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# NONLINEAR BALL AND BEAM CONTROL SYSTEM IDENTIFICATION

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***Abstract.*** *The Ball and Beam system is a common didactical experiment in control laboratories that can be used to illustrate many different closed-loop control techniques. The plant itself is subjected to many nonlinear effects, which the most common comes from the relative motion between the ball and the beam (assumed normally as rolling without slipping). However, many other nonlinear effects, such as beam flexibility, ball slip, actuator elasticity, shock at the end of the beam, to name a few, are present. Besides that, the system is naturally unstable.*

*The actuator consists of a (rubber made) belt attached at the free ends of the beam and connected to a DC motor. For heavy balls and fast beam movements, the elasticity of the actuator (belt) starts to influence the dynamical behavior of the system, impairing the performance of closed-loop control systems (that do not consider these effects at the design phase). In this work, the ball and beam system is modeled considering the nonlinear actuator dynamics (nonlinear elasticity of the belt). Also, some considerations about other nonlinearities, such as the effect of collisions at the end of the beam, are made. The elastic coefficients of the belt are experimentally identified, as well as the collision coefficients. The nonlinear behavior of the system is studied and a control strategy is proposed.*

## 1 INTRODUCTION

The ball and beam is a common didactical plant in many control laboratories around the world, as it is very nonlinear, unstable, which means that it is difficult to control, and can be a benchmark for testing several advanced control techniques [2]. In Figure 1, we present a photo of the complete ball and beam system, with the mechanical plant and the power module / control [1]. The ball's position is measured by a camera and a microprocessor, that calculates, based on the image acquired in real time, the distance from the left end of the beam. Other possible implementations include ultrasonic sensors to measure the ball's position [17]. The beam's angular position is also measured by an optical encoder, but a tacho generator could be also used. A current signal is sent to the DC motor, that is the system actuator and causes the beam to rotate around its center by means of an elastic belt. The system is then a single-input-single-output (SISO) system. All the signals are voltage signals between zero to 10 volts. The paper is organized as follows: In section 2, a mathematical model is obtained by Newton's method [6], which needs to know the friction forces between the ball and the beam In section 3, the model's parameters are identified. In section 4, we present a control design techniques for the ball and beam system, which is the H infinity Robust Loop Shaping. Experimental results for a pole placement design, which is very simple to obtain, are also presented for a comparative analysis and in section 5, conclusions and future works are presented.

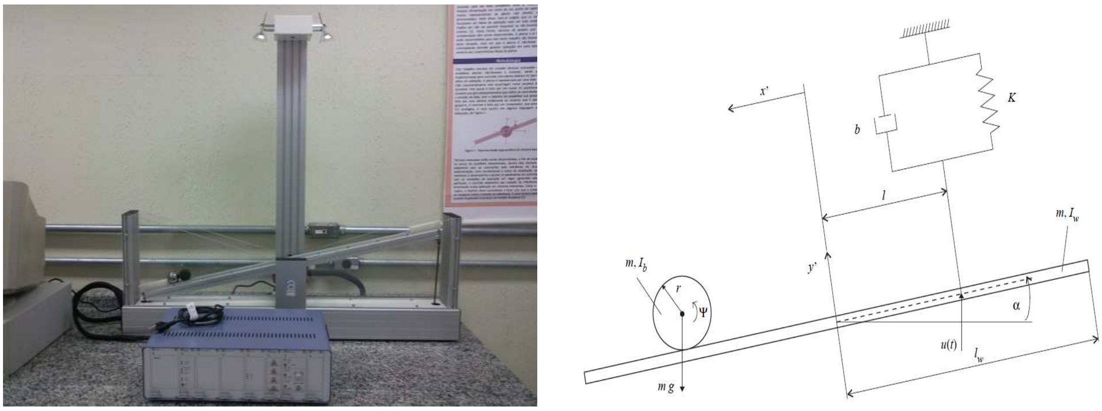


Figure 1. Ball and Beam system.

## 2 SYSTEM MODELING

The modeling of the system presented in Figure 1, in the right side, is presented in this section. We present the modeling based in the Newton's method, that is, by separating the bodies and applying the balance of forces in each body (a alternative formulation is presented in [10]). It is also possible that the ball rolls and slip at the same time for large α, but for the most part of the time, the rolling is supposed to be without slipping. In this case, the system can be considered as non-holonomic [9], which allows a reduction in the number of degrees of freedom from three to two. In the Newton's method, it is necessary to model the ball as developing its movement in a *non-inertial system*, that is fixed in the beam [7], [8]. The contact forces need to be calculated in this case.

### 2.1 Modelling by Newton's method

In a non-inertial reference frame, the Newton's Law must be properly modified, by using the concept of *total derivative* (or *covariant derivative*) which accounts for the variations of the frame of reference itself their effect in the total movement. The coordinate transformation matrix between the body fixed frame and the inertial frame is:

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \tag{1}$$

and the rotation matrix is given by $\Omega=R^{-1}\dot{R}$. This matrix represents the variation of the moving frame expressed in the moving frame base. The covariant (total) derivative can be written as:

$$D_t = \frac{\partial}{\partial t} + \vec{\Omega}\times = \frac{\partial}{\partial t} + \Omega \tag{2}$$

where $\vec{\Omega}$ is the vector equivalent to $\Omega$. The Newton's law for rotating body can be simply written as:

$$\vec{T} = D_t\vec{H} \tag{3}$$

and the Newton equation for linear motion is:

$$\vec{F} = mD_tD_t\vec{S}' = m\left(\frac{\partial}{\partial t} + \vec{\Omega}\times\right)\left(\frac{\partial}{\partial t} + \vec{\Omega}\times\right)\vec{S}' \tag{4}$$

where $S'$ is the ball's center of mass position in the moving reference frame. This is the covariant second Newton's law, that is valid for any reference frame [11]. After some manipulation, the second Newton's law can be written as:

$$m\ddot{S}' = F - m\Omega^2 S' - m\dot{\Omega}S' - 2m\Omega\dot{S}' \tag{5}$$

where $S'$ is the ball's center of mass position, $m$ is the ball's mass, and $F$ is the total force applied in the ball (expressed in the non-inertial frame of reference). Let $R$ be the orthogonal matrix transforming the non-inertial reference frame to the inertial frame (fixed in the laboratory). That is:

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \tag{6}$$

and

$$\Omega = \begin{bmatrix} 0 & -\dot{\alpha} \\ \dot{\alpha} & 0 \end{bmatrix} \tag{7}$$

**Note:** The matrix $\Omega=\dot{R}R^{-1}$ represents a different velocity, which is the angular velocity of the non-inertial frame expressed in the inertial frame. The change between non-inertial frame to the inertial (or vice-versa) is a similarity transformation, also known as adjunct transformation.

Consider now the decomposition of the system in two independent bodies (in contact by friction and normal forces), as shown in Figure 2.

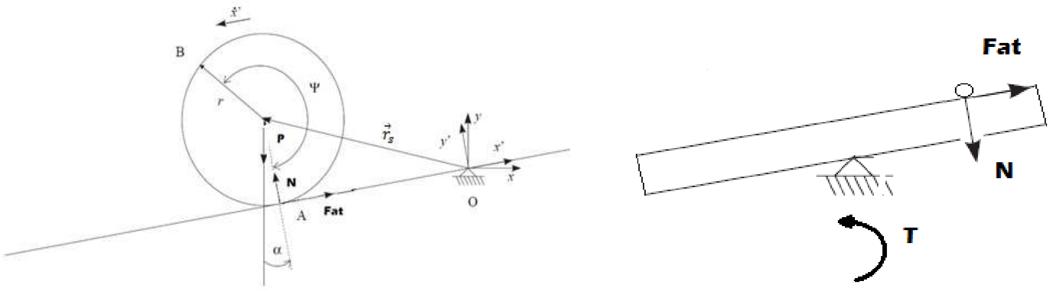


Figure 2. Decomposition of the system.

Applying the second Newton's Law in the ball alone (in the moving reference frame), we have the equation:

$$m\ddot{S}' = [F_{at} - mg\sin\alpha \quad N - mg\cos\alpha]^t - m\begin{bmatrix} 0 & -\dot{\alpha} \\ \dot{\alpha} & 0 \end{bmatrix}\begin{bmatrix} 0 & -\dot{\alpha} \\ \dot{\alpha} & 0 \end{bmatrix}S' - m\begin{bmatrix} 0 & -\ddot{\alpha} \\ \ddot{\alpha} & 0 \end{bmatrix}S' - 2m\begin{bmatrix} 0 & -\dot{\alpha} \\ \dot{\alpha} & 0 \end{bmatrix}\dot{S}' \tag{8}$$

In any situation, the relative position of the ball's center of the mass is always constant and $y' = r$, so that the $S'$ can be written as:

$$S' = [x' \quad r]^t \;,\; \dot{S}' = [\dot{x}' \quad 0]^t \text{ and } \ddot{S}' = [\ddot{x}' \quad 0]^t \tag{9}$$

so that we have:

$$[m\ddot{x}', 0]^t = [F_{at} - mg\sin\alpha, N - mg\cos\alpha]^t + [m\dot{\alpha}^2 x', m\dot{\alpha}^2 r]^t - [-m\ddot{\alpha}r, m\ddot{\alpha}x']^t - 2m[0, \dot{\alpha}\dot{x}']^t \tag{10}$$

which implies that the normal force must be:

$$N = m(g\cos\alpha - r\dot{\alpha}^2 + \ddot{\alpha}x' + 2\dot{x}'\dot{\alpha}) \tag{11}$$

and the movement of the center of mass is

$$m\ddot{x}' = -F_{at} - mg\sin\alpha + m\dot{\alpha}^2 x' + m\ddot{\alpha}r \tag{12}$$

In order to determine the relative rotational movement of the ball, described by the angle $\Psi$, we have to apply the rotational second Newton Law, in a non-inertial reference frame, which is expressed as:

$$\vec{T} = \frac{d\vec{H}}{dt} = \frac{\partial\vec{H}}{\partial t} + \vec{\Omega}\times\vec{H} \tag{13}$$

where $\vec{H}$ is the angular momentum, $\vec{T}$ is the total torque applied in the system and $\vec{\Omega}$ the angular velocity vector, that is related to the matrix $\Omega$ by a Lie Algebra isomorphism [9] and [12]. In fact, it is possible to represent all the vector quantities as anti-symmetric matrices, with the vector product substituted by the matrix Lie Bracket. As the vectors $\vec{H}$ and $\vec{\Omega}$ have the same directions, which means that their vector product is null, the equation is reduced to

$$\vec{T} = \frac{\partial \vec{H}}{\partial t} = I_b \ddot{\Psi} = rF_{at} \tag{15}$$

Considering now the torque applied by the motor in the beam (in Figure 2), and the contact normal force that opposes the movement, we have

$$I_w \ddot{\alpha} = lu(t) - x'N - bl\dot{\alpha} - k_1 l\alpha - k_2 l\alpha^3 \tag{16}$$

In which $u(t)$ is the input control and all the other torques are opposing the movement, where $k_1$ and $k_2$ are the elastic coefficient of the rubber belt (that transmit the force to the beam), $b$ is the viscous friction coefficient of the beam and $I_w$ is the beam moment of inertia.

**Note:** some details about the geometry of the beam are ignored here, and the interested reader should consult [10] for a more thorough modeling.

### 2.2 State-Space Representation

The state-space representation of the system is: defining the state variables as $x_1 = x'$, $x_2 = \dot{x}'$, $x_3 = \alpha$, $x_4 = \dot{\alpha}$, $x_5 = \Psi$, $x_6 = \dot{\Psi}$ and writing the normal force not depending on accelerations, we have:

$$N = \frac{m\left(g\cos x_3 - r\dot{x}_4 + 2x_2 x_4 - lx_1 u(t)/I_w\right)}{1 + m{x_1}^2/I_w} \tag{17}$$

the state space representation is then:

$$\dot{x}_1 = x_2 \tag{18}$$

$$\dot{x}_2 = F_{at} - mg\sin x_3 + mx_1{x_4}^2 + mr\left(\frac{lu(t) - x_1 N - blx_4 - klx_3}{I_w}\right) \tag{19}$$

$$\dot{x}_3 = x_4 \tag{20}$$

$$\dot{x}_4 = \frac{lu(t) - x_1 N - blx_4 - k_1 lx_3 - k_2 lx_3^3}{I_w} \tag{21}$$

$$\dot{x}_5 = x_6 \tag{22}$$

$$\dot{x}_6 = \frac{1}{I_w} F_{at} \tag{22}$$

### 2.3 Friction and Collision Models

The friction force $\vec{F}_{at}$ normally depends on the normal force in the ball and a friction coefficient, but this is a superior limit, as the movement really starts if the other forces (those that put the body in movement) are higher than the static friction force (that depends on the normal contact force, but with s different friction). This is the Coulomb Law. For the case of rolling movement, this friction coefficient (rolling friction coefficient) is very small (for example, for steel contact the value is around 0.003).

There was a great controversy in those kinds of friction models for rigid body dynamics. In [14], there is a complete discussion on the nature and models of friction and collision in rigid bodies. In general, all the rigid bodies involved in this didactical kit must deform, as they are elastic bodies. When the ball touches the walls at the end of the beam, the point of contact changes its velocity smoothly, and not discontinuously, as the rigid body models assumes. On the other hand, considering those bodies as such would imply in solving a set of partial differential equations, which is not reasonable in the numerical point of view.

In the rigid body approach, the changes in velocities must be discontinuous, which implies in accelerations (and contact forces) of impulsive nature. The Coulomb's law says that the friction force is limited by μN, where N is the normal force of contact, but can be lower [14]. In general, the dynamic equations of the complete system will be a discontinuous function of the very state variables, and there is no mathematical theorem that guarantees the existence of solutions. If, on the other hand, we consider the problem as a differential inclusion, in the sense of Filippov (that is, the system velocities must belong to a convex set in each point), the numerical solutions will be more realistic. In fact, the approximation of the Coulomb law by a smooth function renders not observable physical realities. It was proven that if we allow that friction forces could be infinite in some instants, there would be no inconsistence in the Coulomb model.

The impact models are the result of forces acting in a short period of time (these forces could be represented by Dirac impulse functions). On the other hand, we do not know in advance the instants of time when those impulses would occur. By using energy analysis, we can say that some impacts dissipates very little energy (that is, they are very *elastic*), while others dissipates almost all the energy (very *inelastic* impacts). So there is a *coefficient of restitution* ε, that expresses those characteristics (0 = pure inelastic, 1 = pure elastic). The impulse of the force (time integral of the force) after the impact is the impulse before the impact multiplied by ε, but this model has drawbacks. In fact, the dissipation of energy in the impact occurs in a not well known molecular/atomic mechanism. It was shown, in experiment, that ε depends on the angle in which the bodies collide.

Determining the real nature of the friction force $F_{at}$ is then a very hard task, as it involves a very complex microscopic dynamic (between the molecules of the contact surfaces). Different modes of vibration, both in microscopic and macroscopic level, could be activated.

## 3 PARAMETERS IDENTIFICATION

The following values for the model parameters were measured experimentally: 1) ball's mass = 0.267 kg, 2) length between the center of the beam and the belt = 0.50; radius of the ball = 0.017 m. In order to focus on the determination of the belt's elastic coefficients, we assume that the viscous friction in the beam's bearings is 1.0 Ns/m, that is the value determined by the kit's manufacturer [1].

By applying different control signals in the plant (force in the CC motor that moves the belt), one can estimate the value of the elastic coefficients $k_1$ and $k_2$. The initial idea was to use a PRBS type signal (pseudo-random binary sequence) which is a *persistently exciting* signal [5]. On the other hand, a signal of this type could have a mean value not zero, and the beam collides most of the time. The only possible solution is to apply a periodic signal, that is not so persistently exciting. By comparing the experimental results, one could estimate the values of the elastic coefficients as $k_1 = 0.00092$ N/m and $k_2 = 0.00000332$ .

By using the kit itself, we can estimate the ball's velocity right before and right after a collision between the ball and the beam's end-of-course (acrylic). After some time (exciting the DC motor with a variable sinusoidal current) several collisions occurred, and the mean value of the coefficient (that is the ratio between the right after and right before velocities) determined was ε = 0.611.

## 4 H INFINITY LOOP SHAPING CONTROL DESING

There are a vast number of control design techniques that could be used in this system, from the simple classic controllers [19] till advanced nonlinear controllers [3], [4]. We chose, on the other hand, to use a control design technique of intermediary complexity (speaking of design phase). This technique is the H infinity loop shaping design, presented for example in [18], [20]. In this technique, the performance requirements are also specified in the open loop singular values plots. The robustness index $\gamma$ for this design technique have the same role as the phase margin for a SISO classical design, but does not suffer from the same drawbacks. We have to fix this parameter in order to start the design. The chosen robust stability index must satisfy:

$$\left\| \begin{bmatrix} K_L(s) \\ I \end{bmatrix} (I - G(s)K_L(s))^{-1} M_L^{-1} \right\|_{\infty} \leq \gamma \tag{17}$$

The plant's transfer matrix $G(s) = M_L(s)^{-1}N_L(s)$ must be decomposed in a product of two co-prime transfer matrices, and it must be multiplied by the weight transfer function $W_1$ e $W_2$ that give the desired shape for the open loop singular values plots. Then the following constant is calculated:

$$\gamma_{min} = (1 - \| [N_L(s) M_L(s)] \|_H^2)^{\frac{1}{2}} \tag{18}$$

and if $\gamma_{min}$ is too high, the open loop singular value plots are unattainable (at the same time that robustness is guaranteed) so that the weight functions have to be modified. The less the value of $\gamma$, the better the robust stability margin, and if the minimum value is achieved, we have optimal robust H infinity control.

Typical closed loop specifications for loop shaping design are in terms of the sensibility function $S(s)$ and complementary sensibility function $T(s)$, which is equal to one when added. As we want good disturbance rejection properties, the low frequency gain of $S(s)$ must be low and as we want good tracking (of reference signals) properties and rejection of noise, the $T(s)$ at low frequencies must be around one. Also, the roll-of frequency must be -20 dB/dec. The weight functions chosen were $W_1(s) = (1/s)I_{2x2}$ $W_2(s) = 2$, $W_2(s) = 5$. In order to compare the performance of the robust control with the less advanced design technique, we utilized a placement design technique [13] with discrete closed-loop poles equal to 0.9512, 0.7788, 0.4724 e 0.4724. In Figure 3, we can see the ball position for a square wave reference signal. In Figure 4, we see the force applied by the DC motor. We clearly see the oscillation pattern due to nonlinear effects (limit cycle).

In Figure 5, we see the reference signal (the same as in the previous case) and the system response. It can be seen that the performance improved due to the fact that the system is less sensitive to plant uncertainties (in this case, the uncertainties in the identified elastic coefficients). On the other hand, in some cases, the oscillatory behavior appears, but with less amplitude. In Figure 6, we see the force applied by the motor.

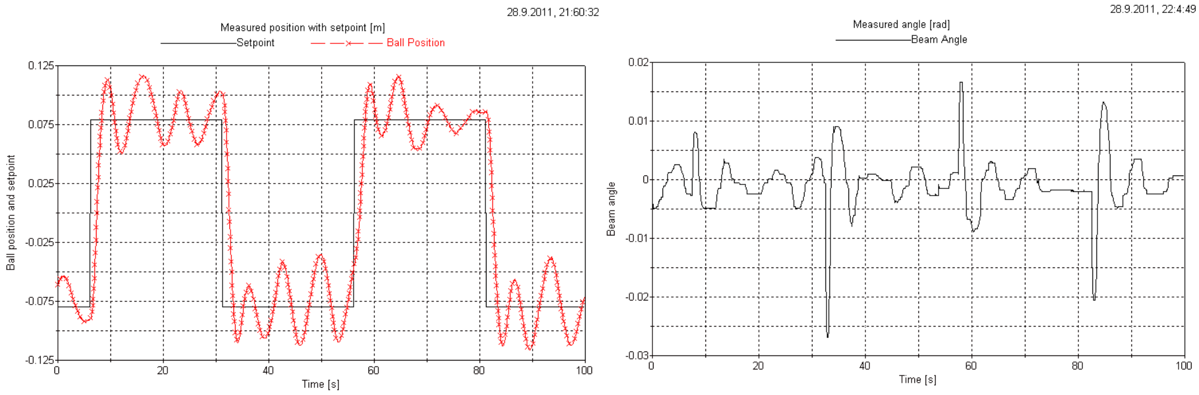


Figure 3. Ball position (real and reference) and beam's angle (in radians).

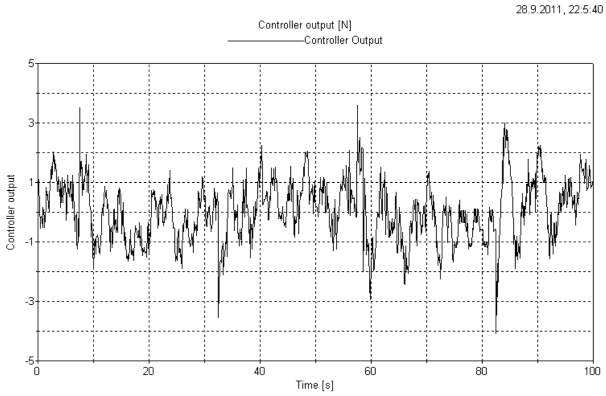


Figure 4. Force applied by the DC motor.

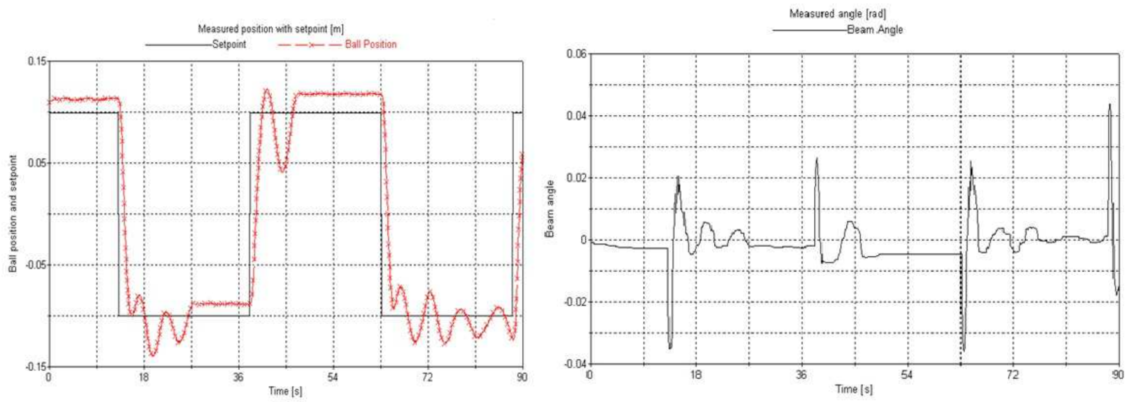


Figure 5. Ball position (real and reference) and beam's angle (in radians).

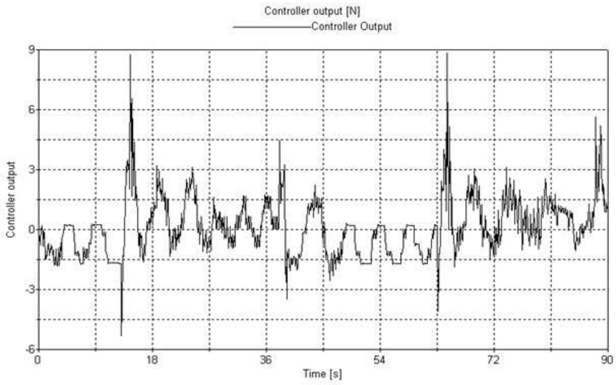


Figure 6. Force applied by the CC motor.

## 5 CONCLUSIONS

It was presented a model for the ball and beam system, obtained by the Newton's method, in which the friction forces appear explicitly. The mathematical formulation of this force is complex and frequently is discontinuous in nature, even in the case of rolling friction. The correct modeling should use elasticity of the "rigid bodies", what increases significantly the complexity of the algorithms (and the computational time). Elasticity coefficients were experimentally identified, as well as restitution coefficients from the metallic ball and the end-occourse (acrylic). It was also designed different linear control techniques for the ball and beam system. One of them did not considered robustness properties and presented poor performance (limit cycles) due to the nonlinear behavior. The other controller (H Infinity), that is robust, could deal better than the other in reducing the oscillations.

Future works includes a more thorough analysis of the friction model, including the differential inclusion numerical technique [14], and more advanced nonlinear control techniques like the Feedback Linearization [16], [4] (or even other linear robust techniques, like LQG/LTR [15]) with a second loop to provide robustness properties and the Sliding Modes technique, that is nonlinear and robust at the same time.

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