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THE NULL FIELD AND THE INTERIOR FIELD METHODS FOR NEUMANN PROBLEMS OF LAPLACE'S EQUATIONS

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Abstract. *Recently, the null fields method (NFM) is proposed by Chen with his groups [8]. In NFM, the fundamental solutions with the field nodes outside of the solution domain are used in the Green formulas. The Fourier expansions of the known and the unknown boundary conditions on the circular boundaries are chosen, so that the explicit discrete collocation equations are easily contained by means of orthogonality of Fourier functions. Recently, a new interior field method (IFM) is proposed in [16, 13], which is the special case of the null field method (NFM) when the field nodes are located exactly on the domain boundary. This paper is devoted to Neumann problems of Laplace's equation by NFM and IFM. The IFM is derived by the Trefftz method, based on the harmonic functions satisfying the Neumann boundary conditions, called the second kind Field Equations (FE). In fact, the explicit collocation equations for Dirichlet problems can also be used for Neumann problem, called the first kind FE. Accuracy and stability are the most important criteria. Convergence of the solution error is optimal by two kinds FE. Stability is also our main goal, which is measured by the traditional condition number and the new effective condition number of discrete matrices. From stability the first kind FE is superior to the second FE. Based on our best knowledge, this is the first time to deal with Neumann problems of Laplace's equation by two kinds field equations of NFM and IFM, accompanied with analysis, computation and comparisons.*

1 INTRODUCTION

For circular domains and holes, there exist a number of papers. In Babone and Caulk [4, 5] and Caulk [7] the Fourier functions are used for the circular holes for boundary integral equations, and in Bird and Steele [6] the simple algorithms as the collocation Trefftz method as in [12] are used. In Ang and Kang [1], complex boundary elements are studied. Recently, Chen and his research group have developed the null-filed method (NFM), in which, the resource nodes are located outside of the solution domain S, the fundamental solutions can be expanded as the convergent series. The Fourier functions are also used to approximate the known or unknown the Dirichlet and Neumann boundary conditions, numerous papers have been published for different physical problems. Since the algorithms, errors and stability are our main concern, we only cite [8] for Laplace's equation.

This paper is organized as follows. In the next section, the first kind field equations (FE) are introduced, and in Section 3, the new second kind field equations (FE) are introduced. In Section 4, numerical experiments are carried out for Neumann problem of Laplace's equation,

2 THE FIRST KIND FIELD EQUATIONS

2.1 Basic Algorithms

Consider Laplace's equation in the circular domain with one circular hole. Denote the disks S_R and S_{R_1} with radii R and R_1, respectively. Let $S_{R_1} \subset S_R$, and the eccentric circular domains S_R and S_{R_1} may have different origins. Hence $2R_1 < R$. The annular solution domain $S = S_R \backslash S_{R_1}$ with the exterior and the interior boundaries ∂S_R and ∂S_{R_1}, respectively. On the exterior boundary ∂S_R, there exist the approximations of Fourier expansions,

$$u = u_0 := a_0 + \sum_{k=1}^{M} \{a_k \cos k\theta + b_k \sin k\theta\} \;\; on \;\; \partial S_R, \tag{1}$$

$$\frac{\partial u}{\partial \nu} = q_0 := p_0 + \sum_{k=1}^{M} \{p_k \cos k\theta + q_k \sin k\theta\} \;\; on \;\; \partial S_R, \tag{2}$$

where a_k, b_k, p_k and q_k are coefficients. On the interior boundary ∂S_{R_1}, similarly

$$\bar{u} = \bar{u}_0 := \bar{a}_0 + \sum_{k=1}^{N} \{\bar{a}_k \cos k\bar{\theta} + \bar{b}_k \sin k\bar{\theta}\} \;\; on \;\; \partial S_{R_1} \tag{3}$$

$$\frac{\partial \bar{u}}{\partial \bar{\nu}} = -\frac{\partial \bar{u}}{\partial \bar{r}} = \bar{q}_0 := \bar{p}_0 + \sum_{k=1}^{N} \{\bar{p}_k \cos k\bar{\theta} + \bar{q}_k \sin k\bar{\theta}\} \;\; on \;\; \partial S_{R_1}, \tag{4}$$

where $\bar{a}_k, \bar{b}_k, \bar{p}_k$ and $\bar{q}_k$ are coefficients. In (1) – (4), θ and $\bar{\theta}$ are the polar coordinates of S_R and S_{R_1} with the origins $(0, 0)$ and $(-R_1, 0)$ respectively, and ν and $\bar{\nu}$ are the outer normal of ∂S_R and ∂S_{R_1} respectively. For the Dirichlet, the Neumann conditions, and their mixed types on ∂S_R is given with the known coefficients.

In S, denote two notes $\mathbf{x} = Q = (x, y) = (\rho, \theta)$, and $\mathbf{y} = P = (\xi, \eta) = (r, \phi)$, where $x = \rho \cos\theta, y = \rho \sin\theta, \xi = R\cos\phi$, and $\eta = R\sin\phi$. Then $\rho = \sqrt{x^2 + y^2}$ and $r = R$ with $r = \sqrt{\xi^2 + \eta^2}$. The FS of Laplace's equation is given by $\ln |\overline{PQ}| = \ln \sqrt{\rho^2 - 2\rho R \cos(\theta - \phi) + R^2}$. From the BEM theory, we have different formulas for different locations of the field notes $Q(\mathbf{x})$:

$$\int_{\partial S} \{\ln |PQ| \frac{\partial u(\mathbf{y})}{\partial \nu} - u(\mathbf{y}) \frac{\partial \ln |PQ|}{\partial \nu}\} d\sigma_{\mathbf{y}} = \begin{cases} -2\pi u(Q), & Q \in S, \\ -\pi u(Q), & Q \in \partial S, \\ 0, & otherwise, \end{cases} \tag{5}$$

where $P(\mathbf{y}) \in (S \cup \partial S)$, and the series expansions of the FS $\ln|PQ|$ are given by (see [3])

$$\ln|PQ| = \ln|P(\mathbf{y}) - Q(\mathbf{x})| = \ln|P(r,\phi) - Q(\rho,\theta)|$$
$$= \begin{cases} U^i(\mathbf{x},\mathbf{y}) = \ln r - \sum_{n=1}^{\infty} \frac{1}{n}(\frac{\rho}{r})^n \cos n(\theta-\phi), & \rho < r, \\ U^e(\mathbf{x},\mathbf{y}) = \ln \rho - \sum_{n=1}^{\infty} \frac{1}{n}(\frac{r}{\rho})^n \cos n(\theta-\phi), & \rho > r, \end{cases} \tag{6}$$

where $\mathbf{x} = (\rho, \theta)$ and $\mathbf{y} = (r, \phi)$. Then we have two kinds of derivative expansions of FS,

$$\frac{\partial U^i(\mathbf{x},\mathbf{y})}{\partial r} = \frac{1}{r} + \sum_{n=1}^{\infty}(\frac{\rho^n}{r^{n+1}})\cos n(\theta-\phi), \quad \rho < r, \tag{7}$$

$$\frac{\partial U^e(\mathbf{x},\mathbf{y})}{\partial r} = -\sum_{n=1}^{\infty}(\frac{r^{n-1}}{\rho^n})\cos n(\theta-\phi), \quad \rho > r, \tag{8}$$

where the superscripts ”e“ and ”i“ designate the exterior and interior field nodes $\mathbf{x}$, respectively.

To distinguish the boundary element method (BEM) which is based on the first equation of in the Green formula (5), the NFM is based on the third equation by using the FS expansions, we have

$$\int_{\partial S_R \cup \partial S_{R_1}} U(\mathbf{x},\xi)\frac{\partial u(\mathbf{y})}{\partial \nu} d\sigma_{\mathbf{y}} = \int_{\partial S_R \cup \partial S_{R_1}} u(\mathbf{y})\frac{\partial U(\mathbf{x},\mathbf{y})}{\partial \nu} d\sigma_{\mathbf{y}}, \quad \mathbf{x} \in \bar{S}^c, \tag{9}$$

where $\bar{S}^c$ is the complementary domain of $S \cup \partial S$. Substituting the Fourier expansions (6) –(8) into (9) yields the basic algorithms of NFM, where the exterior normal of ∂S_{R_1} is given by $\frac{\partial U(\mathbf{x},\mathbf{y})}{\partial \nu} = -\frac{\partial U(\mathbf{x},\mathbf{y})}{\partial r}$.

2.2 Explicit Equations

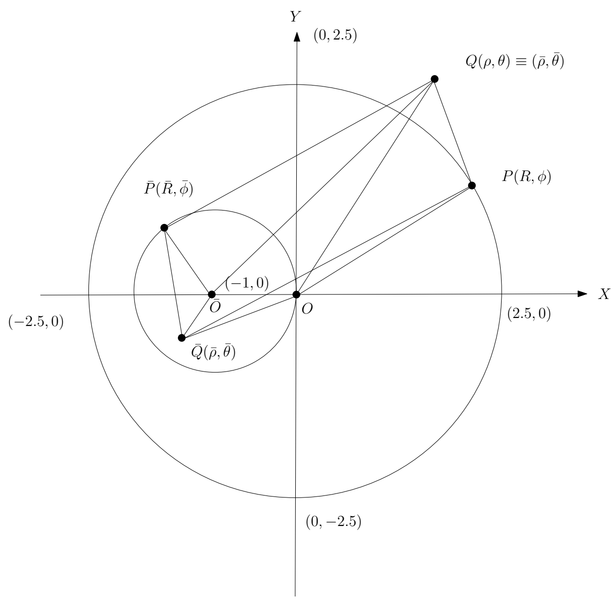

Figure 1: The polar coordinate systems with $\mathbf{x} = Q(\rho,\theta)$ (or $\mathbf{x} = \bar{Q}(\bar{\rho},\bar{\theta})$), $\mathbf{y} = P(r,\theta)$ (or $\mathbf{y} = \bar{P}(\bar{r},\bar{\phi})$)

Denote two systems of polar coordinates by (ρ, θ) and $(\bar{\rho}, \bar{\theta})$ with the origins $(0, 0)$ and (x_1, y_1) for S_R and S_{R_1}, respectively, see Figure 1. There exist the following conversion relations,

$$\rho = \sqrt{(\bar\rho\cos\bar\theta + x_1)^2 + (\bar\rho\sin\bar\theta + y_1)^2}, \quad \tan\theta = \frac{\bar\rho\sin\bar\theta + y_1}{\bar\rho\cos\bar\theta + x_1}, \tag{10}$$

$$\bar\rho = \sqrt{(\rho\cos\theta - x_1)^2 + (\rho\sin\theta - y_1)^2}, \quad \tan\bar\theta = \frac{\rho\sin\theta - y_1}{\rho\cos\theta - x_1}. \tag{11}$$

First, consider the exterior field nodes $\mathbf{x} = (\rho, \theta)$ with $\rho > r = R$. The first explicit algebraic equations of the NFM are obtained in [13] for the exterior field nodes,

$$\begin{aligned}
&\mathcal{L}_{ext}(\rho,\theta;\bar\rho,\bar\theta) := \\
&-R\pi\sum_{k=1}^{M}(\frac{R^{k-1}}{\rho^k})(a_k\cos k\theta + b_k\sin k\theta) + R_1\pi\sum_{k=1}^{N}(\frac{R_1^{k-1}}{\bar\rho^k})(\bar a_k\cos k\bar\theta + \bar b_k\sin k\bar\theta)\\
&-\{2\pi R(\ln\rho)p_0 - R\pi\sum_{k=1}^{M}\frac{1}{k}(\frac{R}{\rho})^k(p_k\cos k\theta + q_k\sin k\theta)\\
&+2\pi R_1(\ln\bar\rho)\bar p_0 - R_1\pi\sum_{k=1}^{N}\frac{1}{k}(\frac{R_1}{\bar\rho})^k(\bar p_k\cos k\bar\theta + \bar q_k\sin k\bar\theta)\} = 0.
\end{aligned}\tag{12}$$

Next, consider the interior field nodes $\mathbf{x} = (\bar\rho, \bar\theta)$ with $\bar\rho < \bar r = R_1$. The second explicit algebraic equations of the NFM are obtained in [13] for the interior field nodes,

$$\begin{aligned}
&\mathcal{L}_{int}(\rho,\theta;\bar\rho,\bar\theta) := -2\pi\bar a_0 - R_1\pi\sum_{k=1}^{N}(\frac{\bar\rho^k}{R_1^{k+1}})(\bar a_k\cos k\bar\theta + \bar b_k\sin k\bar\theta)\\
&+2\pi a_0 + R\pi\sum_{k=1}^{M}(\frac{\rho^k}{R^{k+1}})(a_k\cos k\theta + b_k\sin k\theta)\\
&-\{2\pi R_1\ln R_1\bar p_0 - R_1\pi\sum_{k=1}^{N}\frac{1}{k}(\frac{\bar\rho}{R_1})^k(\bar p_k\cos k\bar\theta + \bar q_k\sin k\bar\theta)\\
&+2\pi R\ln Rp_0 - R\pi\sum_{k=1}^{M}\frac{1}{k}(\frac{\rho}{R})^k(p_k\cos k\theta + q_k\sin k\theta)\} = 0.
\end{aligned}\tag{13}$$

Eqs. (12) and (13) are called the *explicate* algebraic equations of the NFM in this paper.

By the Green formula (5) as used in the BEM, the solution at the interior nodes: $\mathbf{x} = (\rho, \theta) \in S$ is expressed by

$$\begin{aligned}
u(\mathbf{x}) = u(\rho,\theta) &= -\frac{1}{2\pi}\int_{\partial S_R\cup\partial S_{R_1}}\{U(\mathbf{x},\mathbf{y})\frac{\partial u(\mathbf{y})}{\partial\nu} - u(\mathbf{y})\frac{\partial U(\mathbf{x},\mathbf{y})}{\partial r}\}d\sigma_{\mathbf{y}} \qquad (14)\\
&= -\frac{1}{2\pi}\{\int_{\partial S_R}\{U^i(\mathbf{x},\mathbf{y})\frac{\partial u(\mathbf{y})}{\partial\nu} - u(\mathbf{y})\frac{\partial U^i(\mathbf{x},\mathbf{y})}{\partial r}\}d\sigma_{\mathbf{y}}\\
&+\int_{\partial S_{R_1}}\{U^e(\mathbf{x},\mathbf{y})\frac{\partial u(\mathbf{y})}{\partial\bar\nu} + u(\mathbf{y})\frac{\partial U^e(\mathbf{x},\mathbf{y})}{\partial\bar r}\}d\sigma_{\mathbf{y}}\}, \quad \mathbf{x}\in S.
\end{aligned}$$

For $(\rho, \theta) \in S$, from (1)–(4) and (6)–(8), Eq. (14) leads to for $(r, \theta) \in S$,

$$\begin{aligned}
&u_{M-N} = u_{M-N}(\rho,\theta;\bar\rho,\bar\theta) = a_0 - R\ln Rp_0 - R_1\ln\bar\rho\bar p_0 \qquad (15)\\
&+\frac{R}{2}\sum_{k=1}^{M}\frac{1}{k}(\frac{\rho}{R})^k(p_k\cos k\theta + q_k\sin k\theta) + \frac{R}{2}\sum_{k=1}^{M}(\frac{\rho^k}{R^{k+1}})(a_k\cos k\theta + b_k\sin k\theta)\\
&+\frac{R_1}{2}\sum_{k=1}^{N}\frac{1}{k}(\frac{R_1}{\bar\rho})^k(\bar p_k\cos k\bar\theta + \bar q_k\sin k\bar\theta) + \frac{R_1}{2}\sum_{k=1}^{N}(\frac{R_1^{k-1}}{\bar\rho^k})(\bar a_k\cos k\bar\theta + \bar b_k\sin k\bar\theta).
\end{aligned}$$

3 THE SECOND KIND FIELD EQUATIONS

3.1 Explicit Collocation Equations

For the Neumann problem, we should use the second kind Green formula of null field nodes from (9),

$$\int_{\partial S_R \cup \partial S_{R_1}} \frac{\partial}{\partial \nu_{\mathbf{x}}} U(\mathbf{x},\mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial \nu_{\mathbf{y}}} d\sigma_{\mathbf{y}} = \int_{\partial S_R \cup \partial S_{R_1}} u(\mathbf{y}) \frac{\partial^2 U(\mathbf{x},\mathbf{y})}{\partial \nu_{\mathbf{x}} \partial \nu_{\mathbf{y}}} d\sigma_{\mathbf{y}}, \ \ \mathbf{x} \in S^c. \quad (16)$$

which lead to the second kind null-field equations. The following derivatives hold from (7) and (8),

$$\frac{\partial^2 U^i(\mathbf{x},\mathbf{y})}{\partial \rho \partial r} = \sum_{n=1}^{\infty} n(\frac{\rho^{n-1}}{r^{n+1}}) \cos n(\theta - \phi), \ \ \rho < r, \quad (17)$$

$$\frac{\partial^2 U^e(\mathbf{x},\mathbf{y})}{\partial \rho \partial r} = \sum_{n=1}^{\infty} n(\frac{r^{n-1}}{\rho^{n+1}}) \cos n(\theta - \phi), \ \ \rho > r, \quad (18)$$

where $r \neq 1$, $\rho \neq 1$ and $U(\mathbf{x},\mathbf{y}) = \ln \sqrt{\rho^2 - 2\rho r \cos(\theta - \phi) + r^2}$.

We derive the second kind of field equations from (16) similarly to Section 2, but rather propose a simple approach. For the smooth integrand in (16), we may exchange the integration and the derivation $\frac{\partial}{\partial \nu_{\mathbf{x}}}$, to give

$$\frac{\partial}{\partial \nu_{\mathbf{x}}} \{ \int_{\partial S_R \cup \partial S_{R_1}} U(\mathbf{x},\mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial \nu_{\mathbf{y}}} d\sigma_{\mathbf{y}} - \int_{\partial S_R \cup \partial S_{R_1}} u(\mathbf{y}) \frac{\partial U(\mathbf{x},\mathbf{y})}{\partial \nu_{\mathbf{y}}} d\sigma_{\mathbf{y}} \} = 0, \ \ \mathbf{x} \in S^c. \quad (19)$$

From calculus, the exchange integration and differentiation in (16) is valid if the series terms with the continuous differentiable functions under the integration are uniformly convergent. This exchange is guaranteed if the solution $u \in H^p(\partial S_R \cup \partial S_{R_1})$ and $u_\nu \in H^{p-1}(\partial S_R \cup \partial S_{R_1})$ with $p \geq 2$, thus to verify (19).

In the derivation, the directional derivatives are needed, and are given in the following lemma.

Lemma 3.1 *For the coordinate systems (ρ, θ) and $(\bar{\rho}, \bar{\theta})$, there exist the derivative formulas,*

$$\frac{\partial f(\bar{\rho},\bar{\theta})}{\partial \rho} = \frac{\partial f(\bar{\rho},\bar{\theta})}{\partial \bar{\rho}} \cos(\theta - \bar{\theta}) + \frac{\partial f(\bar{\rho},\bar{\theta})}{\bar{\rho} \partial \bar{\theta}} \sin(\theta - \bar{\theta}), \ \ (\rho, \theta) \in \partial S_R, \quad (20)$$

$$\frac{\partial g(\rho,\theta)}{\partial \bar{\rho}} = \frac{\partial g(\rho,\theta)}{\partial \rho} \cos(\theta - \bar{\theta}) - \frac{\partial g(\rho,\theta)}{\rho \partial \theta} \sin(\theta - \bar{\theta}), (\bar{\rho}, \bar{\theta}) \in \partial S_{R_1}. \quad (21)$$

From Lemma 3.1, the directional derivative of the external field equation is given as,

$$\begin{aligned} &\frac{\partial}{\partial \rho} \mathcal{L}_{ext}(\rho, \theta; \bar{\rho}, \bar{\theta}) := R\pi \sum_{k=1}^{M} k(\frac{R^{k-1}}{\rho^{k+1}})(a_k \cos k\theta + b_k \sin k\theta) \\ &-R_1 \pi \sum_{k=1}^{N} k(\frac{R_1^{k-1}}{\bar{\rho}^{k+1}})(\bar{a}_k \cos[(k+1)\bar{\theta} - \theta] + \bar{b}_k \sin[(k+1)\bar{\theta} - \theta]) \\ &-\{2\pi R \frac{1}{\rho} p_0 + R\pi \sum_{k=1}^{M} \frac{R^k}{\rho^{k+1}} (p_k \cos k\theta + q_k \sin k\theta) + 2\pi R_1 \frac{1}{\bar{\rho}} \bar{p}_0 \cos(\theta - \bar{\theta}) \\ &+R_1 \pi \sum_{k=1}^{N} \frac{R_1^k}{\bar{\rho}^{k+1}} (\bar{p}_k \cos[(k+1)\bar{\theta} - \theta] + \bar{q}_k \sin[(k+1)\bar{\theta} - \theta])\} = 0, \end{aligned} \quad (22)$$

Similarly, the directional derivative of the interior field equation can given as,

$$\frac{\partial}{\partial\bar{\nu}}\mathcal{L}_{int}(\rho,\theta;\bar{\rho},\bar{\theta}) = -\frac{\partial}{\partial\bar{\rho}}\mathcal{L}_{int}(\rho,\theta;\bar{\rho},\bar{\theta}) :=$$
$$-R_1\pi\sum_{k=1}^{N}k(\frac{\bar{\rho}^{k-1}}{R_1^{k+1}})(\bar{a}_k\cos k\bar{\theta}+\bar{b}_k\sin k\bar{\theta})$$
$$+R\pi\sum_{k=1}^{M}k(\frac{\rho^{k-1}}{R^{k+1}})(a_k\cos[(k-1)\theta+\bar{\theta}]+b_k\sin[(k-1)\theta+\bar{\theta}])$$
$$+R_1\pi\sum_{k=1}^{N}\frac{\bar{\rho}^{k-1}}{R_1^k}(\bar{p}_k\cos k\bar{\theta}+\bar{q}_k\sin k\bar{\theta})$$
$$+R\pi\sum_{k=1}^{M}\frac{\rho^{k-1}}{R^k}(p_k\cos[(k-1)\theta+\bar{\theta}]+q_k\sin[(k-1)\theta+\bar{\theta}])=0. \tag{23}$$

The solution in S is sill given in (15) with an arbitrary constant c.

4 NUMERICAL EXPERIMENTS BY TWO KINDS FE

In this section, numerical experiments are carried out for Neumann problems by the NFM in two kinds of field equations (FE), to explore the intrinsic characteristics of Neumann Problems. Similarly, choose $\epsilon = \bar{\epsilon} = 0$, and the IFM in two kinds FE are used. The pseudo singularity as in [14] is again discovered in numerical solutions, and the overdetermined system (ODS) of linear algebraic equations is applied to overcome the pseudo singularity introduced in [14].

4.1 By First Kind FE

In computation, choose $R = 2.5$ and $R_1 = 1$ and the origins of S_R and S_{R_1} are located at $(0,0)$ and $(-R_1, 0)$, respectively, called Model Problem in this paper. For simplicity, we choose the simple Neumann conditions $u_\nu = p_0 = \frac{2}{5}$ on S_R and $\bar{u}_{\bar{\nu}} = \bar{p}_0 = -1$ on S_{R_1}. By noting the symmetry, we obtain two equations from (12) and (13)

$$\mathcal{L}_{ext}^{Neum}(\rho,\theta;\bar{\rho},\bar{\theta}) \equiv -\frac{R}{2}\sum_{k=1}^{M}(\frac{R^{k-1}}{\rho^k})a_k\cos k\theta+\frac{R_1}{2}\sum_{k=1}^{N}(\frac{R_1^{k-1}}{\bar{\rho}^k})\bar{a}_k\cos k\bar{\theta} \tag{24}$$
$$-Rp_0(\ln\rho)-R_1\bar{p}_0\ln\bar{\rho}=0.$$

and where the coefficient a_0 may be arbitrary for the Neumann problem. Hence we may remove a_0.

$$\mathcal{L}_{int}^{Neum}(\rho,\theta;\bar{\rho},\bar{\theta}) \equiv -\frac{R_1}{2}\sum_{k=1}^{N}(\frac{\bar{\rho}^k}{R_1^{k+1}})\bar{a}_k\cos k\bar{\theta}+\frac{R}{2}\sum_{k=1}^{M}(\frac{\rho^k}{R^{k+1}})a_k\cos k\theta \tag{25}$$
$$-(\bar{a}_0+Rp_0\ln R+R_1\bar{p}_0\ln R_1)=0,$$

where the coefficient a_0 may be arbitrary constant a_0 is removed. Then there are $(M+N)+1$ unknown coefficients.

We choose the IFM with $\epsilon = \bar{\epsilon} = 0$, and obtain the collocation equations as the following:

$$\mathcal{L}_{ext}^{Neum}(R, i\Delta\theta;\bar{\rho}_i,\bar{\theta}_i)=0,\ \ i=0,1,...,M, \tag{26}$$
$$\mathcal{L}_{int}^{Neum}(\rho_i,\theta_i;R_1,(i+\frac{1}{2})\Delta\bar{\theta})=0,\ \ i=0,1,...,N-1, \tag{27}$$

where $\Delta\theta = \frac{2\pi}{2M+1}$ and $\Delta\bar{\theta} = \frac{\pi}{N}$. The interior solution is also given by from (15)

$$u_{M-N}^{Neum}(\rho,\theta) = a_0 - Rp_0 \ln R - R_1\bar{p}_0 \ln\bar{\rho} \tag{28}$$
$$+\frac{R}{2}\sum_{k=1}^{M}(\frac{\rho^k}{R^{k+1}})(a_k\cos k\theta + \frac{R_1}{2}\sum_{k=1}^{N}(\frac{R_1^{k-1}}{\bar{\rho}^k})\bar{a}_k\cos k\bar{\theta},\ \ (r,\theta)\in S,$$

where the coefficient a_0 is an arbitrary constant. When the total number of equations is larger than the number of unknown coefficients:

$$\bar{M} + \bar{N} > (M+N)+1, \tag{29}$$

we obtain the overdetermined system

$$\mathbf{F}\mathbf{x} = \mathbf{b}, \tag{30}$$

where $\mathbf{F} \in R^{m\times n}$ with $m(=\bar{M}+\bar{N}) > n(=(M+N)+1)$. We may use the QR or the singular value decomposition to solve (30). We compute the traditional condition number and the effective condition number, defined as

$$\text{Cond} = \frac{\sigma_{\max}}{\sigma_{\min}},\ \ \text{Cond\_eff} = \frac{\|\mathbf{b}\|}{\sigma_{\min}\|\mathbf{x}\|}, \tag{31}$$

where $\sigma_{\max}$ and $\sigma_{\max}$ are the maximal and the minimal singular values of the matrix $\mathbf{F}$ in (30), respectively.

By using the first FE of IFM (i.e., (24) and (25) with $\epsilon = \bar{\epsilon} = 0$), the errors and condition numbers are listed in Table 1, where $\delta = u - u_{M-N}$. From Table 1, we can find the following asymptotes,

$$\|\delta_\nu\|_{\infty,\partial S} = O(0.515^M),\ \ \|\delta_\nu\|_{0,\partial S} = O(0.516^M), \tag{32}$$
$$\text{Cond} = O(2.05^M),\ \ \text{Cond\_eff} = O(2.05^M). \tag{33}$$

In Table 1, we choose the good pairs $(M:N) = (2:1)$ as used in [13, 14]. Although the errors in (32) are exponentially convergent, the condition numbers in (33) grow exponentially. Note that the fields nodes of the IFM are located on the domain boundary, the stability as shown in [15] is very good due to Cond $= O(M)$. However, after checking the singular values σ_i of matrix $\mathbf{F}$ in Table 1, we find that only the minimal singular value $\sigma_{\min}$ is very small, but the next minimal singular value $\sigma_{min-next}$ is not,

$$\sigma_{\min} \ll \sigma_i,\ \ i = 1,2,...,n-1. \tag{34}$$

This is exactly the pseudo-singularity discovered in [14]. Two techniques can be solicited to achieve good stability: (1) overdetermined system (ODS) and (2) the truncation singular value decomposition (TSVD). Here we just reveal the results from ODS.

First by adding add one more equation in (27), we have

$$\hat{\mathcal{L}}_{int}^{Neum}(\rho_i,\theta_i;R_1,i\Delta\bar{\theta}^+) = 0,\ \ i = 0,1,2,...,N, \tag{35}$$

where $\Delta\bar{\theta}^+ = \frac{2\pi}{2N+1}$. By using (26) and (35), the overdetermined system (30) is obtained with $m = n+1$. By the ODS, the results are given in Table 2. From Table 2, the errors are almost the same as those in Table 1, but the condition numbers are nearly constants

$$\text{Cond} = O(1),\ \ \text{Cond\_eff} = O(1)! \tag{36}$$

(M,N)	$(10,5)$	$(20,10)$	$(30,15)$	$(40,20)$
$\|\delta_\nu\|_{\infty,\partial S}$	4.45(-3)	7.32(-6)	1.03(-8)	1.36(-11)
$\|\delta_\nu\|_{0,\partial S}$	7.16(-3)	1.21(-5)	1.74(-8)	2.35(-11)
$\sigma_{\max}$	3.27	4.62	5.66	6.54
$\sigma_{\min}$	4.88(-5)	4.23(-8)	3.60(-11)	3.05(-14)
$\sigma_{min-next}$	6.37(-1)	8.96(-1)	1.10	1.27
Cond	6.70(4)	1.09(8)	1.57(11)	2.14(14)
Cond_eff	4.03(4)	6.56(7)	9.43(10)	1.29(14)

Table 1: Errors and condition numbers of the Neumann problem by the first kind FE.

(M,N)	$(10,5)$	$(20,10)$	$(30,15)$	$(40,20)$
$\|\delta\|_{\infty,\partial S}$	6.66(-4)	7.02(-7)	6.97(-10)	6.70(-13)
$\|\delta_\nu\|\|_{0,\partial S}$	9.72(-4)	1.02(-6)	1.01(-9)	9.97(-13)
Cond	5.22	5.20	5.19	5.19
Cond_eff	3.12	3.11	3.11	3.11

Table 2: Errors and condition number of Neumann problem by the first kind FE via ODS.

4.2 By Second Kind FE

Based on symmetry of Model Problem, Eqs. (22) and (23) are simplified as

$$\frac{\partial}{\partial\rho}\mathcal{L}^{Neum}_{ext}(\rho,\theta;\bar\rho,\bar\theta) := R\pi\sum_{k=1}^{M}k(\frac{R^{k-1}}{\rho^{k+1}})a_k\cos k\theta \tag{37}$$
$$-R_1\pi\sum_{k=1}^{N}k(\frac{R_1^{k-1}}{\bar\rho^{k+1}})\bar a_k\cos[(k+1)\bar\theta-\theta]-2\pi R\frac{1}{\rho}p_0-2\pi R_1\frac{1}{\bar\rho}\bar p_0\cos(\theta-\bar\theta)=0,$$

and

$$\frac{\partial}{\partial\bar\nu}\mathcal{L}^{Neum}_{int}(\rho,\theta;\bar\rho,\bar\theta) = -\frac{\partial}{\partial\bar\rho}\mathcal{L}_{int}(\rho,\theta;\bar\rho,\bar\theta) := \tag{38}$$
$$-R_1\pi\sum_{k=1}^{N}k(\frac{\bar\rho^{k-1}}{R_1^{k+1}})\bar a_k\cos k\bar\theta + R\pi\sum_{k=1}^{M}k(\frac{\rho^{k-1}}{R^{k+1}})a_k\cos[(k-1)\theta+\bar\theta]=0.$$

Note that the number of unknown in (37) and (38) are only $M+N$. We choose the following collocation equations,

$$\frac{\partial}{\partial\bar\nu}\mathcal{L}^{Neum}_{ext}(R,(i+\frac{1}{2})\Delta\theta;\bar\rho_i,\bar\theta_i)=0,\ \ i=0,1,...,M-1, \tag{39}$$
$$\frac{\partial}{\partial\bar\nu}\mathcal{L}^{Neum}_{int}(\rho_i,\theta_i;R_1,(i+\frac{1}{2})\Delta\bar\theta)=0,\ \ i=0,1,...,N-1, \tag{40}$$

where $\Delta\theta=\frac{\pi}{M}$ and $\Delta\bar\theta=\frac{\pi}{N}$.

By using the second kind FE (i.e., (39) and (40)), the errors and condition numbers are listed in Table 3. From Table 3, we can find the following asymptotes,

$$\|\delta_\nu\|_{\infty,\partial S}=O(0.515^M),\ \ \|\delta_\nu\|_{0,\partial S}=O(0.516^M), \tag{41}$$
$$\text{Cond}=O(2.74^M),\ \ \text{Cond\_eff}=O(2.42^M), \tag{42}$$

where $\delta = u - u_{M,N}$, and ν is the exterior normal to ∂S. Looking at Table 3, both the minimal singular value $\sigma_{\min}$ and the next minimal singular value $\sigma_{min-next}$ are very smaller, but others are not,

$$0 < \sigma_{\min} < \sigma_{min-next} \ll \sigma_{min-next-next}. \tag{43}$$

This also falls into the pseudo-singularity discovered in [14]. Two technique given in the first kind FE above can still be employed, to achieve good stability, such as the overdetermined system (ODS) by adding two more collocation equations. We will not give explicit numerical output here due to limited space. By Table 4, the condition numbers are small, and the effective condition numbers are constants as shown in the following:

$$\text{Cond} = O(M), \;\; \text{Cond\_eff} = O(1).$$

(M,N)	$(10,5)$	$(20,10)$	$(30,15)$	$(40,20)$
$\|\delta_\nu\|_{\infty,\partial S}$	4.87(-3)	7.96(-6)	1.12(-8)	1.46(-11)
$\|\delta_\nu\|_{0,\partial S}$	7.85(-3)	1.31(-5)	1.88(-8)	2.52(-11)
$\sigma_{\max}$	5.69	1.70(1)	3.18(1)	4.93(1)
$\sigma_{\min}$	7.63(-4)	4.92(-7)	9.47(-11)	1.53(-14)
$\sigma_{min-next}$	1.66(-3)	1.30(-6)	1.64(-9)	1.85(-12)
$\sigma_{min-next-next}$	4.25(-1)	6.02(-1)	7.37(-1)	8.52(-1)
Cond	7.47(3)	3.46(7)	1.35(11)	3.23(15)
Cond_eff	6.29(2)	1.38(6)	8.77(9)	6.27(13)

Table 3: Errors and condition number of the Neumann problem by the second kind FE.

(M,N)	$(10,5)$	$(20,10)$	$(30,15)$	$(40,20)$
$\|\delta_\nu\|_{\infty,\partial S}$	7.70(-4)	7.53(-7)	7.29(-10)	6.92(-13)
$\|\delta_\nu\|_{0,\partial S}$	1.06(-3)	1.07(-6)	1.04(-9)	1.02(-12)
Cond	1.57(1)	2.95(1)	4.43(1)	5.92(1)
Cond_eff	1.11	1.12	1.12	1.12

Table 4: Errors and condition number of Neumann problem by the second kind FE via ODS.

Let us compare the Neumann problems by the first and the second kind FE. Interestingly, the unknown constant $\bar{a}_0$ is given by the first kind FE, but not by the second kind FE, although constant $\bar{a}_0$ as an axillary constant disappears in the final solution (14).

Since the convergence rates are all optimal, the stability and the complexity of algorithms are the main concern. Moreover, the ODS is simpler in programming. Therefore, the first kind FE using ODS is strongly recommended. There exist many reports by the first FE, but few of them reported the computation by the second FE. In this section, the numerical experiments of Neumann problems by the first and the second FE, and their comparisons are provided. This is significant for Neumann problems by the IFM.

Finally, let us compare (32) and (42) with those of the Dirichlet conditions in [13, 14]. The very small errors remain almost the same, since the optimal convergence rates are achieved from both Dirichlet and Neumann problems. For the IFM with ODS, the stability is excellent for both the Dirichlet and Neumann problems.

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