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USING CLENSHAW SUMMATION FOR RECURSIVE COMPUTATION OF HIGH ORDER AND DEGREE GEOPOTENTIAL FOR SPACE APPLICATIONS

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Abstract. *An artificial satellite subject to the attraction of the Earth is disturbed due to non-spherical distribution and non-symmetrical Earth mass. This uneven distribution of mass is expressed by the so-called spherical harmonic coefficients of the Earth potential. For a faster computation, the acceleration derived from the potential is obtained by a series expansion in terms of these harmonics, the fully normalized Legendre polynomials and their derivatives, and several recursions associated with the longitude, geocentric latitude and altitude of the center of mass of the satellite. This paper analyzes the detailed aspects of disturbances in artificial satellites, related with the modeling of the Earth's gravitational potential as well as numerical implementation of a recursive algorithm to calculate the acceleration of the geopotential based on the Clenshaw summation. In general, one uses recursive equations of high degree and order to calculate the Legendre polynomials in order to obtain faster processing and numerical accuracy. However, the recursions can yield numerical errors at each step of the recursion so that higher orders and degrees of harmonics, the accumulated error may be quite pronounced. The computational implementation of the algorithm is carried out by a PC computer. With the implementation of this algorithm it is possible to calculate the geopotential acceleration for different orbits and different situations. Such approach aims at mitigating the numerical problems arising from usage of extended series expansion when computing recursively the Legendre polynomials. Once the favorable numerical properties are proven, the algorithm can be used in the solution of practical problems of orbital space mechanics and for the Brazilian Space Mission.*

1. INTRODUCTION

The motion of artificial satellites orbiting Earth results in ellipse with constant size and eccentricity in a fixed plan. If the orbital motion was not disturbed, the satellite remains in that orbit. The main effects that change the orbit with time are inhomogeneous mass of Earth, atmospheric drag and gravitational perturbations from other bodies, notably the Moon and the Sun.

An artificial satellite subjected to the attraction of the Earth has perturbations due to non-spherical distribution and non symmetrical mass of the Earth. This irregular distribution of mass is expressed by the spherical harmonic coefficients of the potential Earth. For a faster calculation of the acceleration derived from the potential can be obtained by the series expansion in terms of these harmonics completely normalized and Legendre polynomials, their derivatives, and variants recursive associated with altitude, longitude, and geocentric latitude of the satellite's center of mass (CM).

The acceleration of the geopotential in a body, which has uniform distribution of mass and simple geometry, usually has an exact mathematical representation. However, for a body that has non-spherical distribution and non-symmetric mass, this acceleration can only be obtained with approaches expresses in the form of series, and the number of terms depends on the available data and indicates the degree approach.

The goal of this paper is to present a recursive algorithm to calculate the acceleration of the geopotential based on the sum of Clenshaw through the spherical harmonic coefficients, in order to obtain more accurate results for the trajectory of the satellite. The use of recursive formulas for calculating enables a fast computation and economy of storage of results. The use of normalized parameters allows a greater numerical accuracy, because they avoid the emergence of large numbers during calculations.

2. GEOPOTENTIAL

The potential of the body in relation with the distribution of non-spherical and non-symmetrical Earth mass can be expressed by the coefficients of the spherical harmonics and calculated in a generic form [1] by:

$$V = \frac{GM}{r}\sum_{n=0}^{\infty}\sum_{m=o}^{n}\left(\frac{a}{r}\right)^{n}\left[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right]P_n^m(\sin\varphi) \tag{1}$$

where V is the potential, G is the universal gravitational constant, M is the Earth mass, r is the distance to body from the Earth center, a is the Earth equatorial radius, λ and φ are the longitude and latitude of body, respectively, $P^n{}_m$ are the Legendre polynomials of order n and degree m and C_{nm} and S_{nm} are the spherical harmonics coefficients.

2.1. The spherical harmonics coefficients.

According to Eq. (1), and the coefficients C_{nm} and S_{nm} should be expressed as closely as possible to the irregular shape of the Earth, so that the potential of the model is represented properly. The determination of these coefficients is obtained experimentally by reduction and data analysis of artificial satellites and also from gravimetric methods. The adopted values in this model correspond to the model called GEM10 - Goddard Earth Model 10 [2]. The GEM10 contains the coefficients of the harmonics up to degree and order 30 and uses the value of $3.9860047\ x10^{14} m^3/s^2$ for *GM* parameters and *6378139m* for the equatorial radius of the Earth .

The harmonics coefficients are listed in its fully normalized form and the relation with the non-normalized ones are:

$$\begin{pmatrix} \overline{C}_{nm} \\ \overline{S}_{nm} \end{pmatrix} = \left[\frac{1}{\varepsilon_m (2n+1)} \frac{(n+m)!}{(n-m)!} \right]^{1/2} \begin{pmatrix} C_{nm} \\ S_{nm} \end{pmatrix} \tag{2}$$

where

$$\varepsilon_m = \begin{cases} 1 & \text{if } m = 0 \\ 2 & \text{if } m \geq 0 \end{cases}$$

and $\overline{C}$ e $\overline{S}$ correspond to the fully normalized coefficients. The *C* coefficients with m = 0 are called *zonal* coefficients and represented by $J_n = -C_{n0}$. The *S* coefficients with $m=0$ are null, i.e. $S_{n0} = 0$. The remaining coefficients *C* and *S* (those for $m \geq 1$) are called *sectoral* coeffcients when $m = n$ and *tesseral* coeffients when $m \neq n$.

Therefore, in this case $m \geq 1$, the Eq. (2) can be deployed as:

$$\overline{J}_n = \left[\frac{1}{2n+1} \right]^{1/2} J_n, \tag{3}$$

$$\begin{pmatrix} \overline{C}_{nm} \\ \overline{S}_{nm} \end{pmatrix} = \left[\frac{1}{4n+2} \frac{(n+m)!}{(n-m)!} \right]^{1/2} \begin{pmatrix} C_{nm} \\ S_{nm} \end{pmatrix}, \tag{4}$$

where $\overline{J}_n$ correspond to the fully normalized *zonal* coefficients J_n, and $\overline{C}_{nm}$ and $\overline{S}_{nm}$ correspond to the fully normalized *tesseral* coefficients C_{nm} e S_{nm}.

2.2. Fully normalized Legendre polynomial recursion

When one is considered a fully normalized harmonic coefficients, the associated Legendre polynomial P_n^m should also be fully normalized, so that Eq. (1) modeling the geopotential is compatible. Thus it takes the following form [2]:

$$V = \frac{GM}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r} \right)^n \left[\overline{C}_{nm} \cos m\lambda + \overline{S}_{nm} \sin m\lambda \right] \overline{P}_n^m (\sin \varphi) \tag{5}$$

For the computation of the fully normalized Legendre polynomials it was used the forward columns method described in Holmes and Featherstone [3]. This recursion is the most used for computing $\overline{P}_{nm}(\varphi)$ and it is described below:

$$\overline{P}_{nm}(\varphi) = g_{nm} t \overline{P}_{n-1,m}(\varphi) - h_{nm} \overline{P}_{n-2,m}(\varphi), \qquad \forall n > m \tag{6}$$

where $t = \cos\varphi$ and

$$g_{nm} = \sqrt{\frac{(2n-1)(2n+1)}{(n-m)(n+m)}} \qquad \text{and} \qquad h_{nm} = \sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(n-m)(n+m)(2n-3)}} \tag{7}$$

When $n = m$, the recursion is described by:

$$\overline{P}_{mm}(\varphi) = u\sqrt{\frac{2m+1}{2m}}\overline{P}_{m-1,m-1}(\varphi), \qquad \forall m > 1 \tag{8}$$

With $u = \sin\varphi$ and the initial values are $\overline{P}_{0,0}(\varphi) = 1$, $\overline{P}_{1,0}(\varphi) = \sqrt{3}t$ e $\overline{P}_{1,1}(\varphi) = \sqrt{3}u$.

Computationally the transformation produces better numerical accuracy since, after normalization, $\overline{C}$, $\overline{S}$ and $\overline{P}$ have better numerically conditioned values to perform calculations and do not introduce factorials. Such calculations can produce values that make computational representation problematic (large positive or negative expoents, prone to overflow/underflows, and therefore susceptible to numerical errors.

3. CRENSHAW METHOD

In order to compute the acceleration derived from the potential it is used the Clenhaw method. The geopotencial V , given by Eq. (5) can be put in a generic form [4]:

$$V = \frac{GM}{r}\sum_{m=o}^{N}\left(\upsilon_m^{(1)}\cos m\lambda + \upsilon_m^{(2)}\sin m\lambda\right), \tag{9}$$

Where N is the higher degree of the spherical harmonic expansion, with $\upsilon_m^{(1)}$ and $\upsilon_m^{(2)}$ are defined as:

$$\upsilon_m^{(i)} = \sum_{n=\mu}^{N} c_n^{(i)}\overline{p}_n \text{ , i = 1,2,} \tag{10}$$

with μ equal 2 or *m*, which is bigger and:

$$\overline{p}_n = \left(\frac{a}{r}\right)^n \overline{P}_n^m(\varphi) \qquad \text{and} \qquad c_n^{(i)} = \begin{cases} \overline{C}_{nm} \to i = 1, \\ \overline{S}_{nm} \to i = 2. \end{cases} \tag{11}$$

For simplification the index *m* is supplied in the functions $\overline{p}_n$ and $\overline{c}_n^{(i)}$.

Then the recursion formula to compute the fully normalized can be express as:

$$\overline{P}_n^m(u) = ug_{nm}\overline{P}_{n-1}^m(u) - h_{nm}\overline{P}_{n-2}^m(u), \tag{12}$$

with $0 \le m \le n-1$, n = 1,2,...,N,

$$g_{n,m} = \sqrt{\frac{(2n+1)(2n-1)}{(n+m)(n-m)}} \quad \text{and} \quad h_{n,m} = \sqrt{\frac{(2n+1)(n-m-1)(n+m-1)}{(2n-3)(n+m)(n-m)}} \tag{13}$$

Multiply the Eq.(12) by q^n, with $q = \frac{a}{r}$, it is possible to get a simplified form:

$$\overline{p}_n - \alpha_{nm}\overline{p}_{n-1} + \beta_{nm}\overline{p}_{n-2} = 0 \tag{14}$$

where $\alpha_n = uqg_{nm}$, $\beta_n = q^2 h_{nm}$, the index *m* is also supplied and the initial values are given by $\overline{p}_1 = \alpha_1$ and $\overline{p}_0 = 1$.

The Eq. (20) is the recursion formula to be used in the algorithm for the Clenshaw Sums given by Eq. (10). For each $i = 1,2$, there is a set of coefficients $y_n^{(i)}$:

$$y_{N+2}^{(i)} = 0, \qquad y_{N+1}^{(i)} = 0, \qquad y_N^{(i)} = c_N^{(i)}, \quad y_{N-1}^{(i)} = c_{N-1}^{(i)} + \alpha_N y_N^{(i)} \tag{15}$$

$$\ldots$$

$$y_k^{(i)} = c_k^{(i)} + \alpha_{k+1} y_{k+1}^{(i)} - \beta_{k+2} y_{k+2}^{(i)}, \quad y_{k-1}^{(i)} = c_{k-1}^{(i)} + \alpha_k y_k^{(i)} - \beta_{k+1} y_{k+1}^{(i)}$$

$$\ldots$$

$$y_m^{(i)} = c_m^{(i)} + \alpha_{m+1} y_{m+1}^{(i)} - \beta_{m+2} y_{m+2}^{(i)}$$

That is valid for $m \le n \le N+2$.

Appling the Clenshaw recursive formula [3], it is possible to get [4]:

$$\upsilon_m^{(i)} = y_m^{(i)} \bar{p}_m + y_{m+1}^{(i)} \left(\bar{p}_{m+1} - \alpha_{m+1} \bar{p}_m \right) \tag{16}$$

And after using Eq. (14) it is expressed by:

$$\upsilon_m^{(i)} = y_m^{(i)} \bar{p}_m - \beta_{m+1} y_{m+1}^{(i)} \bar{p}_{m-1}. \tag{17}$$

However $\bar{p}_{m-1} = \bar{P}_{m-1}^{m} \equiv 0$, and Eq. (17) is reduced to:

$$\upsilon_m^{(i)} = y_m^{(i)} \bar{p}_m, \; i = 1,2. \tag{18}$$

Then the potential, given by Eq. (9), is given by [3]:

$$V = \frac{GM}{r} \sum_{m=o}^{N} \left(y_m^{(1)} \cos m\lambda + y_m^{(2)} \sin m\lambda \right) \bar{p}_m. \tag{19}$$

The Eq. (19) shows that only the setoral terms of $\bar{p}_m$ can be calculed and storafged. For $n = m$:

$$\bar{p}_m = f_m \sqrt{1-u^2}\, q \bar{p}_{m-1}, \tag{20}$$

where $f_1 = \sqrt{3}$ and the others coefficients are defined by [5]:

$$f_m = \sqrt{\frac{2m+1}{2m}}, \; m > 1 \tag{21}$$

with initial value $\bar{p}_0 = 1$.

4. ACCELERATION DERIVED FROM THE POTENTIAL

The acceleration derived from the potential V in a spherical coordinates is obtained through:

$$\nabla V = \frac{\partial V}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial V}{\partial \varphi} \hat{e}_\varphi + \frac{1}{r \sin\varphi} \frac{\partial V}{\partial \lambda} \hat{e}_\lambda \tag{22}$$

with the partial derivates given by [4]:

$$\frac{\partial V}{\partial \lambda} = -\frac{GM}{r} \sum_{m=0}^{N} m \left(\upsilon_m^{(1)} \sin m\lambda - \upsilon_m^{(2)} \cos m\lambda \right) \tag{23}$$

$$\frac{\partial V}{\partial \varphi} = \frac{GM \cos\varphi}{r} \sum_{m=0}^{N} \left(\upsilon_{m,u}^{(1)} \cos m\lambda + \upsilon_{m,u}^{(2)} \sin m\lambda \right) \tag{24}$$

$$\frac{\partial V}{\partial r} = -\frac{GM}{r^2} \sum_{m=0}^{N} \left(\upsilon_{m,r}^{(1)} \cos m\lambda + \upsilon_{m,r}^{(2)} \sin m\lambda \right) \tag{25}$$

where

$$\upsilon_m^{(i)} = y_m^{(i)} p_m, \qquad \upsilon_{m,u}^{(i)} = y_m^{(i)} \frac{dp_m}{du} + y_{m,u}^{(i)} p_m, \qquad \upsilon_{m,r}^{(i)} = y_{m,r}^{(i)} p_m, \; i = 1,2. \tag{26}$$

$$\frac{dp_m}{du} = \left[\left(1-u^2\right)\frac{dp_{m-1}}{du} - up_{m-1} \right] \frac{q\, f_m}{\left(1-u^2\right)^{1/2}} \tag{27}$$

$$y_{m,u}^{(i)} = uqg_{n+1,m} y_{m+1,u} + qg_{n+1,m} y_{m+1}^{(i)} - q^2 h_{n+2,m} y_{m+2,u}^{(i)}, \; i = 1,2. \tag{28}$$

$$y_{m,r}^{(i)} = c_m^{(i)}(m+1) + uqg_{n+1,m} y_{m+1,r} - q^2 h_{n+2,m} y_{m+2,r}^{(i)}, \; i = 1,2. \tag{29}$$

and the initial values for the recursion are $y_{N+2,u}^{(i)} = y_{N+1,u}^{(i)} = 0$, $y_{N+1}^{(i)} = 0$ e $y_{N+2,r}^{(i)} = y_{N+2,r}^{(i)} = 0$.

It is also possible to apply recursion formulas for trigonometric functions in order to avoid the explicit evolution of $\cos m\lambda$ and $\sin m\lambda$ [2]:

$$\cos(m\lambda) = \cos(m-1)\lambda \cos\lambda - \sin(m-1)\lambda \sin\lambda \tag{30}$$

$$\sin(m\lambda) = \sin(m-1)\lambda \cos\lambda + \cos(m-1)\lambda \sin\lambda \tag{31}$$

4.1. Acceleration in a body system

In general the acceleration derived from the potential are represented in cartesian co-ordenates (x_b, y_b, z_b) associated with the body . However, the application of the Clenshaw methods produces gradints in spherical coordenates. Then it is necessary to do the following coordenates transformation:

$$\ddot{x} = \frac{\partial V}{\partial x} = \left(\frac{1}{r}\frac{\partial V}{\partial r} - \frac{z_b}{r^2\sqrt{x_b^2 + y_b^2}}\frac{\partial V}{\partial \varphi} \right) x_b - \left(\frac{1}{\left(x_b^2 + y_b^2\right)}\frac{\partial V}{\partial \lambda} \right) y_b \tag{32}$$

$$\ddot{y} = \frac{\partial V}{\partial y} = \left(\frac{1}{r}\frac{\partial V}{\partial r} - \frac{z_b}{r^2\sqrt{x_b^2 + y_b^2}}\frac{\partial V}{\partial \varphi} \right) y_b + \left(\frac{1}{\left(x_b^2 + y_b^2\right)}\frac{\partial V}{\partial \lambda} \right) x_b \tag{33}$$

$$\ddot{z} = \frac{\partial V}{\partial z} = \left(\frac{1}{r}\frac{\partial V}{\partial r} \right) z_b + \frac{\sqrt{x_b^2 + y_b^2}}{r^2}\frac{\partial V}{\partial \varphi} \tag{34}$$

5. CLENSHAW RECURSION FORMULA

The Clenshaw recursion formula is an efficient way to solve the sum of functions that can assume the form [4]:

$$f(x) = \sum_{k=0}^{N} c_k F_k(x) \tag{35}$$

with F_k given by the recursive formula:

$$F_{n+1}(x) = \alpha(n,x) F_n(x) + \beta(n,x) F_{n-1}(x) \tag{36}$$

for some functions $\alpha(n,x)$ and $\beta(n,x)$.

Using now the recursive formula for $k = N, N\text{-}1, \ldots, 1$ [4]:

$$y_{N+2} = y_{N+1} = 0 \tag{37}$$

$$y_k = \alpha(k,x)y_{k+1} + \beta(k+1,x)y_{k+2} + c_k$$

If the eq. (37) is solved for c_k, then the Eq. (35) can be expressed by:

$$\begin{aligned} f(x) = \cdots &+ [y_8 - \alpha(8,x)y_9 - \beta(9,x)y_{10}]F_8(x) + [y_7 - \alpha(7,x)y_8 - \beta(8,x)y_9]F_7(x) \\ &+ [y_6 - \alpha(6,x)y_7 - \beta(7,x)y_8]F_6(x) + [y_5 - \alpha(5,x)y_6 - \beta(6,x)y_7]F_5(x) \\ &+ \cdots + [y_1 - \alpha(1,x)y_2 - \beta(2,x)y_3]F_1(x) + [c_0 + \beta(1,x)y_2 - \beta(1,x)y_2]F_0(x) \end{aligned} \quad (38)$$

After some considerations about Eq. (38) it is possible to get [4]:

$$f(x) = \beta(1,x)F_0(x)y_2 + F_1(x)y_1 + F_0(x)c_0 \quad (39)$$

5.1. APLICATION FOR THE GEOPOTENTIAL

Assuming $m = 2$ and $2 \le n \le 5$ in a Eq. (12), it is possible to obtain [3]:

$$v_2^{(i)} = \sum_{n=2}^{5} c_n^{(i)} p_n = c_2^{(i)} p_2 + c_3^{(i)} p_3 + c_4^{(i)} p_4 + c_5^{(i)} p_5 \quad (40)$$

By the recursion of Eq. (15) it is possible to get the terms p_n of the Eq. (40):

$$p_3 = \alpha_{32} p_2 \quad ; \; p_4 = \alpha_{42} p_3 - \beta_{42} p_2 ; \quad p_5 = \alpha_{52} p_4 - \beta_{52} p_3 \; . \quad (41)$$

After some manipulations of the Eqs. (41), it is possible to get:

$$p_5 = (\alpha_{52}\alpha_{42}\alpha_{32} - \alpha_{52}\beta_{42} - \alpha_{32}\beta_{52}) p_2 \quad (42)$$

Then each term p_n can be expressed in function of the sectoral term p_2 ($n = m$) and the recursion terms α_{nm} and β_{nm} .

6. SOME RESULTS FOR CBERS SATELLITE

The numerical program was developed in C- Code and carried out by a PC computer [6]. Applications are done using the data of satellite CBERS, which has slope 98.504 °, 778 km altitude and period of 100.26 min.

The main function of the algorithm [5] initially loads the data C_{nm} and S_{nm} from gravity model, given by EGM96, and also loads the values of x, y and z of CBERS orbit. After loading the data, the numerical program asks the user to enter the value of N (degree of spherical harmonic expansion), with the maximum value of *N* equals 360. After storing these data, the program calls the function *gradV* that calculates and returns the values of the partial derivatives of the first order. With these calculated values, the main function performs the transformation of the spherical coordinates gradients for acceleration in Cartesian coordinates, using Eqs. (32-34).

The function *gradV* of the algorithm [5] computes the first order partial derivatives of the potential with respect to the longitude (λ), latitude (φ) and distance (r). First this function estimates, within a loop, the values of the terms dp_m/du from Eqs. (21) and (27), and the terms p_m from Eqs (20-21) and stores these values in two vectors. Then, the values of $y_m^{(i)}$, $y_{m,u}^{(i)}$,$y_{m,r}^{(i)}$, $i=1,2$, are calculated in two loops, one ranging from m = 0 to m = N-1 and another ranging from n = N to n = m, using Eqs. (15),(29) and (30). With these values it is estimated, $v_m^{(i)}$, $v_{m,u}^{(i)}$, $v_{m,r}^{(i)}$, $i=1,2$, using Eq. (26) for each m value. Finally, when m = N-1, the first derivates of the geopotential are calculated by Eqs. (23-25). For the calculations of $y_m^{(i)}$, $y_{m,u}^{(i)}$,$y_{m,r}^{(i)}$, $i=1,2$, the loop is reversed, i e, *n* ranges from *N* to *0*, and the initial conditions are

$$y^{(1)}_{N+2} = y^{(2)}_{N+2} = 0\,,\ y^{(1)}_{N+1} = y^{(2)}_{N+1} = 0\,,\ y^{(1)}_{N+2,u} = y^{(2)}_{N+2,u} = 0\,,$$

$$y^{(1)}_{N+1,u} = y^{(2)}_{N+1,u} = 0 \quad y^{(i)}_{N+2,r} = y^{(i)}_{N+2,r} = 0\,.$$

Several tests [5] were performed with different values of N, different situations and orbits in order to prove the numerical properties of the program.

In Figure 1 and Figure 2 are represented the differences between the values of the accelerations on x-axis, y-axis and z-axis for N from 2 to 32 and the reference values with N=360.It is possible to notice that the increase of the N values the diference is smaller. When N is close to 360, this diference is around 10^{-17}, which makes sure the numeric propriety of the algorithm.

In order to compare the acceleration results between the proposed method with N=360 and the the values computed by Satellite Control Center of INPE with N=30, the Figure 3 and Figure 4 show the diference beetween these two values.

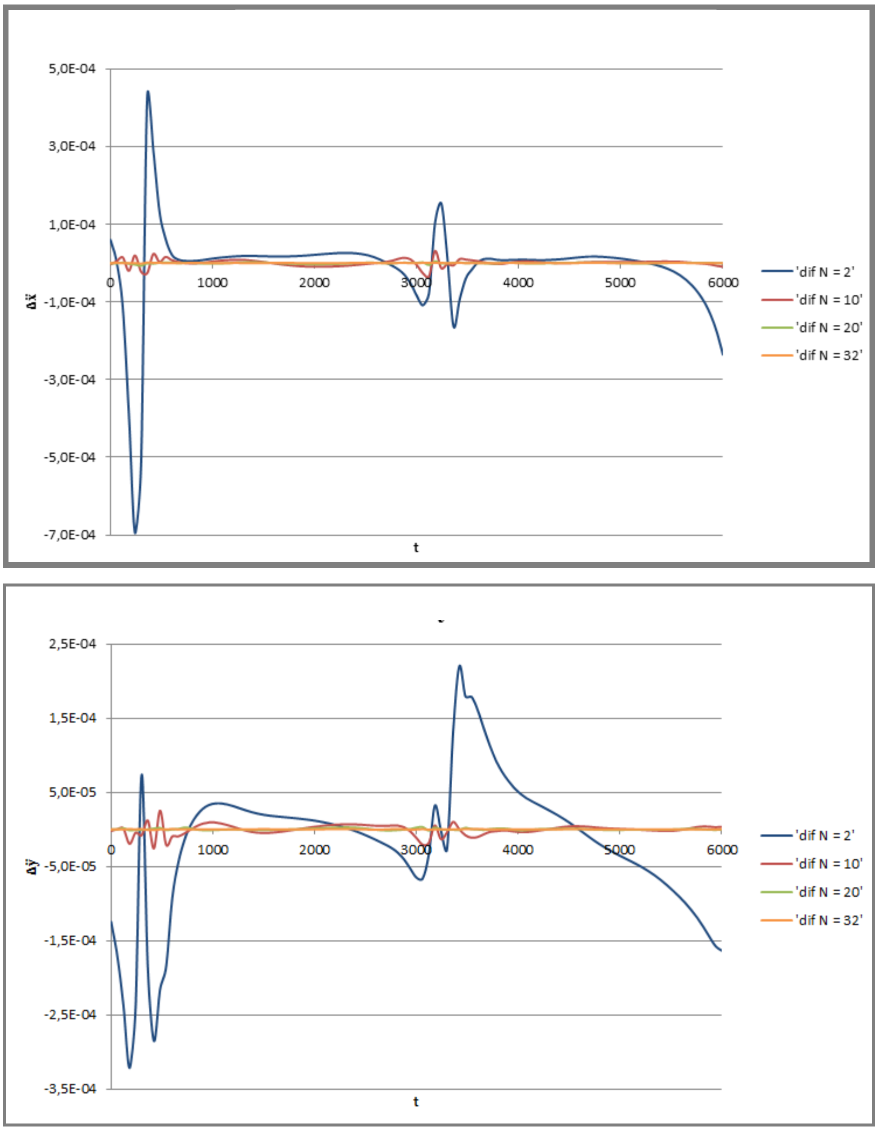

Figure 1 – Temporal variation for the diference of the acceleration components in x-axis and y- axis.

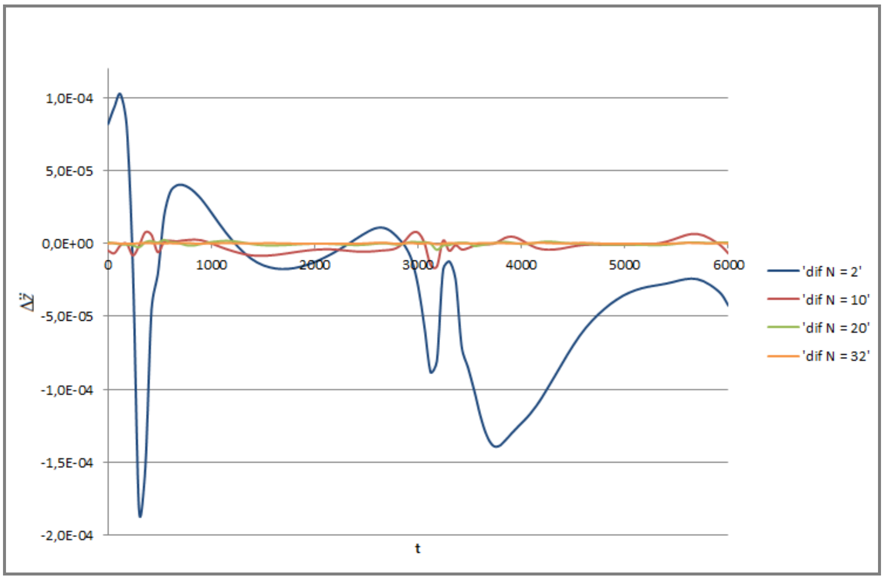

Figure 2 – Temporal variation for the diference of the acceleration z-axis component.

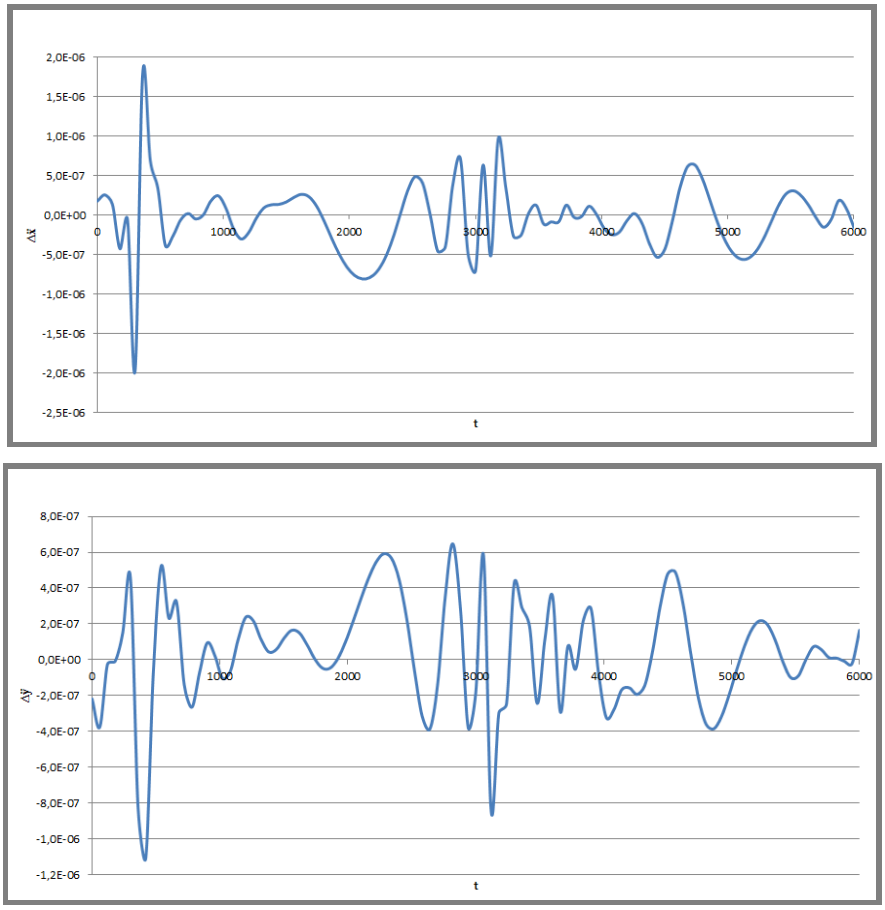

Figure 3 – Comparision between x-axis and y-axis components acceleration for N = 30 and N=360.

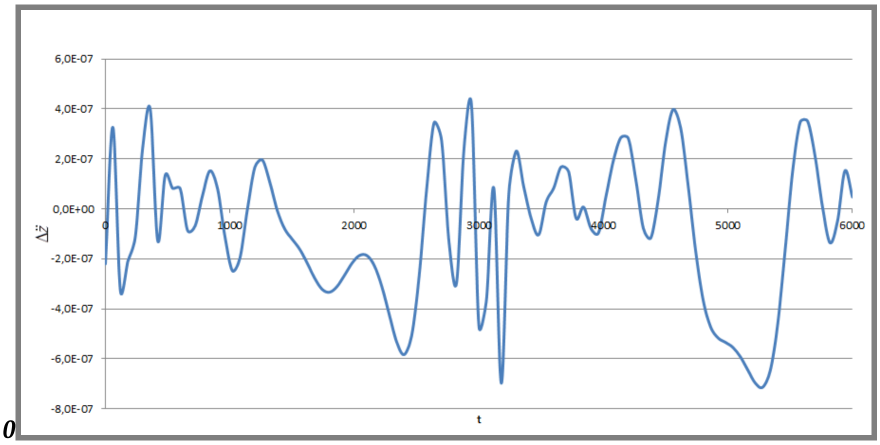

Figure 4 – Comparision between z-axis component acceleration for N = 30 and N=360.

6. CONCLUSIONS

This paper shows some mathematical details of the implemented geopotential computation and the acceleration derived from geopotential. The proposed method for calculating the acceleration is based on the sum of the geopotential Clenshaw and has a significant difference during the calculation. Traditional methods used for this calculation consider the first terms of the spherical harmonic coefficients (C_{nm} and S_{nm}), therefore, the latter terms, which are smaller, end up not being considered, causing a difference in the outcome. In the method proposed in this paper, the calculation uses initially the last terms of the coefficients of spherical harmonics, ensuring that they will be properly considered and that the end result will be closer to the real value.

Comparisons are made with the values obtained by the Satellite Center Control of INPE, which uses $N=30$, and the values obtained by the proposed method with $N = 360$. Through the favorable results obtained in this study, the algorithm can be used in the solution of practical problems of orbital space mechanics and for the Brazilian Space Mission.

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