11th International Conference on Vibration Problems
Z. Dimitrovová *et al.* (eds.)
Lisbon, Portugal, 9-12 September 2013

CHAOTIC AND REGULAR VIBRATIONS IN DISCRETE-CONTINUOUS SYSTEMS TORSIONALLY DEFORMED WITH LOCAL NONLINEARITIES

Amalia Pielorz*[1], Danuta Sado[2]

[1]Kielce University of Technology, Poland
apielorz@tu.kielce.pl

[2]Warsaw University of Technology, Poland
dsa@simr.pw.edu.pl

Keywords: Nonlinear Dynamics, Regular Vibrations, Chaotic Vibrations, Discrete-Continuous Systems, Wave Approach.

Abstract. *The paper deals with nonlinear vibrations in discrete-continuous mechanical systems consisting of rigid bodies connected by shafts torsionally deformed with local nonlinearities having hard and soft characteristics. The systems are loaded by an external moment harmonically changing in time. In the study the wave approach is used leading to solving equations with a retarded argument. Numerical results are presented for three-mass systems. In the study of regular vibrations in the case of a hard characteristic amplitude jumps are observed while in the case of a soft characteristic an escape phenomenon is observed. Irregular vibrations, including chaotic motions, can occur for selected parameters. The possibilities of occurring of such vibrations are discussed on the basis of tools presented in [5-7].*

1 INTRODUCTION

Regular nonlinear vibrations in discrete-continuous mechanical systems are considered e.g. in [1-3]. Such systems consist of rigid bodies connected by elastic elements. Damping in the systems is taken into account by means of an equivalent damping located in selected cross-sections. In systems torsionally deformed, to the first rigid body a discrete element is attached. The spring in this element has nonlinear characteristics described by the polynomial of the third degree. The behavior of the systems in the case of the both types of characteristics, hard and soft, is discussed. The systems are loaded by an external moment represented by the function harmonically changing in time. In the considerations the wave approach is used leading to solving equations with a retarded argument.

Irregular vibrations, including chaos, are studied in the technical literature mainly for discrete models, [4-8]. In [9, 10] some attempts are done for the investigation of irregular vibrations in discrete-continuous systems after the adaptation the approach used in [5-7]. The aim of the present paper is to compare the results presented in [9, 10] for the both types of the characteristics of local nonlinearities in multi-mass systems torsionally deformed.

Exemplary numerical results are presented for three-mass systems. It will be shown that in the case of a hard characteristic amplitude jumps are observed while in the case of a soft characteristic an escape phenomenon is observed. These effects concern regular vibrations with large damping. Irregular and chaotic vibrations can occur for selected parameters. The possibilities of occurring of such vibrations are discussed on the basis of the Poincaré maps, and the maximal exponents of Lyapunov.

2 ASSUMPTIONS AND GOVERNING EQUATIONS

Consider a multi-mass discrete-continuous system consisting of N shafts connected by rigid bodies. The *i*-th shaft, i = 1,2,...,N, is characterized by length l_i, density ρ, shear modulus G and polar moment of inertia I_{0i}, [1, 2]. The mass moment of inertia of rigid bodies, i = 1,2,…N+1, are J_i. The first rigid body J_1 is loaded by the moment $M(t) = M_0 \sin pt$, where M_0 and p are the amplitude and frequency of the external moment, correspondingly. Equivalent external and internal damping, having coefficients d_i and D_i, are taking into account in appropriate cross-sections. The x-axis is parallel to the shafts axis and its beginning corresponds to the cross-sections where the first rigid body is located. The function $\theta_i(x,t)$ represents angular displacements in the *i*-shaft. It is assumed that displacements and velocities of the shaft cross-sections are equal to zero at time instant t = 0.

A nonlinear discrete element is located in the cross-section x = 0. The moment of forces acting in the nonlinear spring is described by the polynomial of the third degree

$$M_{sp} = k_1\theta_1 + k_3\theta_1^3 \tag{1}$$

where k_i are parameters standing by linear and nonlinear terms, correspondingly. The local nonlinearity can have the characteristic of a hard type or of a soft type.

It is convenient to introduce the following nondimensionless quantities

$$\begin{gathered}\bar{x} = x/l_1, \;\; \bar{t} = ct/l_1, \;\; \bar{\theta}_i = \theta_i/\theta_0, \;\; \bar{d}_i = d_i l_1/(J_1 c), \;\; \bar{D}_i = D_i c/l_1, \\ \bar{k}_1 = k_1 l_1^2/(J_1 c^2), \;\; \bar{k}_3 = k_3\theta_0^2 l_1^2/(J_1 c^2), \;\; K_r = I_{01}\rho l_1/J_1, \;\; E_i = J_1/J_i, \\ \bar{M} = M l_1^2/(J_1 c^2\theta_0), \;\; \bar{M}_{sp} = M_{sp} l_1^2/(J_1 c^2\theta_0), \;\; \bar{l}_i = l_i/l_1, \;\; B_i = I_{0i}/I_{01},\end{gathered} \tag{2}$$

where $c = (G/\rho)^{1/2}$ is the speed of the torsional wave in shafts. Then, the determination of angular displacements and velocities leads to solving N equations of motion

$$\theta_{i,tt} - \theta_{i,xx} = 0, \quad i = 1,2,...,N \tag{3}$$

with zero initial conditions and with boundary conditions in appropriate cross-sections

$$\begin{aligned}
&M(t) - \theta_{1,tt} + K_r(D_1\theta_{1,xt} + \theta_{1,x}) - d_1\theta_{1,t} - M_{sp}(t) = 0 \quad \text{for} \quad x = 0, \\
&\theta_i(x,t) = \theta_{i+1}(x,t) \quad \text{for} \quad x = \sum_{k=1}^{i} l_k, \quad i = 1,2,...,N-1, \\
&-\theta_{i,tt} - K_r B_i E_{i+1}(D_i\theta_{i,xt} + \theta_{i,x}) + K_r B_{i+1} E_{i+1}(D_{i+1}\theta_{i+1,xt} + \theta_{i+1,x}) - E_{i+1} d_{i+1}\theta_{i,t} = 0 \\
&\qquad\qquad \text{for} \quad x = \sum_{k=1}^{i} l_k, \quad i = 1,2,...,N-1, \\
&-\theta_{N,tt} - K_r B_N E_{N+1}(D_N\theta_{N,xt} + \theta_{N,x}) - E_{N+1} d_{N+1}\theta_{N,t} = 0 \quad \text{for} \quad x = \sum_{k=1}^{N} l_k
\end{aligned} \tag{4}$$

where comma denotes partial differentiation.

Eqs. (3) are solving using the wave approach, i.e., looking for the solutions in the form

$$\theta_i(x,t) = f_i(t-x) + g_i(t + x - 2\sum_{k=1}^{i-1} l_k), \quad i = 1,2,...,N \tag{5}$$

where the functions f_i and g_i represent the waves caused by the external loading $M(t)$ propagating in the *i*-th shaft in the positive and negative senses of the x-axis, respectively. These functions are continuous and equal to zero for negative arguments.

Substituting (5) into the boundary conditions (4) nonlinear ordinary differential equations with a retarded argument for the functions f_i and g_i are obtained. They can be found in [1, 2].

3 NUMERICAL CALCULATIONS

Exemplary investigations are performed for a three-mass system, shown in Figure 1, assuming the following nondimensionless parameters

$$K_r = 0.05, \quad k_1 = 0.05, \quad N = 2, \quad l_1 = l_2 = 1, \quad E_2 = E_3 = 0.8, \quad B_2 = 1. \tag{6}$$

The damping parameters $d_0 = d_i = D_i$, the parameter k_3 and the amplitude M_0 of the external loading $M(t)$ can vary. The three first natural frequency for linear the system are then $\omega_1 = 0.089$, $\omega_2 = 0.261$ and $\omega_3 = 0.376$.

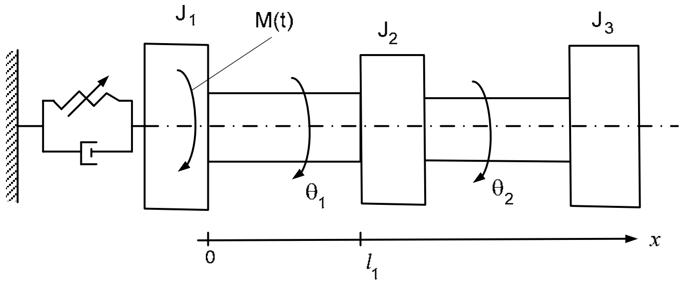

Figure 1: Three-mass torsional system.

Regular vibrations were studied in [1, 2] for the both types of nonlinear characteristic (1), i.e. with $k_3 = 0.005$ and $M_0 = 1$ for the hard characteristic case in [1] while with $k_3 = -0.005$ and $M_0 = 0.1$ for the soft characteristic case in [2]. It appears that in the case of a hard characteristic amplitude jumps are observed while in the remaining case the escape phenomenon occurs, [1, 2, 11]. The examples of these effects for amplitude-frequency curves are shown in Figure 2 for several values of the amplitudes M_0 and with damping coefficients $d_0 = 0.1$.

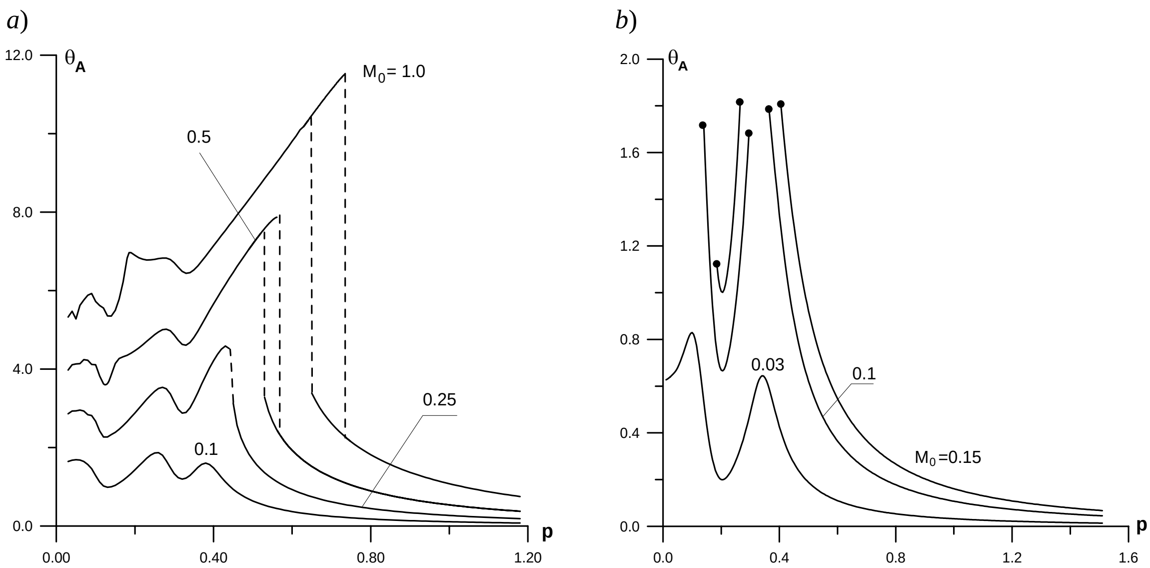

Figure 2: Amplitude-frequency curves for angular displacements: *a*) a hard characteristic case, *b*) a soft characteristic case.

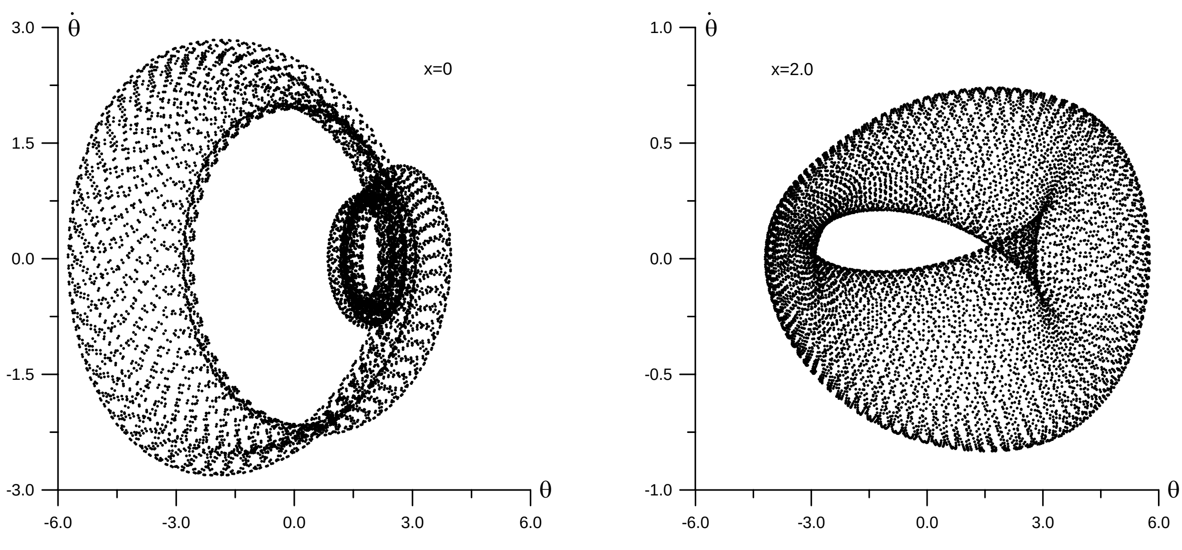

Figure 3: Poincaré maps for $p = 0.83$, cross-sections $x = 0, 2$ and hard characteristic case.

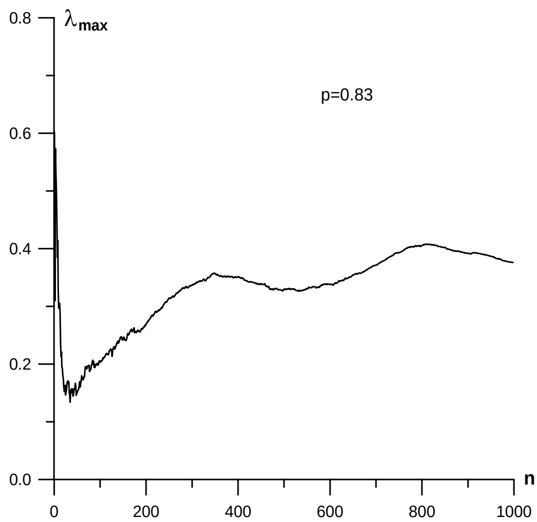

Figure 4: Maximal Lyapunov exponents. for $p = 0.83$, cross-section $x = 0$ and hard characteristic case.

On the basis of the bifurcation curves for the three-mass systems given in [9, 10] it was found that except regular vibrations also irregular vibrations can occur in considered torsional systems assuming damping coefficients equal to $d_0 = d_i = D_j = 0.001$. Namely, such vibrations can be expected with $k_3 = 0.005$ (hardening characteristic) for $p < 1.2$ and $M_0 = 1$ while with $k_3 = -0.005$ (softening characteristic) for $0.730 < p < 0.905$ and $M_0 = 0.4$.

In Figure 3 for the hardening characteristic case Poincaré maps for $p = 0.83$ in cross-sections $x = 0, 2$ while the maximal Lyapunov exponents for angular displacements in $x = 0$ are presented in Figure 4. Maximal Lyapunov exponents are positive, so in the considered frequency p vibrations are chaotic.

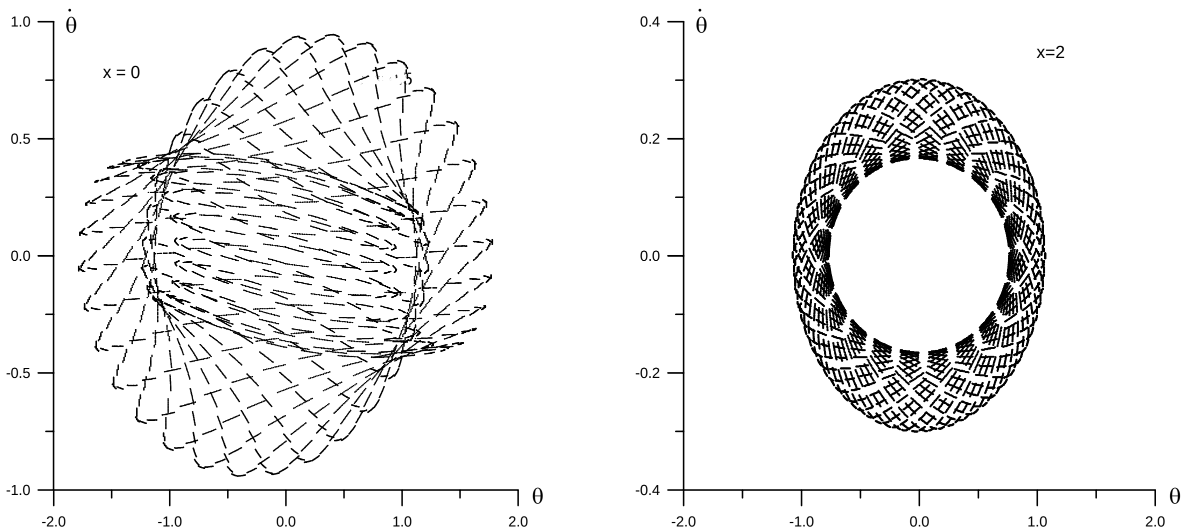

Figure 5: Poincaré maps for $p = 0.8767$, cross-sections $x = 0, 2$ and soft characteristic case.

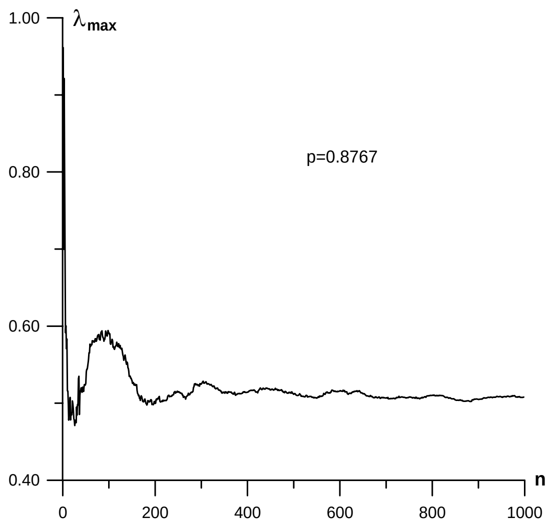

Figure 6: Maximal Lyapunov exponents. for $p = 0.8767$, cross-section $x = 0$ and soft characteristic case.

Diagrams in Figure 5 concern the motion of the three-mass system with the local nonlinearity having the softening characteristic. They show Poincaré maps for $p = 0.8767$ in cross-sections $x = 0, 2$. In Figure 6 the maximal Lyapunov exponents for angular displacements in $x = 0$ are given. Maximal Lyapunov exponents are also positive, so the motion with the assumed frequency p is also chaotic.

4 CONCLUSIONS

In discrete-continuous systems torsionally deformed with a local nonlinearity having a hardening characteristic as well as a softening characteristic and loaded by the external moment harmonically changing in time, regular and irregular vibrations may appear. Different kinds of irregular vibrations including chaotic vibrations can be found in the limited range of the change of the parameters representing the systems and the external moment. Presented numerical calculations concern the three-mass system, however governing equations allow us to widen considerations to other discrete-continuous systems. Exemplary diagrams show variety of Poincaré maps in selected cross-sections of the considered three-mass system. Besides, the positive values of the maximal exponents of Lyapunov justify chaotic motions for the assumed frequencies of the external moment.

It should be pointed out that the ranges of the parameters where the chaotic motion can occur are different for the different types of the characteristics of the local nonlinearities. One can see that the Poincaré maps are different not only for hard and soft characteristics but also for the cross-sections of the considered systems.

5 REFERENCES

[1] A. Pielorz, Dynamic analysis of a nonlinear discrete-continuous torsional system by means of wave method, *ZAMM*, 75, 691-698, 1995.

[2] A. Pielorz, Non-linear vibrations of a discrete-continuous torsional system with non-linearities having characteristic of a soft type, *Journal of Sound and Vibration*, **225**(2), 375-389, 1999.

[3] A. Pielorz, Vibrations of rod-rigid element systems with a local non-linearity, *International Journal of Non-Linear Mechanics*, 45, 888-894, 2010.

[4] C. Moon, *Chaotic Vibrations*. New York, John Wiley and Sons Inc., 1987.

[5] D. Sado and K. Gajos, Note on chaos in three degree of freedom dynamical system with double pendulum, *Meccanica,* 38, 719-729, 2003.

[6] D. Sado, and M. Kot, Chaotic vibration of an autoparametrical system with a non-ideal source of power, *Journal of Theoretical and Applied Mechanics*, 45, 2007, 119-131.

[7] D. Sado, *Regular and chaotic vibrations in selected systems with pendulum* (in Polish), Wydawnictwa Naukowo-Techniczne, Warszawa, 2010.

[8] W. Szemplińska-Stupnicka, *Chaos, Bifurcations and Fractals Around Us*, World Scientific, London, 2003.

[9] A. Pielorz and D. Sado, On Regular and Irregular Nonlinear Vibrations in Torsional Discrete-Continuous Systems, *International Journal of Bifurcation and Chaos*, **21**, 3073-3082, 2011.

[10] A. Pielorz and D. Sado, Iregular Vibrations in Multi-Mass Discrete-Continuous Systems Torsionally Deformed, *Latin American Journal of Solids and Structures*, **10**, 139-146, 2013.

[11] H. B. Stewart, J. M. T. Thompson, Y. Ueda and A. N. Lansbury, Optimal escape from potential wells – patterns of regular and chaotic bifurcation, *Physica D*, **85**, 259-295, 1995.