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RESPONSE ANALYSIS OF A CURVED RAIL SUBJECT TO A MOVING LOAD

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Abstract. *In this paper, an analytical solution to the response of a curved railway track resting on a viscoelastic foundation subject to a moving load is presented. Trigonometric functions are employed as the trial functions in approximating the displacement of the curved rail. The accuracy of the present solution is verified through comparison against available analytical results in the literature. To further examine the accuracy of the proposed analytical method, a computational study of the problem was also carried out using the moving element method (MEM) based on piecewise straight beam elements. Good agreement was found between all these results.*

1 INTRODUCTION

The rapid advancement in train technology resulting in higher operational train speed has vastly improved the popularity of train travel as a viable alternative for long distance transportation needs. However, safety concerns of the train-track system are also significantly heightened due to increased risk and impact of failures. In particular, when the train travels over a curved rail, the risk of derailment and vehicle overturning is increased. It is known that horizontal centrifugal force emerges when a train travels over curved rails. High lateral forces on the rail caused by the passage of high-speed trains result in a number of undesirable effects including increased curve resistance, wear of rail and wheel, rail tilting, gauge widening and noise [1]. It is therefore important to understand the dynamic behavior of curved rail subject to moving high-speed trains.

The dynamic response of a railway track subject to moving train loads can be simplified as a beam traversed by moving loads, and has interested researchers over the past century. The case of a straight beam subject to moving loads has been extensively investigated [2-6]. However the dynamic behavior of a curved beam subject to moving loads has not been extensively studied analytically. Earlier studies mainly focus on the out-of-plane free vibration of a curved beam [7, 8]. To include the in-plane forced vibration, Yang et al. [9] analyzed the dynamic response of a simply supported curved beam subject to moving loads. However, the consideration of only the first mode of vibration is not applicable to the response analysis of curved rail where the steady-state response is usually of interest. To investigate the dynamic behavior of a curved beam on an elastic Winkler foundation, S. Nair et al. [10] studied the dynamic stability of a curved rail under a moving load. However, there seems to be inconsistencies in the results obtained for the case of the curved rail as compared to the straight rail. Also, in the aforementioned research works, the systems are assumed to be undamped, which is not realistic as it is expected for some form of damping to exist in any physical system [4].

This paper is concerned with offering an analytical solution to the dynamic response of a curved rail resting on a Winker-type foundation subject to a moving load. As a special case, the response of a straight railway track is obtained, and results are compared with available solutions in the literature. For the purpose of demonstrating the accuracy of the analytical solution to the response of curved rail under moving loads, the moving element method (MEM) [11] based on piecewise straight beam element is employed to generate the solutions for comparison.

2 METHODOLOGY AND FORMULATION

Figure 1 shows the coordinates system of a curved rail. The railway track is modeled as an Euler-Bernoulli beam with a constant radius R, continuously supported by a Winkler-foundation. Figure 2 shows a cross-sectional view of the rail-foundation system. The railhead is subject to a moving load exerted by a moving train mass. The inertia effect of the moving mass, however, is neglected in this study. In view that the superelevation angle θ is usually small for most cases, a reasonable assumption is made such that the rail is resting up-right on the foundation, while the effect of the superelevation angle is taken into account when calculating the moving load acting on the railhead (see Figure 2). By assuming the rail cross-section to be bisymmetric and neglecting warping deformation [14], the linear differential equations of motion for the curved rail may be written as

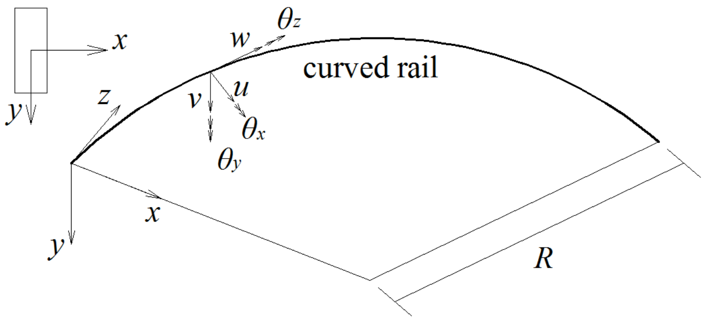

Figure 1. Curved railway track.

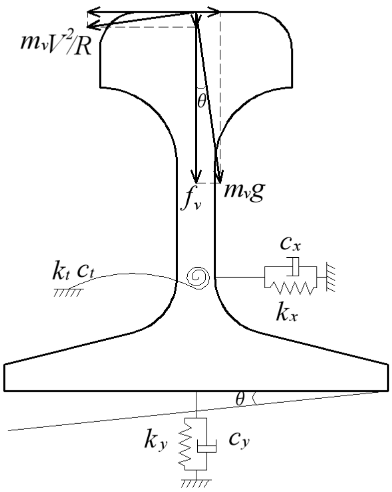

Figure 2. Cross-section of railway track.

$$
\begin{gathered}
m\frac{\partial^2 u}{\partial t^2}+EI_y\left(\frac{\partial^4 u}{\partial z^4}+\frac{2}{R^2}\frac{\partial^2 u}{\partial z^2}+\frac{u}{R^4}\right)+\frac{EA}{R}\left(\frac{u}{R}-\frac{\partial w}{\partial z}\right)+c_x\frac{\partial u}{\partial t}+k_x u=f_h\delta(z-Vt)\\
m\frac{\partial^2 v}{\partial t^2}+EI_x\left(\frac{\partial^4 v}{\partial z^4}-\frac{1}{R}\frac{\partial^2 \beta}{\partial z^2}\right)-\frac{GJ}{R}\left(\frac{1}{R}\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 \beta}{\partial z^2}\right)+c_y\frac{\partial v}{\partial t}+k_y v=f_v\delta(z-Vt)\\
m\frac{\partial^2 w}{\partial t^2}+EA\left(\frac{1}{R}\frac{\partial u}{\partial z}-\frac{\partial^2 w}{\partial z^2}\right)+c_z\frac{\partial w}{\partial t}+k_z w=0\\
\rho J\frac{\partial^2 \beta}{\partial t^2}+\frac{EI_x}{R}\left(\frac{\beta}{R}-\frac{\partial^2 v}{\partial z^2}\right)-GJ\left(\frac{\partial^2 \beta}{\partial z^2}+\frac{1}{R}\frac{\partial^2 v}{\partial z^2}\right)+c_t\frac{\partial \beta}{\partial t}+k_t\beta=f_h d\delta(z-Vt)
\end{gathered}
\tag{1}
$$

where m denotes the mass per unit length of the rail; A denotes the area of the cross-section of the rail; u, v, w and β refer to the displacements of the centroid in x, y, and z directions and a rotation of the cross-section about z-axis, respectively; E denotes the Young's modulus of the rail, G the modulus of rigidity; I_x and I_y refer to the second moment of inertia about x- and y-axes, respectively; J denotes the torsional constant; c_x, c_y, c_z and c_t refer to the lateral, vertical, longitudinal and torsional damping property of the foundation, respectively, while k_x, k_y, k_z and k_t denote the lateral, vertical, longitudinal and torsional stiffness property of the foundation, respectively; V refers to the velocity; δ denotes the direct-delta function, and d the distance between the railhead and the elastic axis of the rail.

In this paper a curved railway track of a finite length L is modeled. The length of the rail should be long enough so that the displacements and forces at the ends of the railway track are negligible when the moving load is located sufficiently far from the boundaries. For this reason, the boundary condition at the two ends of the curved railway track is reasonably assumed to be free.

As can be seen from Eq. (1), the linear governing equations of the curved railway track are partially uncoupled. Therefore, the analysis of the out-of-plane (v and β) and in-plane (u and w) responses of the curved rail is performed separately.

2.1 Out-of-Plane Response of Curved Rail Subject to a Moving Load

This paper mainly focuses on the response of the curved rail when the moving load is away from the truncated ends. For this reason, the deflections of the rail and reaction forces at the boundaries are usually negligible. Thus, the vertical displacement of the curved rail can be reasonably expressed as the summation of a series of sinusoidal functions as

$$v = \sum_{i=1}^{\infty} q_{vi}(t) \sin \frac{i\pi x}{L} \tag{2}$$

where q_{vi} denotes the generalized coordinate of the ith vertical vibration mode. In view of the coupled equations of motion presented in Eq. (1), the expression for the angle of torsional deformation can be written as

$$\beta = \sum_{i=1}^{\infty} q_{\beta i}(t) \sin \frac{i\pi x}{L} \tag{3}$$

where $q_{\beta i}$ denotes the ith generalized coordinate of the torsional angle.

To solve the coupled differential equations, Galerkin's approach is adopted to formulate the weighted residual forms of the governing equations of motion. As the mode of vibration is orthogonal with each other, the governing equations for the out-of-plane response can be rewritten as

$$\begin{aligned} \frac{\partial^2 q_{vi}}{\partial t^2} + \eta_{i1} \frac{\partial q_{vi}}{\partial t} + a_{i1} q_{vi} + a_{i2} q_{\beta i} &= \frac{2 f_v}{mL} \sin \frac{i\pi Vt}{L} \\ \frac{\partial^2 q_{\beta i}}{\partial t^2} + \eta_{i2} \frac{\partial q_{\beta i}}{\partial t} + b_{i1} q_{\beta i} + b_{i2} q_{vi} &= \frac{2 f_h d}{\rho J L} \sin \frac{i\pi Vt}{L} \end{aligned} \tag{4}$$

The general solutions to out-of-plane responses are composed of the complementary and particular solutions, which can be written as

$$\begin{aligned} q_{vi} &= q_{vic} + q_{vip} \\ q_{\beta i} &= q_{\beta ic} + q_{\beta ip} \end{aligned} \tag{5}$$

where the subscripts c and p denote the complementary and particular solutions, respectively. The complementary solutions can be written as

$$\begin{aligned} q_{vic} &= e^{-\xi w_{opi} t} \left(A_{i1} \sin \omega_{opDi} t + B_{i1} \cos \omega_{opDi} t \right) \\ q_{\beta ic} &= e^{-\xi w_{opi} t} \left(A_{i2} \sin \omega_{opDi} t + B_{i2} \cos \omega_{opDi} t \right) \end{aligned} \tag{6}$$

where ω_{opi} and ω_{opDi} denote the ith natural and damped frequencies of out-of-plane vibration of the curved rail, respectively; ξ the damping ratio; A_{ij} and B_{ij} are constants to be determined from the initial conditions. First, we shall consider the undamped case. With ξ set to zero in Eq. (6) and in view of the homogenous form of Eq. (4) we obtain

$$\begin{bmatrix} a_{i1} - \omega_{opi}^2 & a_{i2} \\ b_{i2} & b_{i1} - \omega_{opi}^2 \end{bmatrix} \begin{Bmatrix} q_{vih} \\ q_{\beta ih} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{7}$$

By solving the eigen-value problem, the ith natural frequency of the out-of-plane vibration can be calculated as

$$\omega_{opi} = \sqrt{\frac{a_{i1} + b_{i1} + \sqrt{(a_{i1} - b_{i1})^2 + 4a_{i2}b_{i2}}}{2}} \tag{8}$$

The damped case is next investigated. For the damped case, the damping property of the railway system is expected to be much smaller than the critical damping property. Therefore, only the underdamped situation, where $\xi^2 < 1$, will be considered. The damped frequency of out-of-plane vibration of the curved rail can be calculated as

$$\omega_{opDi} = \omega_{opi}\sqrt{1 - \xi^2} \tag{9}$$

The particular solutions to the out-of-plane response can be written as

$$\begin{aligned} q_{vip} &= G_{vi} \sin\frac{i\pi Vt}{L} + K_{vi} \cos\frac{i\pi Vt}{L} \\ q_{\beta ip} &= G_{\beta i} \sin\frac{i\pi Vt}{L} + K_{\beta i} \cos\frac{i\pi Vt}{L} \end{aligned} \tag{10}$$

where G and K are coefficients to be determined by substituting Eq. (10) into Eq. (4).

The vertical and torsional deformations of the curved rail subject to a vertical moving load can be written as

$$\begin{aligned} v &= \sum_{i=1}^{\infty}\left[e^{-\xi w_{opi}t}\left(A_{i1}\sin\omega_{opDi}t + B_{i1}\cos\omega_{opDi}t\right) + G_{vi}\sin\frac{i\pi V}{L}t + K_{vi}\cos\frac{i\pi V}{L}t\right]\sin\frac{i\pi x}{L} \\ \beta &= \sum_{i=1}^{\infty}\left[e^{-\xi w_{opi}t}\left(A_{i2}\sin\omega_{opDi}t + B_{i2}\cos\omega_{opDi}t\right) + G_{\beta i}\sin\frac{i\pi V}{L}t + K_{\beta i}\cos\frac{i\pi V}{L}t\right]\sin\frac{i\pi x}{L} \end{aligned} \tag{13}$$

The complementary response, also known as the transient response, damps out quickly and is of little practical interest. Thus, the steady-state out-of-plane response may be written as

$$\begin{aligned} v &= \sum_{i=1}^{\infty}\left(G_{vi}\sin\frac{i\pi V}{L}t + K_{vi}\cos\frac{i\pi V}{L}t\right)\sin\frac{i\pi x}{L} \\ \beta &= \sum_{i=1}^{\infty}\left(G_{\beta i}\sin\frac{i\pi V}{L}t + K_{\beta i}\cos\frac{i\pi V}{L}t\right)\sin\frac{i\pi x}{L} \end{aligned} \tag{14}$$

2.2 In-Plane Response of Curved Rail Subject to a Moving Load

Similarly, the trial function for the radial displacement of the rail is assumed to be the summation of a series of sinusoidal functions as

$$u = \sum_{i=1}^{\infty} q_{ui}(t)\sin\frac{i\pi x}{L} \tag{15}$$

where q_{ui} denotes the ith generalized coordinate of the radial vibration mode. In view of the form of the coupled governing equations given in Eq. (1), the trial function for the axial displacement is assumed as

$$w = \sum_{i=1}^{\infty} q_{wi}(t) \cos\frac{i\pi x}{L} \tag{16}$$

where q_{wi} denotes the ith generalized coordinate of the axial displacement.

Note that this expression for the axial displacement satisfies the assumed essential free end boundary conditions. Once again, Galerkin's approach is employed to solve the coupled equations in Eq. (1). By performing a similar procedure as elaborated in the previous section for deriving the out-of-plane responses, the radial and axial displacements of the curved beam can be expressed as

$$\begin{aligned} u &= \sum_{i=1}^{\infty} \left[e^{-\xi w_{ipi} t} \left(C_{i2} \sin \omega_{ipDi} t + D_{i2} \cos \omega_{ipDi} t \right) + G_{ui} \sin \frac{i\pi V}{L} t + K_{ui} \cos \frac{i\pi V}{L} t \right] \sin \frac{i\pi x}{L} \\ w &= \sum_{i=1}^{\infty} \left[e^{-\xi w_{ipi} t} \left(C_{i2} \sin \omega_{ipDi} t + D_{i2} \cos \omega_{ipDi} t \right) + G_{wi} \sin \frac{i\pi V}{L} t + K_{wi} \cos \frac{i\pi V}{L} t \right] \cos \frac{i\pi x}{L} \end{aligned} \tag{17}$$

where ω_{ipi} and ω_{ipDi} denote the ith natural and damped frequencies for the in-plane vibration, respectively; C and D are constants to be determined from the initial conditions.

The steady-state in-plane response can be written as

$$\begin{aligned} u &= \sum_{i=1}^{\infty} \left(G_{ui} \sin \frac{i\pi V}{L} t + K_{ui} \cos \frac{i\pi V}{L} t \right) \sin \frac{i\pi x}{L} \\ w &= \sum_{i=1}^{\infty} \left(G_{wi} \sin \frac{i\pi V}{L} t + K_{wi} \cos \frac{i\pi V}{L} t \right) \cos \frac{i\pi x}{L} \end{aligned} \tag{18}$$

3 NUMERICAL EXAMPLES

In an attempt to examine the convergence and accuracy of the proposed analytical formulas, a comparison is made against available analytical results for the case of a straight beam under a moving load [3]. In the case of a curved beam subject to a moving load, where analytical results are not available, the results obtained by using the proposed formulas are compared against independent numerical results using the MEM.

3.1 Straight Beam on Viscoelastic Foundation

The case of a straight beam of length L resting on a viscoeleastic foundation with 5% damping ratio subject to a moving load is considered. The moving load considered arises from the weight due to a 8500 kg lumped mass moving at a conventional high-speed rail (HSR) speed of 250 km/h. Parameters relating to the properties of the beam and foundation used for this study are listed in Table 1.

Parameter	Value	Parameter	Value
E	2×10^{11} N/m^2	I_x	3.055×10^{-5} m^4
A	7.69×10^{-3} m^2	m	60 kg/m
k_y	1×10^{7} N/m^2		

Table 1. Parameters for beam and foundation.

By setting the curvature of the rail and superelevation to zero, i.e. $1/R = 0$ and $\theta = 0$, the analytical formulas developed in this paper reduces to those for the special case of a straight rail subject to moving loads. A convergence study is first performed to ensure the accuracy of computed results. Figure 3 shows the results of the convergence study, which presents the steady-state response at the instant when the moving load arrives at the midpoint of the rail.

As can be seen in the figure, all results converge steadily with increasing number of modes. It is also noted that lesser number of modes is required for convergence with shorter lengths of truncated rail.

The accuracy of the present solution is next investigated by comparing against available analytical steady-state response solution of a straight beam on a Winkler-foundation obtained by Kenney [3]. Figure 4 shows the converged steady-state vertical rail deflection profiles when the moving load reaches the midpoint of the truncated rail for four lengths of truncated rail, as well as the results obtained by the proposed formula by Kenney [3]. As can be seen in Figure 4, clear disagreement is found between the present solutions for the case of a truncated length of 5.0 m and those by Kenney. For longer truncated lengths of 7.5 and 10.0 m, the results are agreeable with each other and Kenney's solution over most part of the truncated rail except in the vicinity of the boundaries. Good agreement between the present analytical and Kenney's results is achieved when the truncated length of rail beam is increased to 12.5 m.

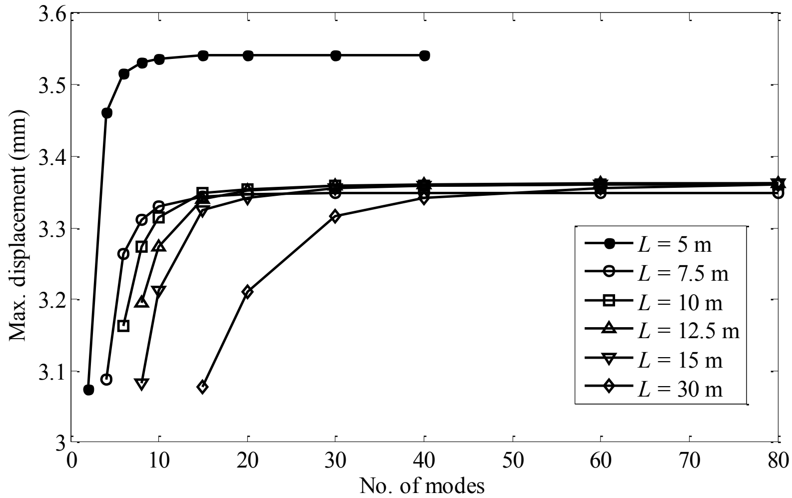

Figure 3. Convergence study on maximum steady-state response of straight rail.

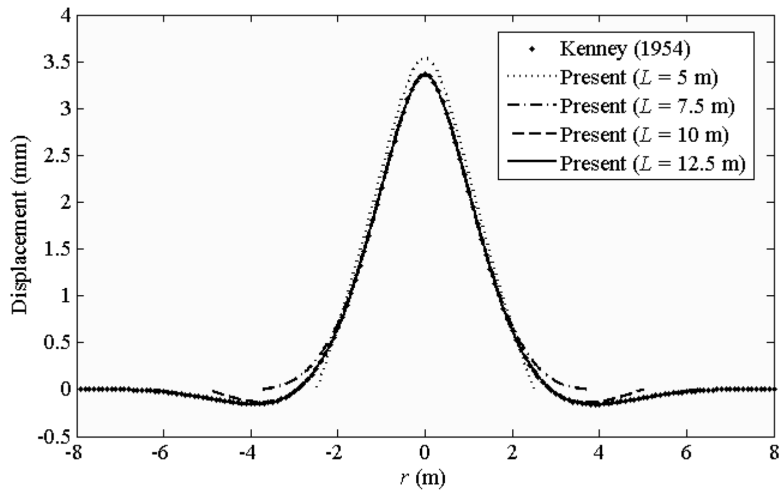

Figure 4. Steady-state vertical deflection of rail.

3.2 Response Analysis of Curved Rail Subject to a Moving Load

The accuracy of the presented analytical formulae for dealing with a curved rail subject to a moving load is next tested. In addition to the parameters adopted in Section 3.1, additional parameters relating to the curved rail are listed in Table 2. The computed steady-state response is compared with numerical results determined via the moving element method (MEM).

Parameter	Value	Parameter	Value
I_y	5.13×10^{-6} m^4	υ	0.2
J	4.23×10^{-6} m^4	ρ	7800 kg/m^3
R	1000 m	θ	5^o
d	0.13 m	k_x	5.5×10^6 N/m^2
k_z	5.5×10^6 N/m^2	k_t	7.1×10^4 N/rad

Table 2. Parameters adopted for the response analysis of curved rail.

Prior to the accuracy test, a convergence study is first performed. The study revealed that convergence for the steady-state out-of-plane and radial displacements is achieved with a minimum truncated length of 30 m and the use of 230 vibration modes in the numerical computation. It is to be noted that convergence requirement for the axial displacement component is not critical in view that its magnitude is found to be negligible.

The accuracy of the proposed solutions for curved rails subject to a moving load at a speed of 250 km/h is investigated by comparing against results obtained via the MEM. Figures 5-7 present the compared results for the steady-state vertical, radial and torsional displacements of the rail, respectively. As can be seen from these figures, excellent agreement is found for all steady-state responses of the curved rail.

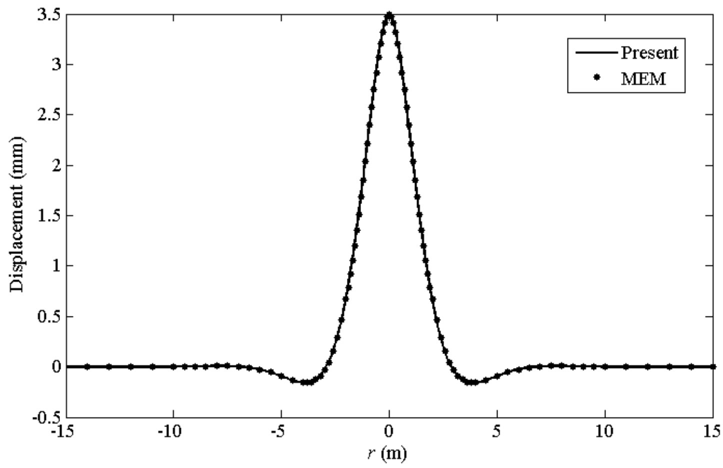

Figure 5. Steady-state vertical deflection of rail under moving load.

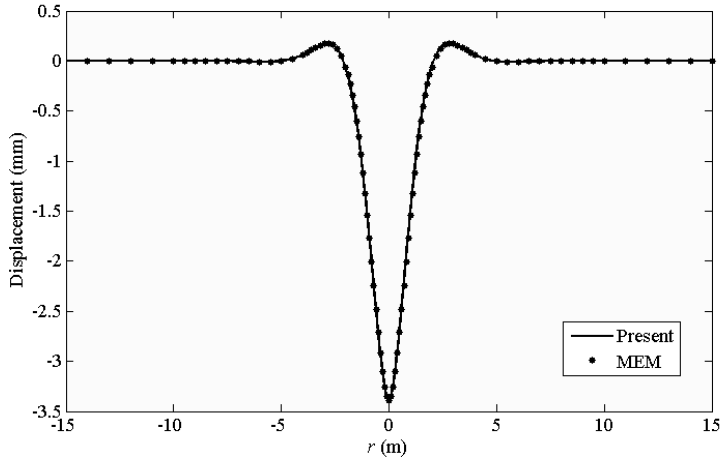

Figure 6. Steady-state radial deflection of rail under moving load.

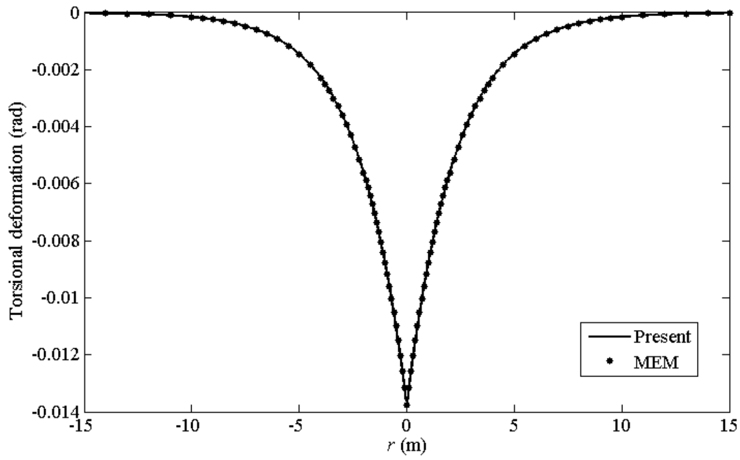

Figure 7. Steady-state torsional displacement of rail under moving load.

4 CONCLUSIONS

In this paper, an analytical solution to the response of a curved rail resting over a Winkler-foundation and subject to a moving load is presented. The boundary conditions of the curved rail are assumed to be free. Trigonometric functions are employed as mode trial functions to obtain the response of the curved rail. All results obtained using the presented method are compared against available solution in the literature or results generated by the moving element method using piecewise straight beam elements. Excellent agreements are found for all numerical comparisons.

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