11th International Conference on Vibration Problems
Z. Dimitrovová *et al.* (eds.)
Lisbon, Portugal, 9-12 September 2013

DIFFRACTION OF PLANE SH WAVES BY A CIRCULAR CAVITY IN QUARTER-INFINITE MEDIUM

Hasan Faik Kara*[1], Hasan Engin[1]

[1] Civil Engineering Department, Istanbul Technical University
faikkara@gmail.com
enginh46@gmail.com

Keywords: SH waves, Quarter infinite medium, Cavity, Addition theorem.

Abstract. *In this study, diffraction of out-of-plane SH waves by a circular cavity in homogeneous, isotropic and linear elastic quarter-infinite medium is investigated. The excitation is taken harmonic so that governing wave equation of the problem turns into Helmholtz wave equation. Separation of variables is applied to Helmholtz wave equation in order to obtain an analytic closed-form solution. Displacement fields are expressed in terms of Fourier-Bessel series. Complex unknowns of Fourier-Bessel series are evaluated by applying boundary conditions. Zero-stress boundary condition at cavity inner surface is satisfied directly when polar coordinate systems are used. Free stress conditions at flat surfaces of quarter infinite medium are satisfied in closed form via imaging method and addition theorems. Numerical examples are compared with early studies and the effect of the cavity is discussed.*

1 INTRODUCTION

Scattering and diffraction of plane SH waves by a circular cavity is investigated by many researchers. Numerical and analytical solution techniques are used to solve these problems. Although analytical techniques can only be applied to simple geometries, they provide exact solutions that can be used to verify numerical methods. A closed-form solution for the problem of scattering and diffraction of plane SH waves by an underground cavity is obtained by Lee [1]. In that problem, stress-free boundary condition of flat surface was satisfied by image technique. Sanches-Sesma [2] investigated the diffraction of elastic SH-waves in wedges. By using the free-field wave solution of Sanches-Sesma, Lee and Sherif [3] determined the behaviour of wedge space with a circular canyon under seismic SH waves.

In this study, diffraction of plane SH waves by a circular cavity in homogeneous, isotropic and linear elastic quarter-infinite medium is investigated. For practical purpose, this investigation may be helpful for the design of underground tunnels near hillsides. In order to obtain exact solutions of this problem, wave function expansion method is used. Displacement fields satisfying wave equation and Sommerfeld radiation condition are expressed in terms of Fourier-Bessel series. Complex unknowns of Fourier-Bessel series are evaluated by applying boundary conditions. Zero-stress boundary condition at cavity inner surface is satisfied directly when polar coordinate systems are used. Free stress conditions at flat surfaces of quarter infinite medium are satisfied in closed form via imaging method since anti-plane waves reflect with same angle from flat surfaces. Addition theorems are used for coordinate transformation during evaluation of constants stage. Numerical examples are compared with early studies and the effect of the cavity is discussed.

2 MODEL AND FORMULATION OF THE PROBLEM

The cross-section of the two-dimensional model is shown in figure 1. The quarter-infinite medium consists of a cylindrical cavity with radius a. Cavity center is at a depth h_1 below the ground surface and the distance between cavity center and hillside is h_2. The medium is subjected to harmonic plane SH wave with wavelength λ which makes an angle γ_i with the y axis. When all lengths are normalised with a, variables of this problem would be γ_i, h_1/a, h_2/a and λ/a.

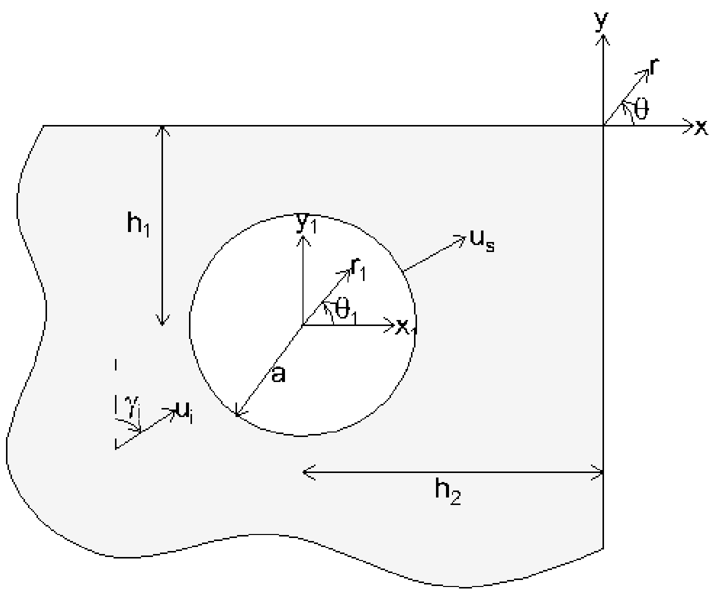

Figure 1: Geometry of the problem

Because out-of-plane waves reflect with same angle from flat surfaces, wave characteristics have to be symmetric with respect *x* and *y* axis. Using this symmetry condition, solution of equivalent model shown in figure 2 is also a solution of the main problem. Here, u_{i1} stands for incoming SH wave and u_{s1} stands for scattered waves from the cavity. $(u_{s2}+u_{i2})$ represents reflected waves from the hillside and $(u_{s4}+u_{i4})$ represents reflected waves from the ground. Similary $(u_{i3}+u_{s3})$ represents reflection of reflected waves.

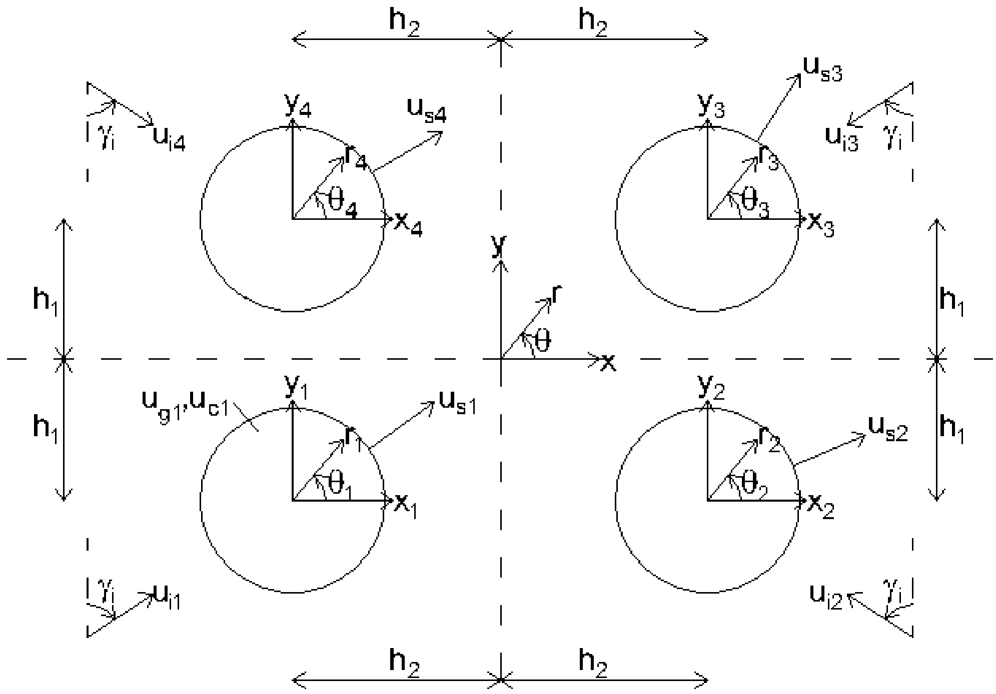

Figure 2: Geometry of equivalent model

Governing equation of this problem is the well-known wave equation:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)U(r,\theta,t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}U(r,\theta,t) \tag{1}$$

Here, *c* denotes for the shear wave speed. Since excitation is taken harmonic, displacements will also be harmonic.

$$U(r,\theta,t)=u(r,\theta)e^{-i\omega t} \tag{2}$$

Here ω is angular frequency. When equation 2 is substituted into equation 1 and wave number ($k=\omega/c$) is introduced, Helmholtz wave equation is obtained.

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+k^2\right)u(r,\theta)=0 \tag{3}$$

By using separations of variables, displacement field can be expressed in terms of two functions:

$$u(r,\theta)=R(r)\Theta(\theta) \tag{4}$$

When equation (4) is placed into equation (3), the following equation is obtained:

$$\frac{r^2}{R(r)}\frac{\partial^2 R(r)}{\partial r^2}+\frac{r}{R(r)}\frac{\partial R(r)}{\partial r}=-\frac{1}{\Theta(\theta)}\frac{\partial^2\Theta(\theta)}{\partial\theta^2} \tag{5}$$

Since two sides of equation (5) is a function of different variables, it holds only for both sides have to be equal to a constant. This constant is chosen to be n^2 for convenience. When the differential equation is solved for each variable separately, there follows:

$$\Theta(\theta)=\beta e^{in\theta}+\overline{\beta}e^{-in\theta} \tag{6}$$

$$R(r)=\sum_{n=-\infty}^{\infty}A_n C_n(kr) \tag{7}$$

Because of periodicity, n has to be an integer. In equation (6), β is a complex constant and $\overline{\beta}$ is complex conjugate of β. In equation (7), A_n's are complex constants and C_n's are Bessel functions of order n. When separate solutions are combined:

$$u(r,\theta)=R(r)\Theta(\theta)=\sum_{n=-\infty}^{\infty}A_n C_n(kr)\left(\beta e^{in\theta}+\overline{\beta}e^{-in\theta}\right) \tag{8}$$

A simpler form of solution can be obtained by removing duplications:

$$u(r,\theta)=\sum_{n=-\infty}^{\infty}A_n C_n(kr)e^{in\theta} \tag{9}$$

In these series, outgoing waves will be represented by $H_n^{(1)}$. Incoming waves, which could be represented by $H_n^{(2)}$, are omitted because of Sommerfeld Condition. Thus, displacement fields that corresponds to scattered waves from the cavity and its images will be in this form:

$$u_{s1}(r_1,\theta_1)=\sum_{n=-\infty}^{\infty}A_{s1,n}H_n^{(1)}(kr_1)e^{in\theta_1} \tag{10}$$

$$u_{s2}(r_2,\theta_2)=\sum_{n=-\infty}^{\infty}A_{s2,n}H_n^{(1)}(kr_2)e^{in\theta_2} \tag{11}$$

$$u_{s3}(r_3,\theta_3)=\sum_{n=-\infty}^{\infty}A_{s3,n}H_n^{(1)}(kr_3)e^{in\theta_3} \tag{12}$$

$$u_{s4}(r_4,\theta_4)=\sum_{n=-\infty}^{\infty}A_{s4,n}H_n^{(1)}(kr_4)e^{in\theta_4} \tag{13}$$

Incident SH waves which have an angle of γ_i with respect to y axis with unit amplitude apart from time-dependent part can simply be expressed as:

$$u_{i1}(r,\theta)=e^{ikrSin(\theta+\gamma_i)} \tag{14}$$

By using the identity (15), equation (14) can also be expressed in terms of a Fourier-Bessel series:

$$e^{\frac{1}{2}z(t-1/t)}=\sum_{k=-\infty}^{\infty}t^k J_k(z) \tag{15}$$

$$u_{i1}(r,\theta)=\sum_{n=-\infty}^{\infty} e^{in\gamma_i} J_n(kr)e^{in\theta} \tag{16}$$

Combination of incident waves and its reflections gives free field waves as follows:

$$u_{ff}(r,\theta)=\sum_{n=-\infty}^{\infty}\left(e^{in\gamma_i}+e^{in(-\gamma_i)}+e^{in(\pi+\gamma_i)}+e^{in(\pi-\gamma_i)}\right)J_n(kr)e^{in\theta} \tag{17}$$

Total displacement field is a sum of free field waves and waves scattered from cavity and its images:

$$u=u_{ff}+u_{s1}+u_{s2}+u_{s3}+u_{s4} \tag{18}$$

3 BOUNDARY CONDITIONS

Displacement function shown in equation (18) must satisfy stress-free boundary conditions at surfaces of the cavity and its images:

$$\sigma_{rz}\big|_{r_1=a}=\mu\frac{\partial u}{\partial r_1}\bigg|_{r_1=a}=0 \tag{19}$$

$$\sigma_{rz}\big|_{r_2=a}=\mu\frac{\partial u}{\partial r_2}\bigg|_{r_2=a}=0 \tag{20}$$

$$\sigma_{rz}\big|_{r_3=a}=\mu\frac{\partial u}{\partial r_3}\bigg|_{r_3=a}=0 \tag{21}$$

$$\sigma_{rz}\big|_{r_4=a}=\mu\frac{\partial u}{\partial r_4}\bigg|_{r_4=a}=0 \tag{22}$$

In these equations, μ represents shear modulus. Because the total displacement is a function of different coordinate systems, addition theorem is used to unify them. As shown in figure 3, a Fourier-Bessel serial expressed in terms of polar coordinates r_i, θ_i can be displaced to a new coordinate system r_j,θ_j by using the identity (23).

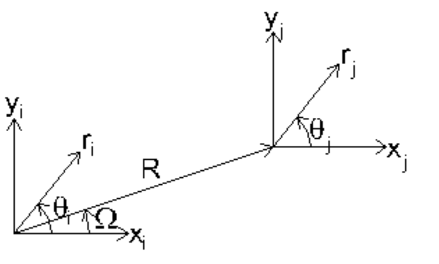

Figure 3

$$H_n^{(1)}(kr_i)e^{in\theta_i}=\sum_{m=-\infty}^{\infty} H_{n-m}^{(1)}(kR)J_m(kr_j)e^{i(n-m)\Omega}e^{im\theta_j} \tag{23}$$

4 NUMERICAL RESULTS

Because Fourier-Bessel Series and Addition Series are both convergent, it is possible to truncate them into a finite sum with *N* terms. Although accuracy of the results depends how great the value of *N* is, it is a necessity to take *N* finite to construct systems of linear equations. However, the Fourier-Bessel series converge quickly so the results can be obtained at intended accuracy. In each boundary condition, *2N+1* linear independent equations can be obtained because of periodicity condition. Hence, totally *8N+4* equations are obtained to evaluate *8N+4* unknown constants. In numerical examples, *N* is taken *16* and variables in length dimension are normalised with respect to cavity radius. For given values of γ_i, λ/a, h_1/a and h_2/a, displacement amplitudes can be determined at any point. In the following figures, displacement amplitudes at surface, when $y=0$, are plotted. In figures 4-8, a comparison with Sánchez-Sesma's free field solution [2] for quarter infinite medium is made. In figure 4, when wave length is relatively large, displacement profile is very similar to free field solution for all incoming wave angles. For shorter wave lengths, displacement profile differs dramatically. In other words, effect of cavity disappears for long wave lengths. Similarly cavity depth is more effective in surface displacements for short wave lengths. In figure 9, surface displacement amplitudes for different wave lengths is shown. It is seen that for short wave lengths there is fluctuation whereas for long wave lengths the profile is simpler.

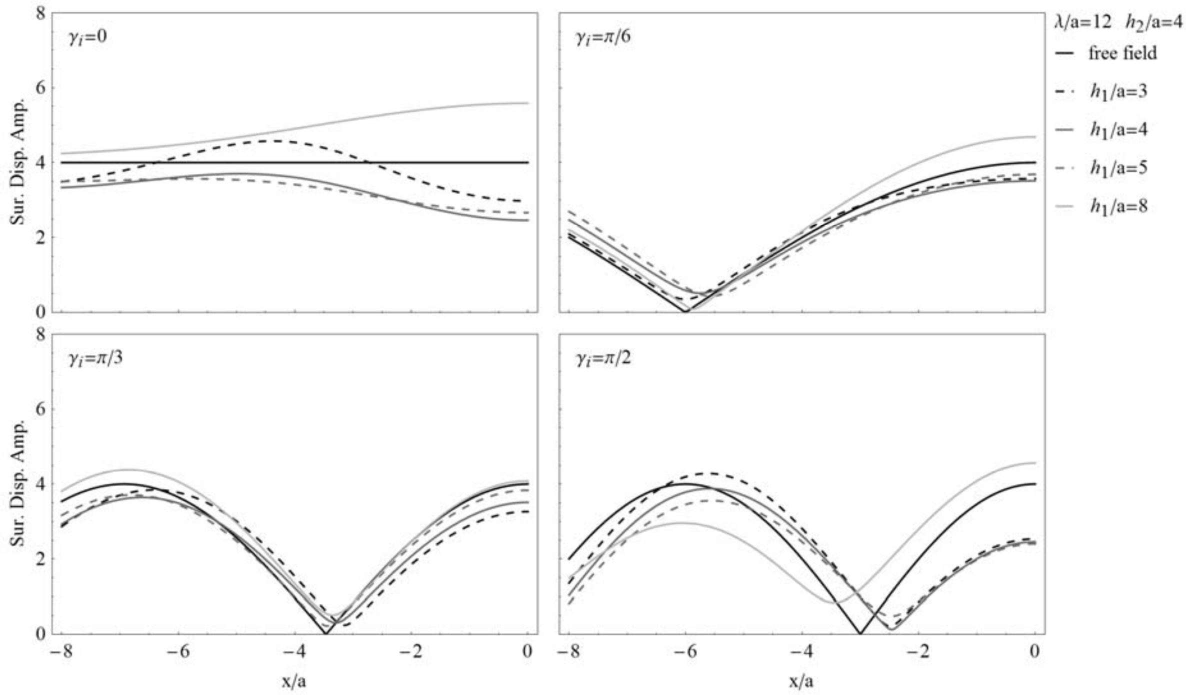

Figure 4

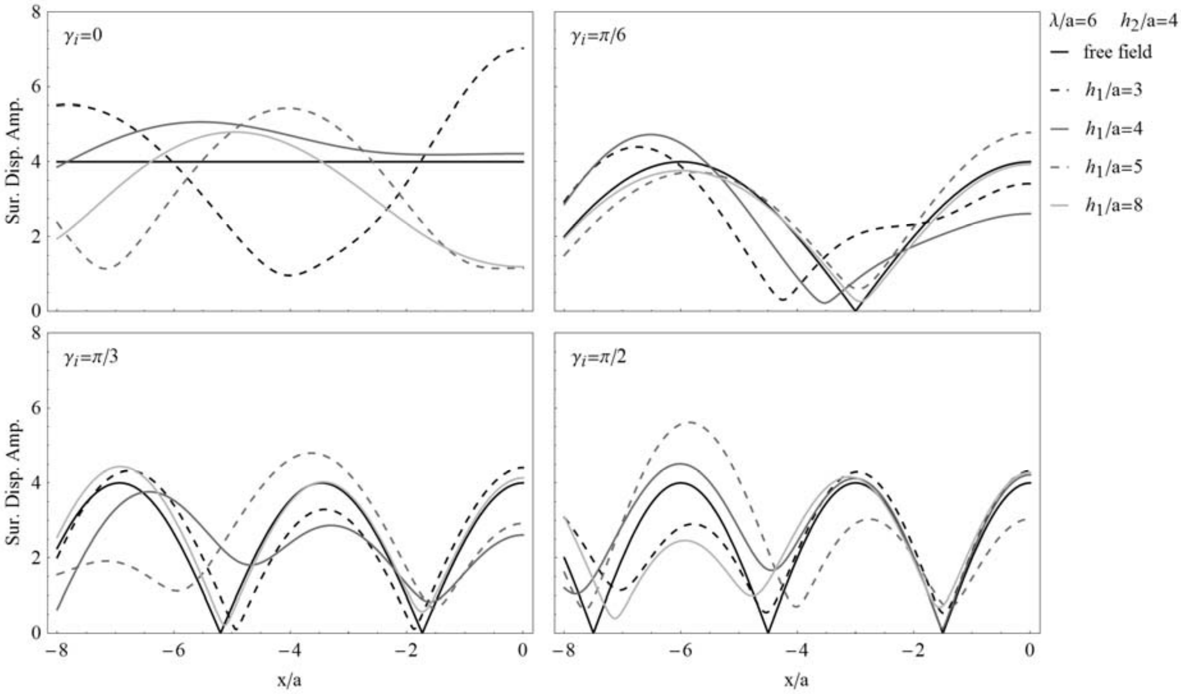

Figure 5

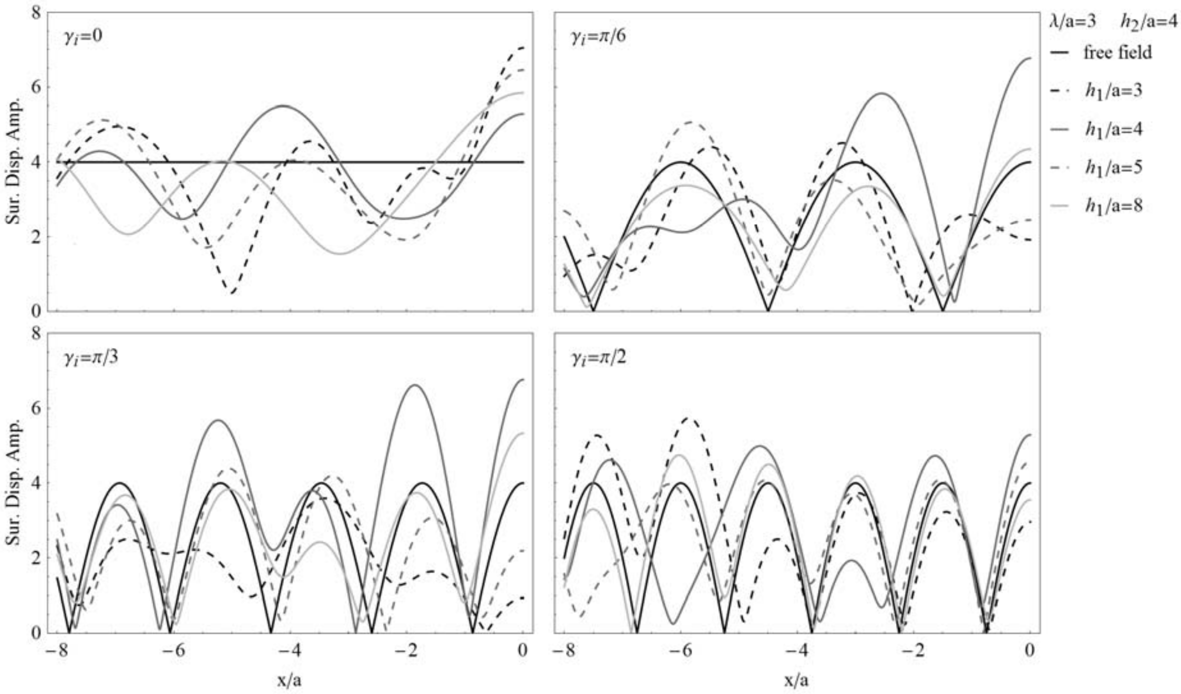

Figure 6

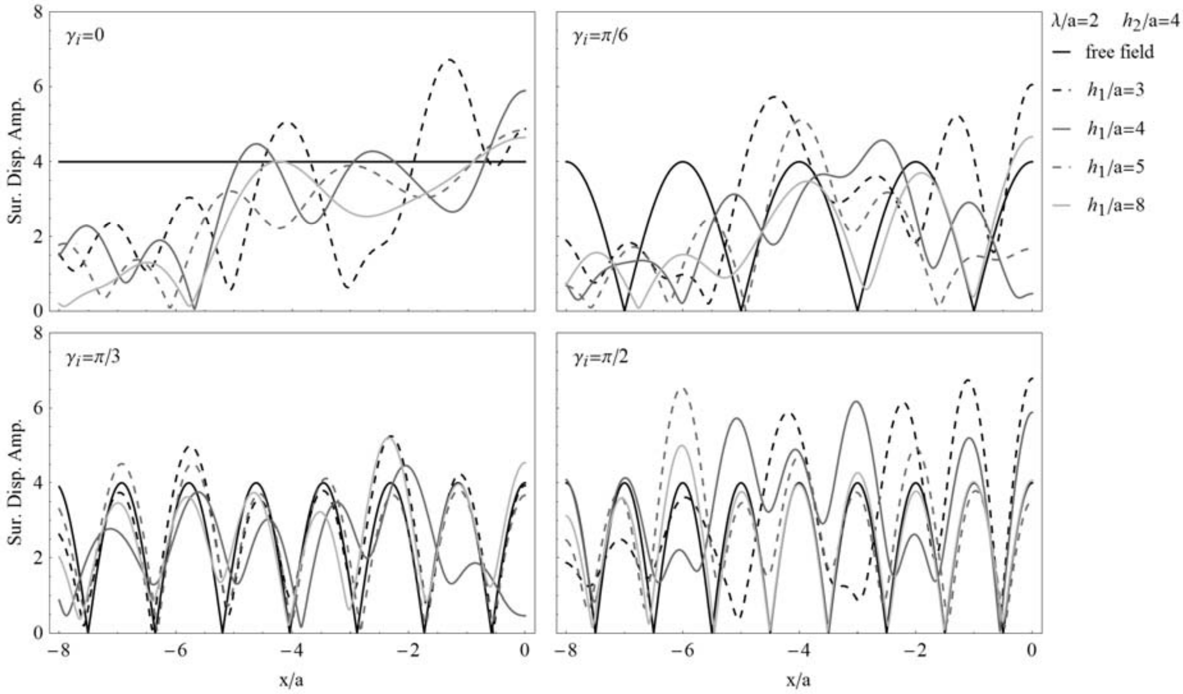

Figure 7

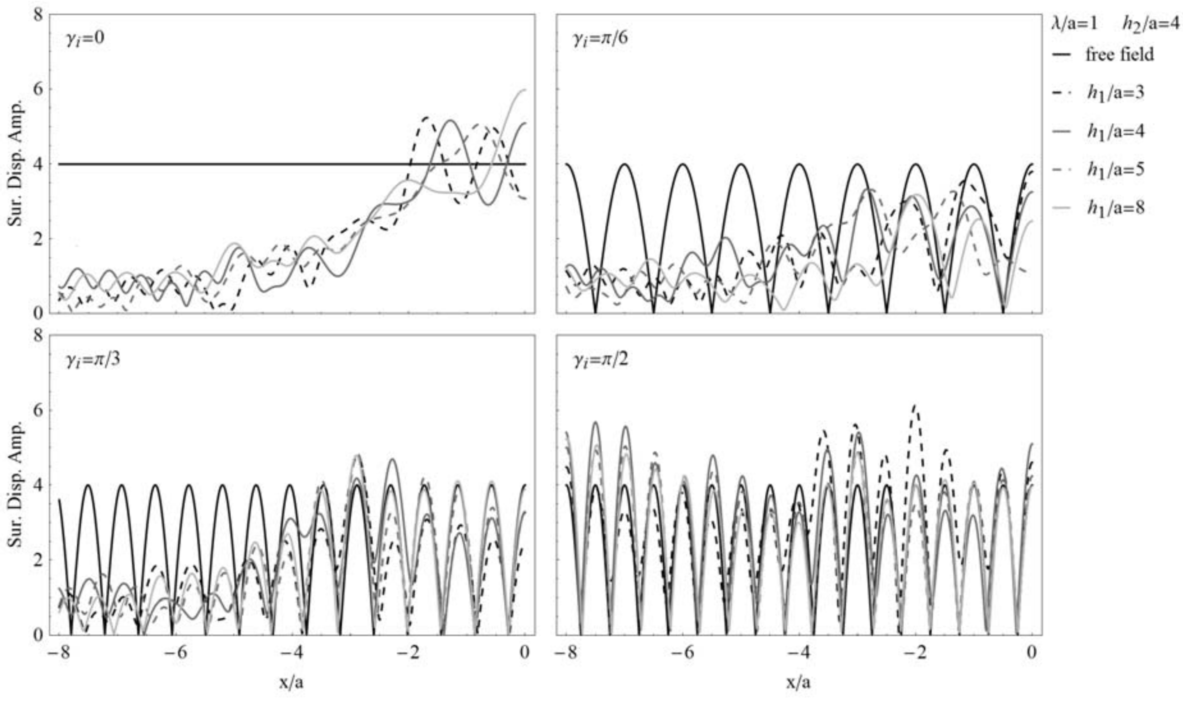

Figure 8

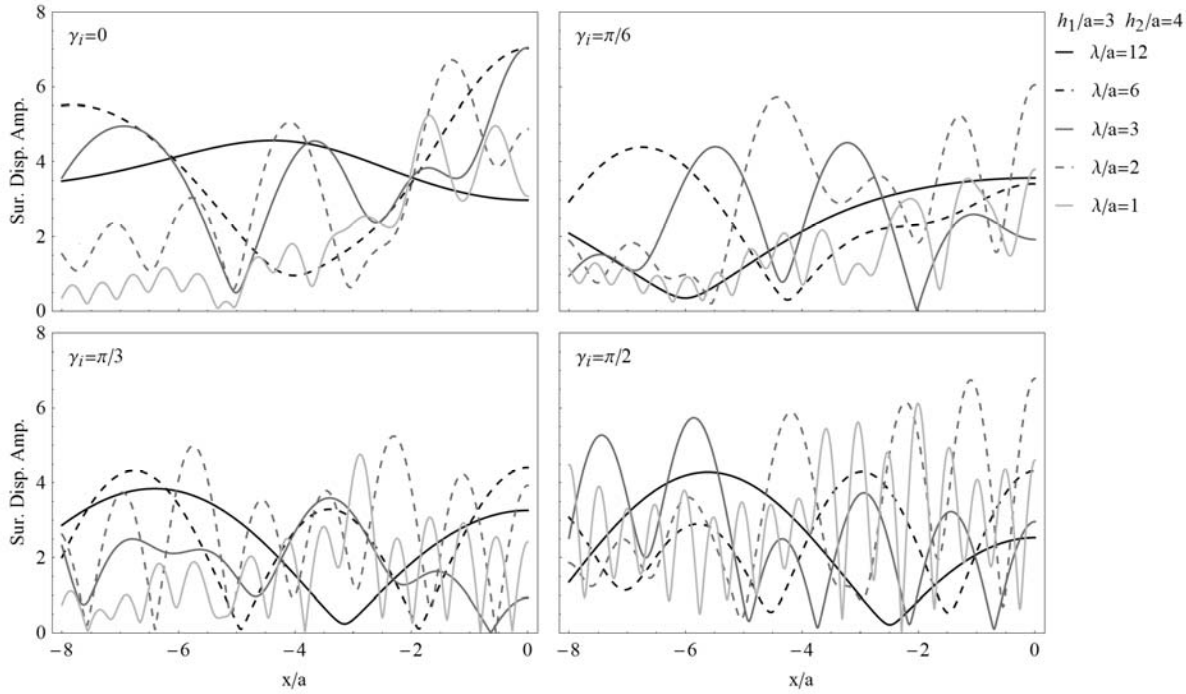

Figure 9

5 CONCLUSIONS

The examples shown in this study are very limited, only surface displacement amplitudes are plotted. However, displacement amplitudes can be calculated at any point in the medium in terms of the variables of this problem. Also, velocities and accelerations can be derived easily from displacements. Geometry of this problem is very simple but this solution technique is also valid for some more complicated geometries. As an example, material inhomogeneities can be considered for future studies.

REFERENCES

[1] Lee, V. W. "On deformations near circular underground cavity subjected to incident plane SH waves." *Proceedings of the Application of Computer Methods in Engineering Conference*. Vol. 2. Los Angeles, 1977.

[2] Sánchez-Sesma, Francisco J. "Diffraction of elastic SH waves by wedges."*Bulletin of the seismological society of America* 75.5 (1985): 1435-1446.

[3] Lee, V. W., and R. I. Sherif. "Diffraction around circular canyon in elastic wedge space by plane SH-waves." *Journal of engineering mechanics* 122.6 (1996): 539-544.

[4] Lin, Chi Hsin, et al. "Zero-stress, cylindrical wave functions around a circular underground tunnel in a flat, elastic half-space: Incident P-waves." *Soil Dynamics and Earthquake Engineering* 30.10 (2010): 879-894.

[5] Hasheminejad, Seyyed M., and Reza Avazmohammadi. "Harmonic wave diffraction by two circular cavities in a poroelastic formation." *Soil Dynamics and Earthquake Engineering* 27.1 (2007): 29-41.

[6] Abramowitz, Milton, and Irene A. Stegun. "Handbook of Mathematical Functions with Formulas, Graph, and Mathematical Tables." *Applied Mathematics Series* 55 (1965): 1046.

[7] Mow, Chao-Chow, and Yih-Hsing Pao. "The diffraction of elastic waves and dynamic stress concentrations." (1971).

[8] Ivanov, Ye A. "Diffraction of electromagnetic waves on two bodies." (1970).