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RELIABLE EVALUATION OF COMPUTATIONAL ERRORS IN DYNAMIC ANALYSIS OF PLATE STRUCTURES

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Abstract. *Considering the rapid progresses in the engineering world, evaluation of computational errors becomes more important everyday. In this paper, with attention to the importance of plates dynamic behaviours in structural, mechanical, and aeronautical engineering, a recent error evaluation method is briefly extended for application to computations with several algorithmic parameters and the performance of the resulting method is studied in application to dynamic analysis of plates. Besides the theoretical explanation, the numerical results reveal the good performance and reliability of the new error evaluation method.*

1 INTRODUCTION

In view of the complicatedness of recent engineering systems, approximate computational methods are in everyday improvement; and accordingly, also because of the lack of sufficient engineering intuition, study of computational errors is essential in engineering practice. In the area of structural engineering, specially, dynamic analysis of structural systems, the analyses are involved in approximations. To say better, in the conventional approach of structural dynamic analysis, the mathematical models are first being discretized in space, using methods, such as finite elements,

$$\begin{aligned}&\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{f}_{\text{int}} = \mathbf{f}_{ext} \qquad 0 \le t < t_{end}\\&\mathbf{u}(t=0) = \mathbf{u}_0 \;\;,\;\; \dot{\mathbf{u}}(t=0) = \dot{\mathbf{u}}_0 \;\;,\;\; \mathbf{f}_{\text{int}}(t=0) = \mathbf{f}_{\text{int}_0}\\&\mathbf{Q}\end{aligned} \tag{1}$$

($\mathbf{M}$ stands for the mass matrix, $\mathbf{u}$ denotes the displacement vector, each top dot implies once differentiation with respect to time, t and t_{end} respectively introduce the time, which is the independent variable, and the length of the time interval under consideration, $\mathbf{f}_{\text{int}}$ and $\mathbf{f}_{ext}$ respectively represent the internal and external forces, $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, and $\mathbf{f}_{\text{int}_0}$ respectively introduce the initial values of the displacements, velocities, internal forces, $\mathbf{Q}$ denotes the restrictions imposed to the problem because of nonlinearity, and finally the order of matrices and vectors in Eq. (1) are determined by the number of the degrees of freedom set in the discretization stage), and then being analyzed in time, by some time integration method [1-6]. Both of the above stages, i.e. discretization in space, and integration in time, are involved in computational errors [7, 8]. Accordingly, the resulting responses are inexact, and to be implemented in practice, should be studied and guaranteed, at least, for reliable upper bound estimations, of the errors. Different theoretical complicated and some simple practical methods exist, for he errors estimation, in the literature; see [7, 9, 10].

In view of the explanations above and the importance and complicatedness of structural systems with plates as components, specially in the special structures analysis and design, mechanical systems structures, aeronautical engineering, etc., attention, in this paper, is paid to dynamic analysis of plates and the computational errors evaluation by a recent practical method [10]. In the next section, the recent error evaluation method is briefly reviewed; in Section 3, implementation of the method in dynamic analysis of plates is discussed, after extending the application of the error evaluation method to computations with several algorithmic parameters; some numerical examples are studied later in Section 4; and finally, in Section 5, the paper is closed with a brief set of the conclusions.

2 THE RECENT ERROR EVALUATION METHOD

As implied in Section 1, the recent error evaluation method is applicable to a broad range of approximate computations. The key points/bases of the method are as stated below:

- The responses converge properly [11]. Convergence is the major essentiality of approximate computations [12, 13], displayed in Figure 1, where E stands for the error, defined below [14]:

$$E = \left\| \mathbf{R} - \mathbf{R}_{ex} \right\| \tag{2}$$

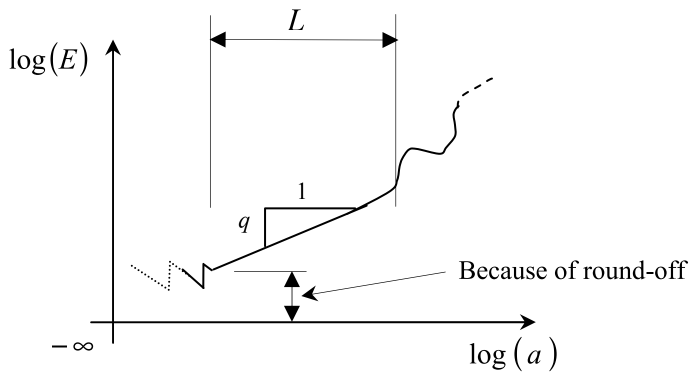

Figure 1: A typical convergence plot.

a is the algorithmic parameter (always nonzero) [15], $\mathbf{R}$ implies the computed response, $\mathbf{R}_{ex}$ is the exact response, $\| \ \|$ introduces an arbitrary norm [16], q denotes the rate of-convergence [17, 18], and L is a length differentiating proper and improper convergences,

$$\begin{array}{lll} L = 0 & \Leftrightarrow & \text{improper convergence} \\ L > 0 & \Leftrightarrow & \text{proper convergence} \end{array} \tag{3}$$

Proper convergence and being located on the line with slope q in Figure 1, is the case for many analyses, considering the practical comments, for assigning values, to the analyses algorithmic parameters, e.g. integration step size, elements sizes, etc.

- The fact that when implementing Richardson extrapolation [19, 20], to two responses, $\mathbf{R}_1$ and $\mathbf{R}_2$, properly converging with the rate q, and obtained, with a_1 and a_2 as the algorithmic parameter, the resultant, $\mathbf{R}_{1,2}$,

$$\mathbf{R}_{1,2} = \frac{a_1{}^q \, \mathbf{R}_2 - a_2{}^q \, \mathbf{R}_1}{a_1{}^q - a_2{}^q} \tag{4}$$

is an approximate response (Richardson extrapolation), converging faster, than the original responses; see Figure 2, where

$$q_R > q \tag{5}$$

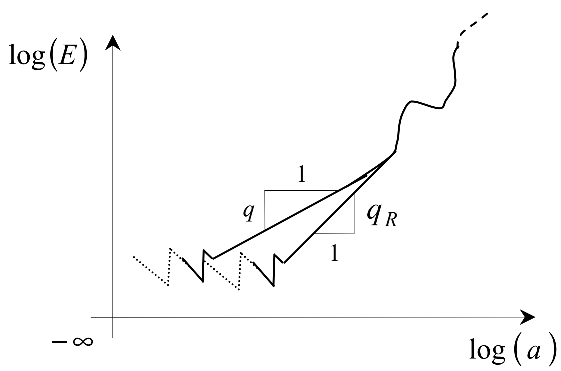

Figure 2: Typical convergence plots for converging responses and their Richardson extrapolations.

- The fact that for responses converging properly, the approximate responses, corresponding to different values of the algorithmic parameter, are either all larger or all smaller than the exact responses [21].
- And finally, the fact that, for approximate responses converging properly, if $\mathbf{R}_1$ is the response corresponding to a_1 and $\mathbf{R}_2$ is the response corresponding to a_2 ($a_1 \neq a_2$), the errors of $\mathbf{R}_1$ and $\mathbf{R}_2$ can be obtained from [10, 22]

$$\begin{aligned} E_{R_2} &\cong \frac{\|\mathbf{R}_2 - \mathbf{R}_1\|}{a_2{}^q - a_1{}^q} a_2{}^q \\ E_{R_1} &\cong \frac{\|\mathbf{R}_2 - \mathbf{R}_1\|}{a_2{}^q - a_1{}^q} a_1{}^q \end{aligned} \tag{6}$$

  based on which, for three responses $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$, corresponding to a_1, a_2, and a_3 ($a_1 \neq a_2 \neq a_3$), as the values of the algorithmic parameter, all corresponding to points on the line with slope q in Figure 1, such that

$$\frac{a_1}{a_2} = \frac{a_2}{a_3} = r > 1 \tag{7}$$

  the error of the most precise Richardson extrapolation, $\mathbf{R}_{2,3}$, can be upper estimated by [10]:

$$E_{R_{2,3}} \leq \frac{\left\| r^q (\mathbf{R}_2 - \mathbf{R}_3) - (\mathbf{R}_1 - \mathbf{R}_2) \right\|}{(r^q - 1)^2} \tag{8}$$

and the computational procedure is stated in [10].

The method is presented for approximate computations with one algorithmic parameter, and its extension to computations with more algorithmic parameters is briefly studied via plates errors analysis, in the next section, for the first time.

3 IMPLEMENTATION IN DYNAMIC ANALYSIS OF PLATES

Algorithmic parameters are parameters, additional to the parameters defining the mathematical problem, with a main role in approximate analyses such that in the limit of zero values, convergence to the exact responses is essential (see Figure 1) [12, 13], i.e.

$$\lim_{a \to 0} \mathbf{R} = \mathbf{R}_{ex} \tag{9}$$

As implied above, while essential in the numerical computations and specifically the computational errors [10], algorithmic parameters are independent, from the physics of the problem, under consideration, and have no role in the exact responses. For instance, solving algebraic equations by the bisection method [14, 23, 24] leads to numerical results depending on the number of iterations, while the number of iterations does not affect the exact solution; i.e. the iteration is not a part of the mathematical problem and model. Therefore, in view of Eq. (9), the inverse of the number of iterations can be considered as the algorithmic parameter of the bisection method. With this description of algorithmic parameter, the number of the algorithmic parameters of an approximate computation can be even more than one; equal to one for the bisection method. Specifically, in approximate analysis of structural dynamic problems, the number of algorithmic parameters is at least two; one controlling the errors originating in space discretization (leading to Eq. (1)), and the other controlling the errors because of time integration.

Considering that dynamic analysis of plates is a special case of structural dynamic analyses, in order to implement Eq. (8), in the plates' dynamic analyses error evaluation, an approach is to link the several algorithmic parameters, i.e. ${}_1a$, ${}_2a$, . . . , via

$$\frac{{}_1a}{{}_1\alpha} = \frac{{}_2a}{{}_2\alpha} = \cdots = a \tag{10}$$

where,

$${}_1\alpha,\ {}_2\alpha,\ {}_3\alpha \ldots \in R^+ \tag{11}$$

(R^+ is the set of positive real numbers) and consider a as the single algorithmic parameter of the problem. This provides the capability of implementing Eq. (8), in an approximate computation originally with several algorithmic parameters. Nevertheless, using conventional values of the algorithmic parameters to result in responses, with adequate accuracy and likely corresponding to points located on the line sloped q in the convergence plot (see Figures 1 and 2), would lose its notion. Consequently, repetition of the analyses is essential not only to arrive at the desired accuracy, but also to check the proper convergence; see the first issue in Section 2; and in this regard, attention to the equivalence between convergence and pseudo convergence plots [25], addressed in Figure 3, where, D_i (i.e. the pseudo error), is defined below:

$$D_{R_i} = \left\| \mathbf{R}_i - \mathbf{R}_{i-1} \right\| \tag{12}$$

is essential.

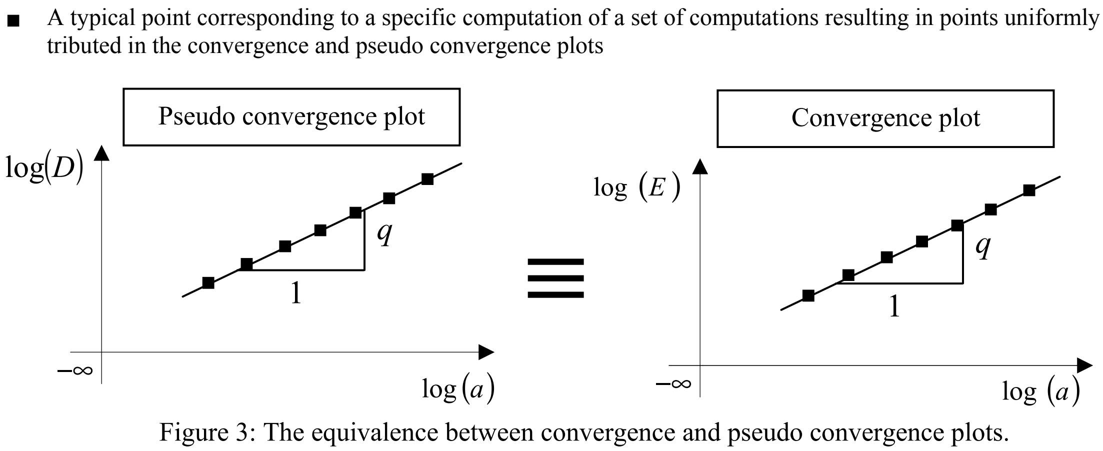

Figure 3: The equivalence between convergence and pseudo convergence plots.

4 ILLUSTRATIVE EXAMPLES

First, consider the rectangular plate (concrete wall) displayed in Figure 4, where the top edge is free and the other edges are simply supported. Considering the plate initially in the quiescent condition, i.e.

$$\mathbf{u}_0 = \dot{\mathbf{u}}_0 = \mathbf{f}_{\text{int}_0} = 0 \tag{13}$$

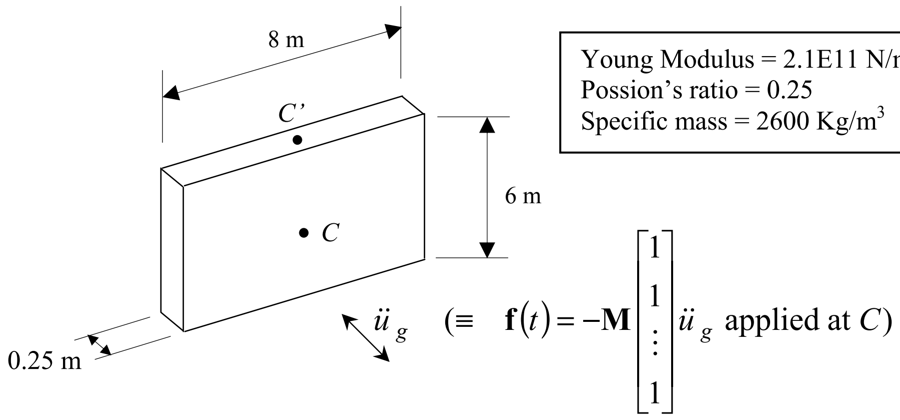

Figure 4: The plate under consideration in the first example.

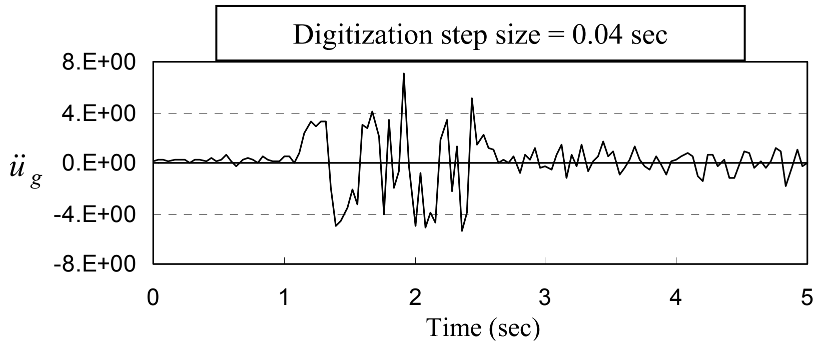

Figure 5: The strong ground motion considered as the excitation applied perpendicular to the plate.

the supports, of the plate, are excited, with the strong ground acceleration, $\ddot{u}_g$, introduced in Figure 5, in the direction perpendicular to the plate. To analyze the plate, the space discrtization is carried out by four-node plate elements [17, 26] and the time integration analysis is handled by means of the average acceleration method of Newmark [27], and for implementing the error evaluation method, the elements sides' lengths, l and w (the thickness is inherently considered in the formulation of the element), and the integration step size, Δt, are considered as the algorithmic parameters and, in view of Eq. (10), are linked by (see also the dimensions of the plate in Figure 4 and the digitization step size in Figure 5)

$$\frac{l}{4} = \frac{w}{3} = \frac{\Delta t}{4} = a \tag{14}$$

Starting from $a = 1$ and repeating the analysis several times, considering $r = 2$ (see Eq. (7)), leads to the error evaluations reported in Table 1, for the maximum displacement of the plate centre and maximum displacement of the middle of the top edge of the plate (both in the direction of the excitation). Comparing Table 1 with Table 2 (where, the exact errors are reported), clearly displays the reliability of the estimation, and hence, is an evidence for the good performance of the error evaluation method, briefly extended in this paper, when applied to plate structures.

As the second example, consider the rectangular plate, in Figure 6, simply supported at four edges and subjected to a 1000 Kgf force at its centre for 10 seconds. The average accel

A	Maximum displacement at C	Maximum displacement at C'
0.25	0.4378	2.707
0.125	0. 5733	0.2578
0.0625	0.0756	0.0600

Table 1: The upper-estimations for some errors in the first example (the right hand side of Eq. (8)).

a	Maximum displacement at C	Maximum displacement at C'
0.25	0.1830	1.5020
0.125	0.2267	0.1800
0.0625	0.0352	0.0420

Table 2: Some of the exact errors in the first example (the left hand side of Eq. (8)).

eration time integration method is implemented for analysis and

$$\frac{l}{5} = \frac{w}{3} = \frac{\Delta t}{4} = a \tag{15}$$

is considered as the relation linking the finite elements sizes to each other and the integration step size, where, a is the algorithmic parameter. Starting from $a = 2$, and sequentially halving a for new analyses result in the exact errors and the errors upper bound estimations for the maximum vertical displacement at the centre of the plate, as noted in Table 3, once again, demonstrating the good performance of the method extended in this paper.

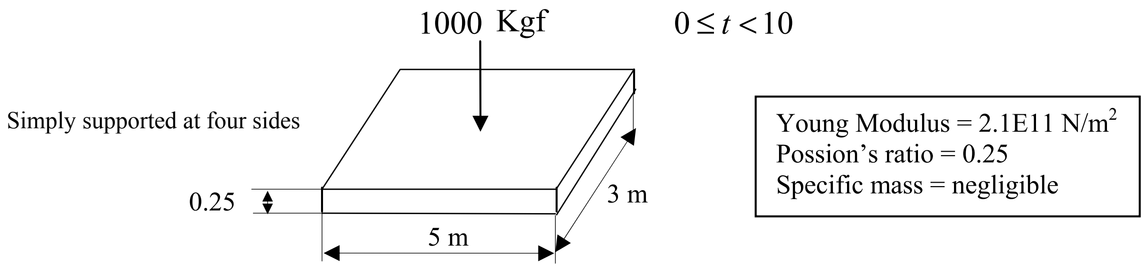

Figure 6: The plate under consideration in the second example.

a	Exact errors	The upper bound estimation of errors
0.5	1.280E-6	2.622E-6
0.25	3.299E-7	8.774E-7
0.125	5.187E-8	2.889E-7
0.0625	1.189E-8	9.018E-8

Table 3: The Upper-estimated and exact errors for the maximum vertical displacement at the centre of the plate in the second example.

5 CONCLUSIONS

Based on a recent error evaluation method proposed for general approximate computations, with one algorithmic parameter, a method applicable to approximate computations with several algorithmic parameters is briefly proposed and implemented in evaluating the responses of plates' dynamic analysis. Specifically,

- Under the assumption of proper convergence, the errors of approximate computations can now be reliably upper-estimated, regardless of the number of the algorithmic parameters.

- The performance of the generalized method proposed in this paper is adequate regarding plates structural dynamic analysis by finite elements and average acceleration time integration.

Further study, specifically regarding, application of the proposed generalization, to different plates structures, analyzed by different discretization and different time integration methods, and meanwhile optimized implementation of the proposed generalization is essential and being recommended here.

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