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VIBRATION OF NON-PRISMATIC SIMPLY SUPPORTED BEAMS UNDER MOVING LOADS: CANCELLATION OF RESONANCES

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Abstract. *The dynamic behaviour of simply supported beams of non-uniform cross-sections using mode superposition method is investigated in this paper. The resonances of simply supported beams subjected to moving loads can be suppressed by varying either the span length, cross section of the beam or the vehicle speed. The present analysis is limited to a single point moving load on simply supported beams of short span. However, the results obtained in the analysis can be applied in the design of railway bridges which are subjected to a series of moving loads. Dynamic responses of both lightly damped and heavily damped structures are studied. The analysis can be extended to predict the dynamic responses of bridges of both uniform and non-uniform cross-sections due to passage of trains when the exact values of the system parameters are known. The objective of the present work is to determine the conditions for cancellation of resonances, thus causing minimum dynamic response and providing maximum safety for a given range of operating speeds and span lengths. Though similar analyses have been done by different researchers, the prediction of the conditions for resonance cancellation in beams with non-uniform cross-sections has not been reported.*

1 INTRODUCTION

The dynamic analysis of railway bridges has been an interesting area of research to structural engineers for several years. With the technological advancements related to high speed trains, the topic continues to be a very significant one as more and more research works are being done in this field. Since most of the railway bridges are subjected to excitation due to high speed trains, the structural integrity of bridges is very important; this will ensure smooth passage of trains, which in turn will guarantee passengers' safety and ride comfort.

A number of research papers have been published in the field of railway bridge dynamics. Many of them model the bridge as a simply supported beam under moving point loads. An analytical solution for an Euler-Bernoulli beam traversed by a succession of massless point loads has been given by Savin [1]. Yeong et al. [2] investigated the vibration of simple beams due to trains moving at high speeds, identified the various factors that govern the dynamic behaviour of bridges and made an optimal design. Yau and Yang [3] studied the vertical acceleration response of a simple beam traversed by a series of equally spaced moving loads at constant speeds. They established that the contribution of higher modes to acceleration cannot be neglected for beams with light damping. The free vibrations caused by a single moving load are found to be maximum for certain speeds and null for some other speeds. A new approximate formula for calculating the maximum vertical acceleration caused by resonances of the fundamental mode has been proposed by Museros et al. [4].

In the present paper, analysis of a simply supported non-uniform beam under a single moving point load is considered. Both free and forced vibration responses are studied and compared with those of a uniform beam. The effect of speed, damping and beam cross-section on the responses is also investigated. Modal analysis of different types of non-uniform beams is done using ANSY to determine their natural frequencies. Similar analysis has been reported by Museros et al. [4], but the effect of damping and non-uniformity of the beam were not considered. The forced vibration analysis of a non-uniform beam using Galerkin method has been done by Fryba [5]. However, the effect of free vibration and the cancellation effects for a non-uniform beam have not been found in the literature.

2 MATHEMATICAL FORMULATION

2.1 Simply supported beam with single moving load

2.1.1 Forced vibration

The equation of motion of a simply supported beam of non-uniform rectangular cross-section traversed by a single force P at constant speed v is given by

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right] + c\frac{\partial w(x,t)}{\partial t} + \mu(x)\frac{\partial^2 w(x,t)}{\partial t^2} = P\delta(x - vt) \tag{1}$$

The depth $h(x)$, mass per unit length $\mu(x)$ and moment of inertia of the cross-section $I(x)$ are given by

$$h(x) = h_o\left(1 + \sin\frac{\pi x}{L}\right)$$

$$\mu(x) = \mu_o \left(1 + \sin\frac{\pi x}{L}\right)$$
$$I(x) = I_o \left(1 + \sin\frac{\pi x}{L}\right)^3$$

where L is the length of the beam, x is the distance from one support in the direction of motion, h_o, I_o and μ_o represent the height, the moment of inertia and the mass per unit length of the beam at $x = 0$. E is Young's modulus, $w(x,t)$ is transverse deflection of the beam, c is the damping coefficient and δ is the dirac delta function.

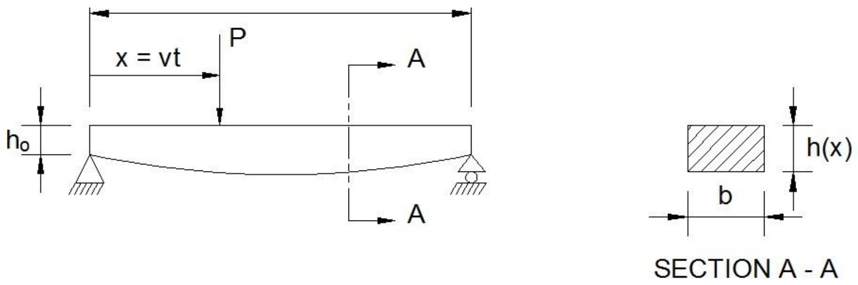


Figure 1: A simply supported beam of non-uniform cross-section with a moving load P

The solution can be assumed in the following form

$$w(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin\frac{n\pi x}{L} \tag{2}$$

where $q_n(t)$ represents the generalized coordinate with n^{th} vibration mode of the beam and $\sin(n\pi x/L)$, the corresponding mode shape. For a simple beam subjected to moving loads, only the first mode is important for the determination of deflection [6]. So, considering only the first mode and applying the orthogonality condition of modal vectors, that is by multiplying both sides of Eq. 1 by $\sin(\pi x/L)$ and integrating from 0 to L, the generalized equation of motion of the beam for 1^{st} mode can be formulated as

$$\ddot{q}_1(t) + 2\zeta_1\omega_1\dot{q}_1(t) + {\omega_1}^2 q_1(t) = \frac{2P}{\mu_o L}\frac{1}{1 + 8/(3\pi)}\sin\frac{\pi v t}{L} \tag{3}$$

where the square of the first natural frequency is given by the expression

$$\omega_1^2 = \frac{3.5\pi^4}{L^4}\frac{EI_o}{\mu_o} \tag{4}$$

and ζ_1 is the damping factor corresponding to the fundamental mode.

Eq. 3 is of the form

$$\ddot{q}_1(t) + 2\zeta_1\omega_1\dot{q}_1(t) + {\omega_1}^2 q_1(t) = F_o \sin\Omega t \tag{5}$$

where $F_o = 2P/\mu_o L(1 + 8/(3\pi))$ and $\Omega = \pi v/L$. For a uniform beam these values are $F_o = 2P/\mu_o L$ and $\omega_1^2 = \pi^4 EI_o/\mu_o L^4$.

Hereafter, the subscript 1 is removed from all the parameters, as the analysis is limited to the fundamental mode of vibration. For zero initial conditions, the solution of Eq. 5 is given by

$$q_f(t) = \frac{F_o}{\omega^2\sqrt{(1-K^2)^2 + (2\zeta K)^2}}\left[\sin(K\omega t) - \frac{K}{1-K^2}e^{-\zeta\omega t}\sin(\omega\sqrt{(1-\zeta^2)}t)\right] \qquad t < L/v \quad (6)$$

where $K = \Omega/\omega$ is the non-dimensional speed and the subscript f indicates forced vibration. From Eq. 2, the forced vibration response is given by

$$w_f(x,t) = \frac{F_o}{\omega^2\sqrt{(1-K^2)^2 + (2\zeta K)^2}}\left[\sin(K\omega t) - \frac{K}{1-K^2}e^{-\zeta\omega t}\sin(\omega\sqrt{(1-\zeta^2)}t)\right]\sin(\pi x/L) \quad (7)$$

for $t < L/v$.

For simulation purposes, a simply supported bridge as mentioned in [4] of span $L = 27$ m, mass $m = 15,000$ kg/m and fundamental natural frequency $\omega = 43.98$ rad/s is considered. The structural damping for this beam is $\zeta = 0.5\%$ as per Eurocode 1. A non-uniform beam of same length and with its height $h(x)$ varying sinusoidally is also considered. The natural frequency of the non-uniform beam is 1.871 times that of uniform beam as per Eq. 4. Figure 2 shows the forced response of beams subjected to a moving point load and compares the responses

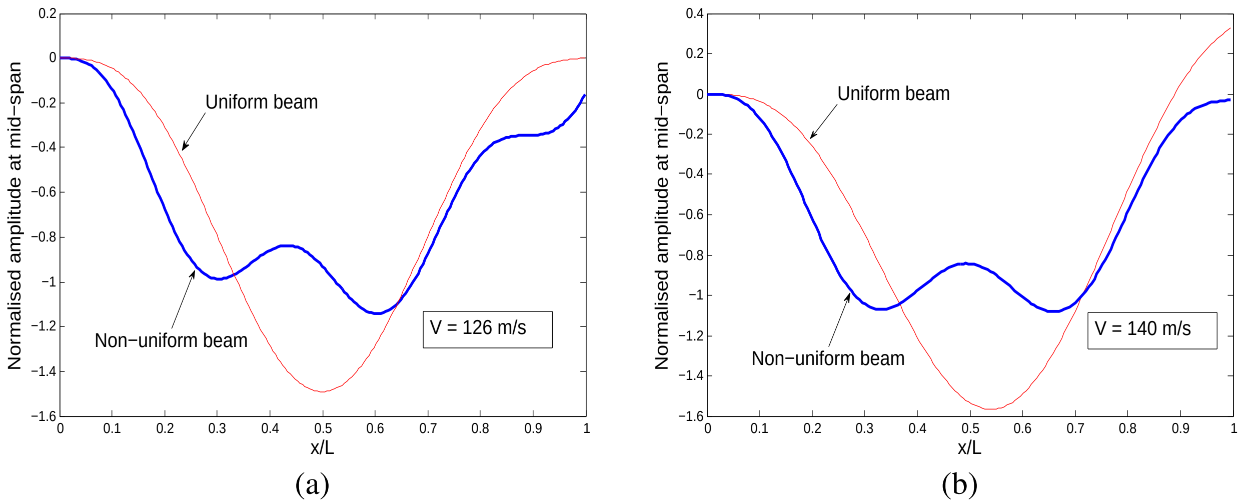


Figure 2: Comparison between the forced responses of a uniform and non-uniform beam at (a) speed,V = 126 m/s and (b) speed,V= 140 m/s.

of uniform and non-uniform beams. At a speed of 126 m/s, the deflection at the mid-span of the uniform beam is found to be maximum when the load position is exactly at the middle of the beam, whereas in the case of non-uniform beam, the maximum response is at $x = 0.6L$ in the direction of motion. However, the uniform beam causes a very small free vibration response as seen from the slope at $t = L/V$ [see Figure 2(a)] , which corresponds to the initial velocity for free vibration to start and is almost zero in this case. Thus, a uniform beam cancels the free vibration amplitudes at this speed. From Figure 2(b) it can be observed that when the moving load speed is 140 m/s, the deflection due to forced vibration is lower in the case of non-uniform beam and it occurs when the load position is at $x = 0.66L$. The maximum deflection for uniform beam is at the load position $x = 0.54L$. Since the slope at $t = L/V$ is almost zero for non-uniform beam, the free vibration response will be minimum in this case. Resonance is cancelled in the case of non-uniform beam, whereas a uniform beam causes higher free vibration amplitudes because of the non-zero initial conditions for free vibration. That is, resonance cancellation depends both on the moving load speed and beam geometry.

2.1.2 Free vibration

In the dynamic analysis of beams, free vibration analysis is very important as it plays a major role in determining the modal parameters such as structural damping. The beam is under forced vibration as long as the moving load is on the beam (for $t < L/v$) and subjected to free vibration once it leaves the beam. Free vibration analysis helps to identify the speed with which the vehicle should traverse the bridge so that there is no free vibration response. The response of the beam governed by free vibration for $t > L/v$ is as follows:

$$q(t) = \left[x_o \cos \omega_d t + \left(\frac{v_o + \zeta \omega x_o}{\omega_d}\right) \sin \omega_d t\right] e^{-\zeta \omega t} \tag{8}$$

where ω_d is the damped natural frequency, x_o is the initial displacement and v_o is the initial velocity. These initial conditions are given by the displacement and velocity values of forced responses at $t = L/v$. That is $x_o = q_f(t = L/v)$ and $v_o = \dot{q}_f(t = L/v)$.

$$x_o = -\frac{F_o}{\omega^2 \sqrt{(1-K^2)^2 + (2\zeta K)^2}} e^{-\zeta \pi / K} \sin(\frac{\pi}{K}\sqrt{1-\zeta^2}) \tag{9}$$

$$v_o = \frac{F_o}{\omega^2 \sqrt{(1-K^2)^2 + (2\zeta K)^2}} \Big\{ K\omega \cos \pi - K\omega e^{-\zeta\pi/K} \cos(\frac{\pi}{K}\sqrt{1-\zeta^2}) + \frac{\zeta K \omega}{\sqrt{1-\zeta^2}} e^{-\zeta\pi/K} \sin(\frac{\pi}{K}\sqrt{1-\zeta^2}) \Big\} \tag{10}$$

Substituting the values of x_o and v_o from Eqs. (9) and (10) in Eq. 8 and using Eq. 2 we get,

$$q(t) = Ae^{-\zeta \omega t} \sin(\omega_d t - \phi) \tag{11}$$

where

$$A = \frac{F_o K e^{-\zeta \omega t} \sqrt{1 + e^{-2\zeta\pi/K} - 2e^{-\zeta\pi} \cos((\pi/K)\sqrt{1-\zeta^2})}}{\omega^2 \sqrt{(1-K^2)^2 + (2\zeta K)^2} \sqrt{1-\zeta^2}} \tag{12}$$

and

$$\phi = \tan^{-1} \left\{ \frac{e^{-\zeta\pi/K} \sin(\frac{\pi}{K}\sqrt{1-\zeta^2})}{\cos \pi - e^{-\zeta\pi/K} \cos(\frac{\pi}{K}\sqrt{1-\zeta^2})} \right\} \tag{13}$$

So, the free vibration response of a beam that is traversed by a point load is given by

$$w(x,t) = \frac{F_o K e^{-\zeta\omega t}\sqrt{1+e^{-2\zeta\pi/K} - 2e^{-\zeta\pi}\cos((\pi/K)\sqrt{1-\zeta^2})}}{\omega^2\sqrt{(1-K^2)^2+(2\zeta K)^2}\sqrt{1-\zeta^2}} \sin(\pi x/L)e^{-\zeta\omega t}\sin(\omega_d t - \phi) \tag{14}$$

The above equation can be used to determine the response of a beam, irrespective of whether it is lightly damped or highly damped, whereas in most of the works reported in the literature only lightly damped beams are considered. Since F_o/ω^2, in the case of beams, is the static deflection, the amplitude can be normalised. For a uniform beam, the normalised amplitude for n^{th} mode as reported in [4] is

$$R_n = \frac{K_n\sqrt{2}}{1-K_n^2}\sqrt{1-\cos n\pi \cos\frac{n\pi}{K_n}} \tag{15}$$

For zero damping and for the fundamental mode, Eq. 11 gets simplified to the same form as given in Eq. 15.

$$A\omega^2/F_o = R = \frac{K\sqrt{2}}{1-K^2}\sqrt{1-\cos\pi\cos\frac{\pi}{K}} \tag{16}$$

Free vibration responses of a simply supported beam of uniform cross-section are shown in Figure 3. The first, second and third cancellation speeds (non-dimensional) for the first mode are found to be 0.333, 0.2 and 0.143 as shown in Figure 3(a). Responses for the first three modes are shown in Figure 3(b). It is seen that the amplitude is largest for the fundamental

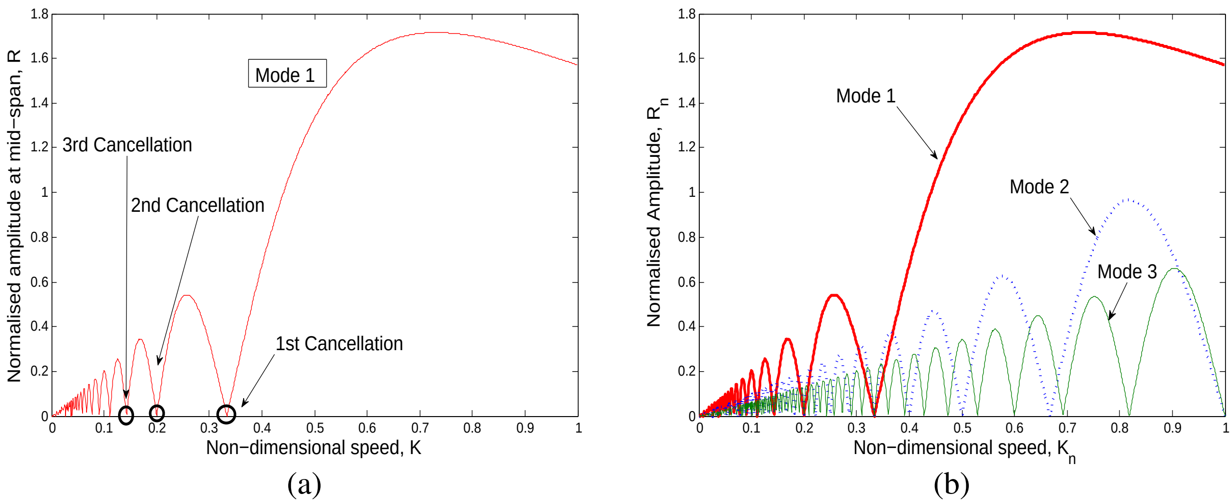
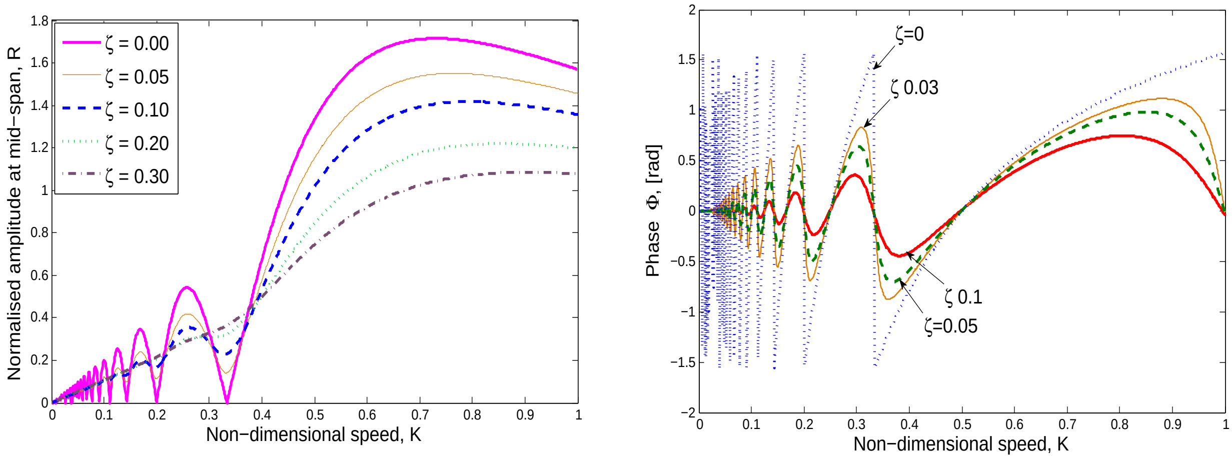


Figure 3: Undamped free vibration response of a uniform beam for (a) fundamental mode and (b) first three modes

mode compared to the second and third modes. Damped free vibration response of the same beam is shown in Figure 4(a). It shows that damping reduces the amplitude at resonance speeds, whereas it increases the same at cancellation speeds. For a damping ratio of 0.3, the response is almost linear upto the first cancellation speed. Figure 4(b) shows the variation of phase angle for free vibration with speed. For undamped free vibration at resonance cancellation speed there is an abrupt phase change, while for damped cases, it is continuous.



(a) (b)

Figure 4: Effect of damping on (a) amplitude and (b) phase of free vibration response

In Figure 5, the free responses of a uniform beam and a non-uniform beam are shown. The uniform beam is the Type-1 beam shown in Figure 7 and the non-uniform beam is the one shown in Figure 1 with sinusoidally varying height $h(x)$. For the non-uniform beam natural frequencies have been computed analytically as well as by finite element method. Eq. 4 developed by Fryba [5] shows that the fundamental natural frequency, ω of the non-uniform beam is

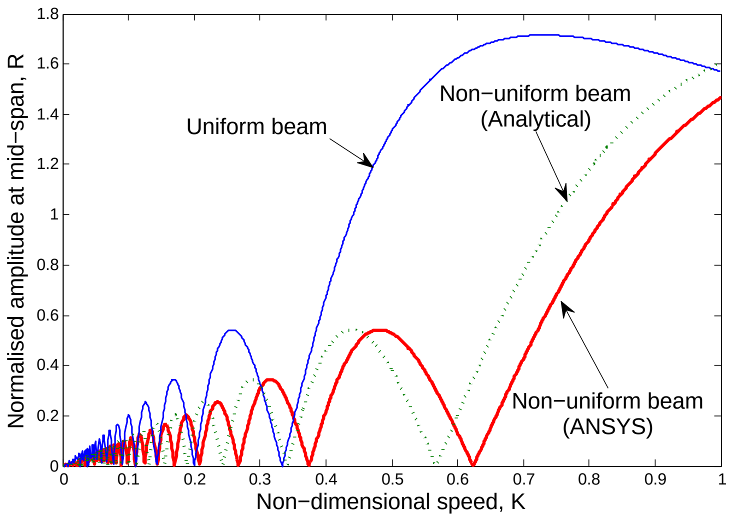
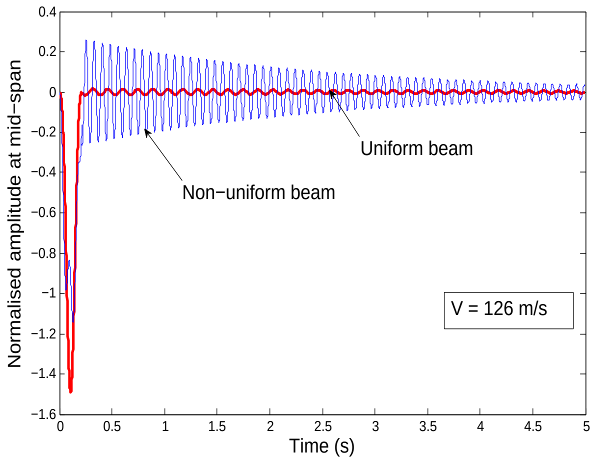


Figure 5: Resonance cancellation at different speed ratios for the fundamental mode

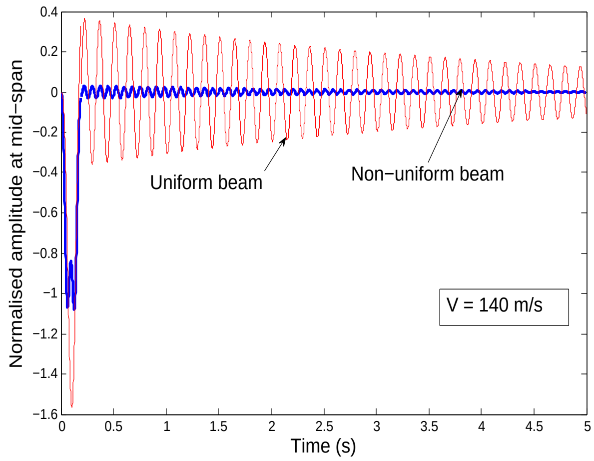
1.871 times that of uniform beam. However, the finite element analysis of a 10 m $\times$ 2 m$\times$ 1 m steel, non-uniform beam on ANSYS using solid 10 node 187 elements shows that ω is only 1.703 times that of the corresponding uniform beam. Hence, free vibration response of the non-uniform beam is determined with the ω obtained from the modal analysis results of ANSYS.

It can be observed that the non-uniformity of the beam shifts the first cancellation speed from K = 0.333 to K = 0.6226 and thus a higher speed with resonance cancellation is possible if a non-uniform beam is used instead of a uniform one. This is highly desirable for the dynamics of railway cars moving at high speeds on bridges.

The time history of forced-free vibration responses of uniform and non-uniform beams are shown in Figure 6. These are the same beams of length 27 m whose forced vibration responses with load position have been described in Figures 2(a)and 2(b). Time history is plotted for two different speeds. The first speed is 126 m/s. This causes free vibration resonances in the non-uniform beam and reduces the same to a great extent in the uniform beam as shown in Figure 6 (a). However, at a speed of 140 m/s, the resonance is almost cancelled in the non-uniform beam, while in the uniform beam there are resonances [see Figure 6(b)]. So, the resonance cancellation depends on the type of beams as well as the speed with which the load is traversed.



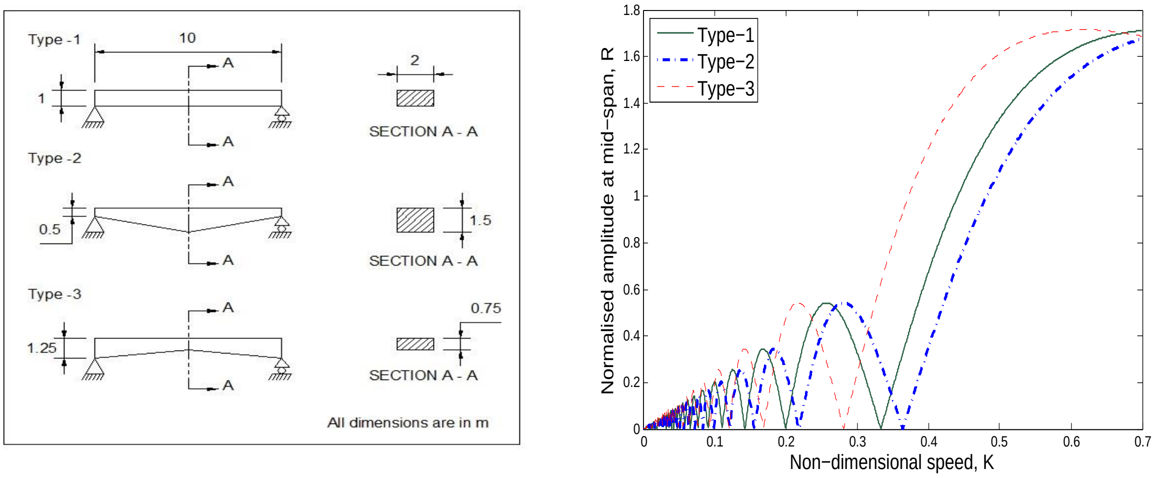
(a)



(b)

Figure 6: Forced and free vibration time history of simply supported uniform and non-uniform beams at two different load speeds (a) V= 126 m/s and (b) V= 140 m/s.

Now, to study the effect of geometry on the free vibration response, three different types of beams made of steel with same length as shown in Figure 7 (a) are considered. The geometry is chosen in such a way that the total volume and hence the mass of all the three types of beams remain the same. Type-1 is a uniform beam, whereas Type-2 and Type-3 are of non-uniform cross-section. Since analytical formulation is difficult for Type-2 and Type-3 beams, finite element analysis using ANSYS is done to determine the natural frequency. Figure 8 shows the modal analysis results of the beams using ANSYS. Solid 10 node 187 elements are used for the analysis. The first natural frequencies obtained for Type-1, Type-2 and Type-3 beams are 21.8748 Hz, 23.7612 Hz and 18.4515 Hz respectively. The fundamental natural frequency is highest for Type-2 beam and lowest for Type-3 beam. With these values of natural frequencies and using Eq. 15, the undamped free vibration responses of the beams for their fundamental modes are plotted as shown in Figure 7 (b). The range of the non-dimensional speed, K used along X-axis is from 0 to 0.7 because, for practical purposes and for short spans, the maximum value of K is not expected to be above 0.55 [4]. It is seen that in the case of Type-2 beam, the first cancellation speed is K = 0.3644 which is higher than that for Type-1 (K = 0.333). For Type-3



(a) (b)

Figure 7: (a) Effect of geometry on the free vibration amplitude, (b) beams of different geometry

beam, the first cancellation speed is K = 0.281, the minimum of the three beams. That is, for the same mass, different geometry provides different natural frequency because the distribution of mass along the span varies with geometry. As the speed ratio depends on the natural frequency, the geometry of the beam affects the resonance and cancellation speeds. Thus, by optimising the geometry of the simply supported beams, higher natural frequencies can be obtained which in turn will enable higher speeds of the moving loads on the beams.

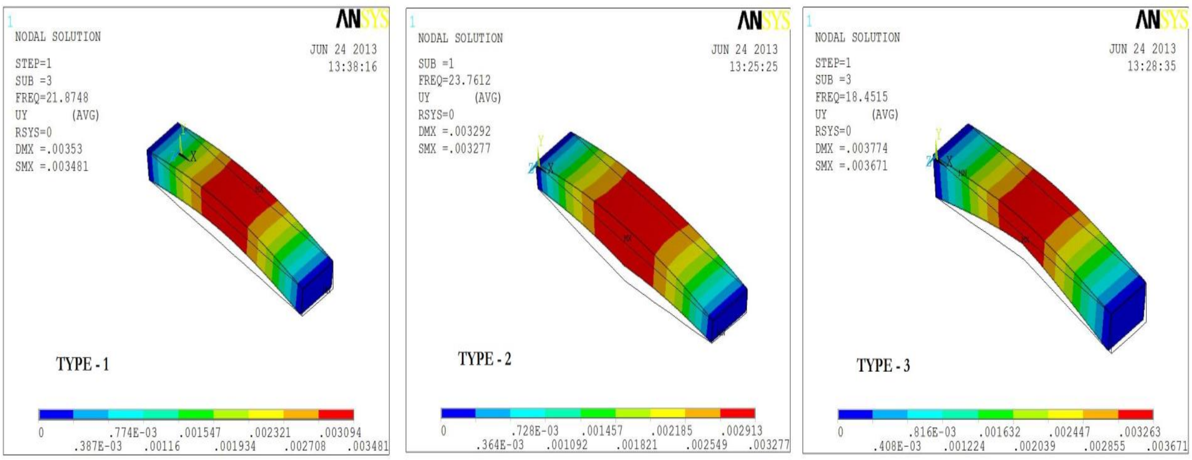


Figure 8: Fundamental modes of vibration of Type-1, Type-2 and Type-3 beams (ANSYS results)

3 CONCLUSIONS

- Free vibration analysis of beams under moving loads helps to identify the speeds at which resonances and cancellation of resonances take place.
- Non-uniformity of the beam cross-section alters the speeds at which resonance cancellation takes place.
- By choosing a geometry that increases the natural frequency of beams, higher operating speeds can be achieved.
- For highly damped structures subjected to moving loads, there is no cancellation of resonances.
- Though the present analysis is only for a single moving point load, the results can be used in railway bridgeanalysis by extending it to a series of moving loads.

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