of nonlinear elements for numerical model. The first one of these two types is the axial element, most used for engineering design of cable structures, and the second one is more complicated elastic catenary element based on analytical solution functions. We explored which of the elements mentioned above is more suitable to compute a nonlinear dynamic analysis of cable structures. The numerical calculation is carried out by the program language "Wolfram Mathematica 8.0" with implemented Predictor-Corrector method, and cross check compliance with known solutions.*

## 1 INTRODUCTION

In modern constructions, simple ones as guy-towers and more complicated ones like cable-supported bridges and various pre-stressed flexible roof structures, cable elements are very often used as basic structural parts. The main advantage of cables is tensile load transfer which ensures better use of material without the risk of stability problems. Even more appealing than material reduction for these systems is diversity of aesthetic shape they produce. On the other hand, high flexibility makes them quite geometrically nonlinear and sensitive to dynamic excitations.

Great flexibility, small mass and weak damping requires taking their nonlinear behaviour into account. For numerical analysis finite element method is a dominant tool and several formulations have been developed to represent nonlinear characteristics of the cable. There are generally two approaches for development of finite cable element [1, 4, 5]. In the first, interpolation functions are adopted to represent displacement field of the element. The most common element in this formulation is two-node straight element. The second approach is based on analytical solutions of catenary. Out of several authors that gave matrix formulation for this element, we will mention few. Thai *et al.* [4] derived flexibility matrix for spatial two-node catenary element and Ahamdi-Kashini [1] obtained tangent stiffness matrix for different types of distributed loading acting on the element which he derived by finite segment approach. Analytical approach has some advantages, such as requiring fewer numbers of elements, but it also has a disadvantage: it needs to employ iterative procedures to obtain stiffness matrix.

Dynamic excitation on cables can come from various sources like wind, snow, rain, ice or earthquakes, but dynamic response of a cable can also result from vibration in other parts of the structure (deck or pylon vibrations etc.). Parametrically excited vibration induced by the support motion is extensively studied in the past years using different approximations depending on the sag of the cable on analytical models [3, 6-8]. For inclined cables, this problem gets more complex because of joined nature of the system response coming from external and parametric induced vibration. External resonance is often predicted in engineering practice, and for vibration caused by support motion it is induced by component that is perpendicular to the cable axes. Resonant response occurs when the exciting frequency is equal to any eigenfrequency of the cable. From studies that have been made in recent years it has become relatively well known that the component of support motion acting along the axis of the cable causes parametric vibration. Takahashi [7] using harmonic balance method investigated influence of axial periodic load for pure parametric vibration, and showed that there were many instability zones around the multiples of cable eigenfrequencies and that the widths of the unstable regions change with the sag to span ratio. Considering only first vibration mode, Sun *et al.* [3] used multiple scales method to obtain unstable regions for parametric resonance of a cable and discussed joined forced vibration and parametric vibration for inclined cable. In his analytical model, Douthe *et al.* [8] included out-of plane mode and analyzed instability areas for external and parametric resonance caused by support motion perpendicular and in the direction of cable axes.

In this paper we will compare numerical solutions obtained from type of predictor-corrector method based on finite element analysis with analytical solution based on parabolic shape of cable that takes into account only first mode of vibration [3].

## 2 NUMERICAL MODEL

### 2.1 Model description

Figure 1 shows a model of an inclined cable. The inclination of the cable axes with respect to the vertical axes is denoted by angle γ. Support motion is situated on node A and supposed to be vertical harmonic motion with amplitude X and frequency Ω. All other external forces, except self weight, are neglected. Mechanical and geometric characteristics are taken to be the same as in [3], so mass per length of the unit equals $m=0,0113$ kg/m, length of cable is $L$=12,0 m, axial elastic stiffness is $EA$ = 194000 N, static tension force $T_0$ = 63 N, inclination angle $\gamma$=68.28° and the cable damping is neglected.

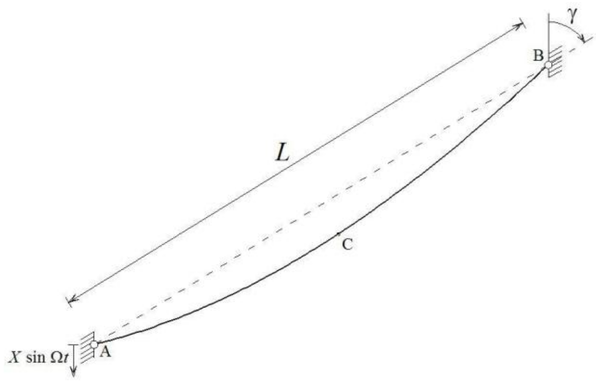


Figure 1: Model of a cable excited by support motion.

### 2.2 Applied finite elements

In numerical analysis two types of finite elements are used: two-node straight bar element and elastic catenary element showed in Figure 2. In the practice of cable modelling, two-node straight element is used the most. For very taut cables with small sag, the straight bar element is a good representation of the cable [5]. A disadvantage is that it takes many of these elements to model cables with large sag and there is a discontinuity of a slope in nodes where there is no external load [5].

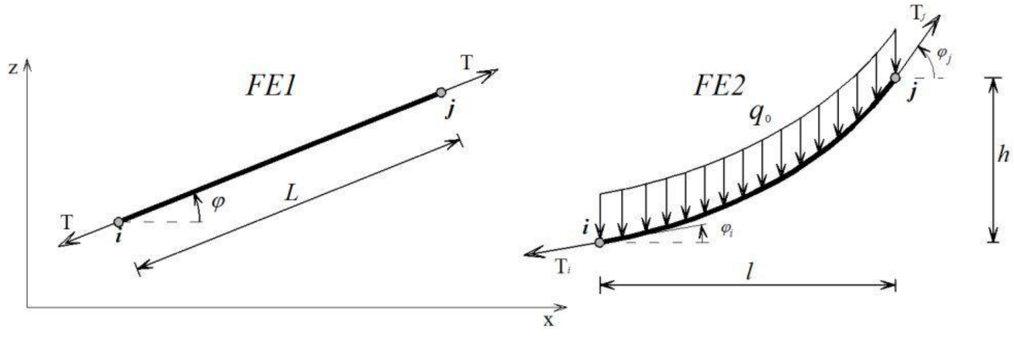


Figure 2: Finite elements used for modelling.

Sub-matrix of tangential stiffness of the first two-node straight element according to [1, 5] is

$$\mathbf{K} = \begin{bmatrix} \mathbf{k}^T & -\mathbf{k}^T \\ -\mathbf{k}^T & \mathbf{k}^T \end{bmatrix}, \qquad \mathbf{k}^T = \frac{AE}{L}\begin{bmatrix} \cos^2\varphi + \dfrac{T}{AE} & \cos\varphi\sin\varphi \\ \cos\varphi\sin\varphi & \sin^2\varphi + \dfrac{T}{AE} \end{bmatrix} \tag{1}$$

The second one, elastic catenary element, takes into account uniformly distributed load along the unstrained length of the element. It is an exact mathematical formulation for a cable loaded by self weight only. Tangent stiffness matrix derived by Ahmadi-Kashani [1] is written in terms of $H$, $\varphi_1$ and $\varphi_2$, where $\varphi_1$ and $\varphi_2$ are angles between the horizontal and the tangent to the cable at the ends, as shown in Figure 2. Tangential sub matrix [1, 5] of an element is

$$\mathbf{k}^T = \alpha \begin{bmatrix} \dfrac{q_0 L_0}{AE} + \sin\varphi_2 - \sin\varphi_1 & \cos\varphi_1 - \cos\varphi_2 \\ \cos\varphi_1 - \cos\varphi_2 & \dfrac{q_0 l}{AE} - \sin\varphi_2 + \sin\varphi_1 \end{bmatrix} \tag{2}$$

in which

$$\alpha = \frac{q_0}{4\sin\dfrac{\varphi_2-\varphi_1}{2}\left[\mu\cos\dfrac{\varphi_1+\varphi_2}{2} - \sin\dfrac{\varphi_2-\varphi_1}{2}\right] + \dfrac{q_0^2 L_0 l}{AE\,H}} \tag{3}$$

where

$$\begin{aligned} \tan\varphi_1 &= \sinh\left[\cosh^{-1}\left(\frac{q_0 L_0}{2H\sinh\mu}\right) - \mu\right] \\ \tan\varphi_2 &= \sinh\left[\cosh^{-1}\left(\frac{q_0 L_0}{2H\sinh\mu}\right) + \mu\right] \end{aligned} \tag{4}$$

Parameter $\mu$ is determined by equation:

$$\mu = \frac{q_0}{2}\left(\frac{l}{H} - \frac{L_0}{AE}\right) \tag{5}$$

For known geometric configuration of nodes, horizontal component of tensile force $H$ is determined by its compatibility function [1, 5]

$$F(H) = \frac{4H^2}{q_0^2}\sinh^2\mu + \frac{h^2}{\left[1 + \left(\dfrac{q_0 L_0}{2AE}\right)\coth\mu\right]^2} - L_0^2 \equiv 0 \tag{6}$$

Eq. (6) is needed to be solved by Newton-Raphson technique for every change of $\mathbf{k}^\mathbf{T}$.

## 3 DIFFERENTIAL EQUATIONS

System of differential equation is formulated adopting d'Alembert principle of dynamic equilibrium. The incremental equation of structure motion for external excitation caused by the motion of a node A is

$$\mathbf{M_n}\,\Delta\ddot{\mathbf{u}}_\mathbf{n} + \mathbf{K_n}\,\Delta\mathbf{u_n} = -\mathbf{K_{n,X}}\,\Delta X \tag{7}$$

where $\mathbf{M_n}$ is mass matrix, $\mathbf{K_n}$ is the stiffness matrix that relates to unknown internal nodes, and the product on the right side of equation is equivalent dynamic excitation of elastic forces generated by support motion.

### 3.1 Numerical solution algorithm

For the nonlinear time-history analysis, an incremental solution based on type of predictor-corrector method is used. In this method, values of variables on the end of the time step are obtained from known values in the beginning of each time step and chosen extrapolation function. As predictor, we are assuming that the vector of acceleration is constant and can be determinate from dynamic equilibrium in the beginning of the time step. Assuming constant acceleration, vector increments of velocity and displacement are

$$\begin{aligned} \Delta\dot{\mathbf{u}}_p &= \ddot{\mathbf{u}}_i\,\Delta t \\ \Delta\mathbf{u}_p &= \dot{\mathbf{u}}_i\,\Delta t + \frac{1}{2}\ddot{\mathbf{u}}_i\,\Delta t^2 \end{aligned} \tag{8}$$

In general, assumption that acceleration is constant is definitely wrong even for very small time steps. Approximated values of variables at the end of time step will be corrected based on consideration that acceleration differs linearly. So from calculated predictor increments on the end of time step from Eq. (7) we can determine increment of acceleration on the end of time step

$$\Delta\ddot{\mathbf{u}} = -\mathbf{K_{n,X}}\,\Delta X - \mathbf{M^{-1}}\,\mathbf{K^T}\,\Delta\mathbf{u}_p \tag{9}$$

Taking the linear change of acceleration into account, we correct the values of variables on the end of time step

$$\begin{aligned} \dot{\mathbf{u}}_{i+1} &= \dot{\mathbf{u}}_i + \Delta\dot{\mathbf{u}}_p + \frac{1}{2}\Delta\ddot{\mathbf{u}}\,\Delta\mathbf{t} = \dot{\mathbf{u}}_i + \ddot{\mathbf{u}}_i\,\Delta t + \frac{1}{2}\Delta\ddot{\mathbf{u}}\,\Delta t \\ \mathbf{u}_{i+1} &= \mathbf{u}_i + \Delta\mathbf{u}_p + \frac{1}{6}\Delta\ddot{\mathbf{u}}\,\Delta t^2 = \mathbf{u}_i + \dot{\mathbf{u}}_i\,\Delta t + \frac{1}{2}\ddot{\mathbf{u}}_i\,\Delta t^2 + \frac{1}{6}\Delta\ddot{\mathbf{u}}\,\Delta t \end{aligned} \tag{10}$$

Now, at the end of the time step, tangent stiffness matrix for new position of nodes and elastic forces is determined. To ensure dynamic equilibrium at the end of interval, vector of residual forces is determined

$$\mathbf{R}_{i+1} = \mathbf{M}\left(\ddot{\mathbf{u}} + \Delta\ddot{\mathbf{u}}\right) + \mathbf{F}_{el_{i+1}} - \mathbf{F}_{q_0} \tag{11}$$

where $F_{el}$ stands for updated elastic forces determined from deformation of elements at the end of time step, and $F_{q0}$ for self-weight vector force.

### 3.2 Approximated analytical model

For very taut cables, while neglecting the initial deflection from self weight, natural frequency of cable’s first mode can be determinate by

$$\omega_1 = \frac{\pi}{L}\sqrt{\frac{T_0}{m}} \tag{12}$$

If initial deflection is assumed to be a parabolic shape, the first mode natural frequency $\omega'_1$ that takes into account initial deflection of cable can be determined from [3]

$$\omega'_1 = \left[1+\frac{1}{2}\left(\frac{2}{\pi}\right)^4 \frac{L\,EA\,\chi^2}{T_0}\right]^{1/2} \omega_1 \tag{13}$$

where $\chi = mg\,\sin\gamma/T_0$ is curvature for parabolic shape static deflection.

According to [3], differential equation governing the motion of first vibration mode amplitude $y(t)$ excited by support motion is

$$\begin{aligned}&\ddot{y}(t)+2\xi\dot{y}(t)+\omega_1^2\left[1+\frac{1}{2}\left(\frac{2}{\pi}\right)^4\frac{L\,EA\,\chi^2}{T_0}+\frac{X\,EA}{T_0}\left(\frac{1}{L}\cos\gamma-\frac{\chi}{2}\sin\gamma\right)\sin\Omega t\right.\\&\left.+\frac{X^2EA}{2L^2T_0}\sin^2\gamma\sin^2\Omega t\right]y(t)-\frac{3EA\chi\pi}{L^2m}y(t)^2+\frac{EA\pi^4}{4L^4m}y(t)^3=\frac{2X^2EA\chi}{L^2m\pi}\sin^2\gamma\sin^2\Omega t\\&+\frac{4X\,EA\chi}{Lm\pi}\cos\gamma\sin\Omega t-\frac{2X\,EA\chi^2}{m\pi}\sin\gamma\sin\Omega t+\frac{2X\Omega^2}{\pi}\sin\gamma\sin\Omega t\end{aligned} \tag{14}$$

Eq. (14) is solved numerically by the same predictor-corrector technique described in section 3.1.

# 4 RESULTS

## 4.1 Static solution results

Static equilibrium shape deflection for self weight of catenary modelled with various number of elements FE1 and FE2 is found using Newton-Raphson technique and compared with exact analytical solutions [2]. Eigenfreqencies from the same elements are compared with analytical solution of cable eigenfrequency obtained from approximation of cable deflection with parabola [2]. The results are shown in Table 1.

| **Number of elements** | **FE1** | | | **FE2** | | |
|---|---|---|---|---|---|---|
| | $u_{Cx}$ | $u_{Cy}$ | $\omega_1$ | $u_{Cx}$ | $u_{Cy}$ | $\omega_1$ |
| 2 | 0,010518 | -0,026414 | 18,5574 | 0,010411 | -0,026146 | 18,4924 |
| 4 | 0,010437 | -0,026212 | 20,2435 | 0,010411 | -0,026146 | 20,2279 |
| 8 | 0,010417 | -0,026163 | 20,6356 | 0,010411 | -0,026146 | 20,6717 |
| 12 | 0,010414 | -0,026154 | 20,7561 | 0,010411 | -0,026146 | 20,7544 |
| 16 | 0,010412 | -0,026150 | 20,7843 | 0,010411 | -0,026146 | 20,7833 |
| Analitical solution [2] | 0,010411 | -0,026146 | 21,8063 | 0,010411 | -0,026146 | 21,8063 |

Table 1: Static deflection of model and first eigenfrequency.

Results for displacement of node C determined by elements FE1, with the increasing number of elements, converge to the analytical solution [2]. Elements FE2 that are taking into account distributed load along element length, are giving results that are the same for elastic catenary regardless of the number of elements, while for dynamical characteristics more finite elements are needed in both cases.

### 4.2 Results of dynamic analysis

Accuracy of described numerical predictor-corrector method used to obtain time history functions depends on time step size. To ensure dynamic equilibrium, the time step must be chosen considering a value of residual forces. Figure 3 shows characteristic diagram for maximum norm value of residual force vector for different values of time step in calculation. Although this diagram shows only maximum norm value of residual forces and this value occurs usually at the very beginning of calculation and tends to decrease as calculation is progressing, chosen time step for further calculation is $\Delta t = 10^{-4}$ *sec*.

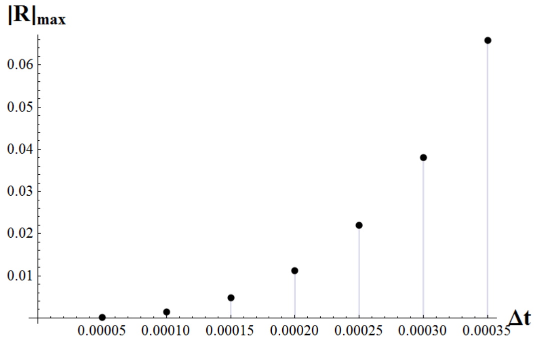


Figure 3: Maximum norm value of vector of residual forces of FE1 and N=8 elements for $\Omega/\omega_1$ = 1.02.

Parametric vibrations of inclined cable are induced by amplitude $X$=0.5 *mm* and varying excitation frequency. Convergence of system response for the increase of number of elements is shown in Figure 4. Figure 5 shows time history response for elements FE1 and FE2. Unlike the results obtained for static equilibrium where FE2 showed as much more superior to FE1, in dynamic time history response analysis both of them are giving almost same results. It is necessary to point out that FE2 is more ″costly″ because it requires an iterative procedure to solve the Eq. (6) for each element in each time step.

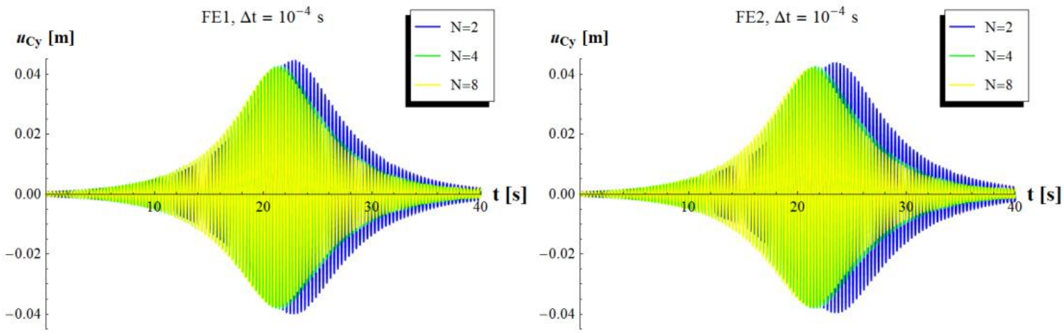


Figure 4: Time history for vertical displacement of point C for $\Omega/\omega_1$ = 2.0.

As we mentioned earlier there are a two main resonant regions caused by support motion. Vertical component of characteristic time history response for point C is shown in Figures 4 and 5. Beating phenomenon and large increase in amplitude in both resonant regions are inherent. Special interests are the regions of frequencies where amplitude increases significantly. The maximum amplitude for the first 40 *sec* of time history response is plotted in Figure 6 for the case is when $\Omega/\omega \approx 1$ and $\Omega/\omega \approx 2$. Comparison of results obtained using finite elements and results that are obtained from Eq. (14) which is solved numerically, shows that finite elements indicate smaller amplitude oscillation.

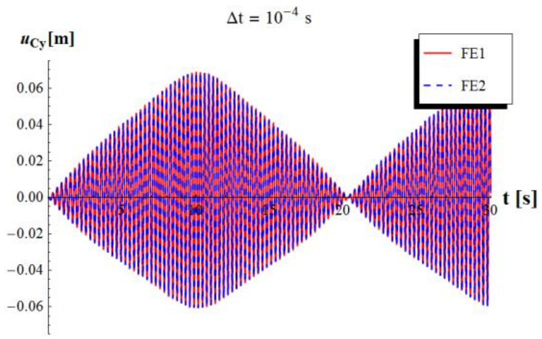


Figure 5: Time history for vertical displacement of point C for $\Omega/\omega_1$ = 1.02.

The reason for this can be found in influence of increased stiffness that is caused by geometrical nonlinearity. This influence is neglected in analytical solution derived in [3].

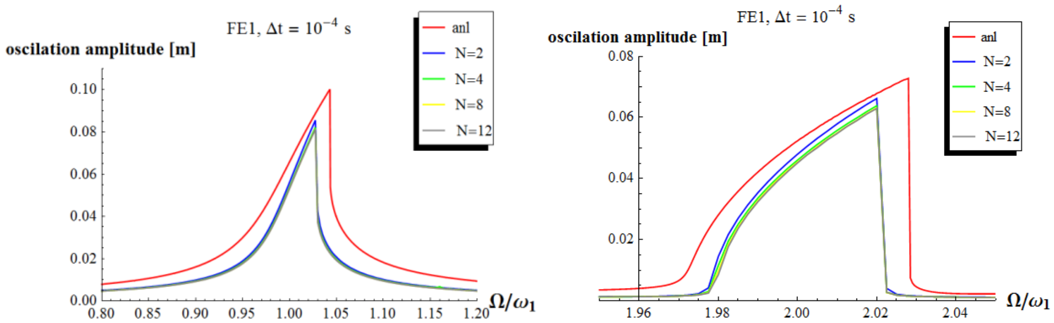


Figure 3: Maximum amplitude regions for different frequency ratio.

The influence of geometric nonlinearity affects not only the amplitude reduction, but also narrows the range of frequencies for higher amplitudes. For given systems the number of finite elements has a slight effect on results, but it has no influence on higher modes. Since this kind of system is very sensitive to parameters change on coupled external/parametric resonance [8], further research is required.

## 5 CONCLUSIONS

- Since FE1 and FE2 are giving almost the same results for dynamic response, and there is no need for iteration procedures to obtain stiffness matrix it is recommended to use simpler elements FE1 because.
- Described predictor-corrector method is applicable for dynamic analysis, and to ensure the small enough step the residual forces can be neglected, so there is no need for further iteration. Although in some cases assumption of linear correction can be inadequate, to avoid further reduction of the time step it is recommended to use multiple corrector
- Maximum amplitude regions can be very well foreseen with finite element analyses.
- With tangential stiffness matrix, it is easier to take into account the effect of geometric nonlinearity of cables.

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