

## STABILITY AND BIFURCATION OF EQUILIBRIA IN THE PRESENCE OF NON-SMOOTH DAMPING DUE TO COULOMB FRICTION

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**Keywords:** Stability, Bifurcation, Self-Excitation, Non-Smooth System, Frictional Damping, Joint Friction.

**Abstract.** *Classical stability problems like divergence (buckling) or oscillatory instability (negative damping, flutter) are usually discussed using models which exhibit smooth, viscous damping forces. However, in practical applications often non-viscous damping forces are observed, which may stem from micro-slip in joints and give rise to non-smooth Coulomb-type damping. In contrast to forced vibrations, investigations on the effects of Coulomb-like joint damping on stability problems are surprisingly rare: therefore, this contribution presents three classical stability problems – buckling, self-excitation due to negative damping and self-excitation due to non-conservative coupling – and discusses the effects of Coulomb-damping on stability and attractivity of equilibria. For all problems, equilibrium points change into equilibrium sets. For the static buckling problem, it is found that the notion of stability is simply replaced by the property of attractiveness – the principal behaviour is not changed. In contrast, it is found that for both oscillatory problems the equilibrium set is always attractive, but the distinction between stable/unstable behaviour is replaced by the distinction between infinite/finite extent of the basin of attraction.*

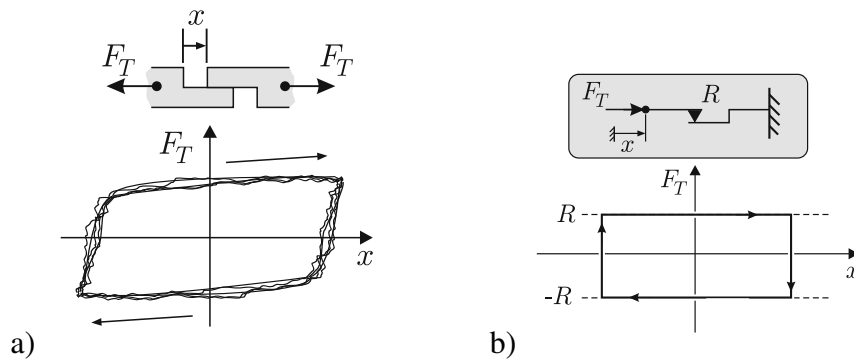


Figure 1: Joint forces: a) Abstract example. b) Modelling using Coulomb friction element.

## 1 INTRODUCTION

### 1.1 Self-excitation in the presence of Coulomb damping

Stability problems occur in many engineering applications as well as situations of everyday life. Instabilities of divergence type often occur as buckling of structures under static load. Oscillatory instabilities (flutter) are also often observed in technical systems and are usually associated with a Hopf-bifurcation of the steady state: important examples stem from the field of friction induced vibrations or from rotordynamics [21], [2], [3]. For all aforementioned instability scenarios the effect of Coulomb damping – which for instance can be used as a simple model for the dissipation stemming from micro-slip in joints – is mostly unclear: reviewing the literature only very few publications can be found so far [20], [22], [7], [9], [8]. In contrast, literature on dynamical effects of joints in the context of forced vibrations is extremely rich (cf. [4], [13], [17], or [6] for instance). Moreover, the stability of non-smooth problems with set-valued force laws is a field of intensive research both in mechanics and control theory (a recent survey can be found in [15] for instance).

### 1.2 Coulomb damping in engineering: dissipation in Joints due to micro-slip

Usually, damping is modelled as proportional to the velocity – however, technical applications often exhibit non-viscous damping mechanisms, which for instance may stem from micro-slip in joints and interfaces between members of a system ([6], [19], [1], [16] for instance). Figure 1 a) shows a typical example of such a joint force. In a first approach such joint forces can be approximated using a Coulomb friction element (Fig. 1 b). With the relative displacement  $x$  the joint force  $F_T$  reads

$$F_T(\dot{x}) \in -R \text{Sign}(\dot{x}) \quad \text{where} \quad \text{Sign}(\dot{x}) = \begin{cases} \text{sign}(\dot{x}) & : |\dot{x}| > 0 \\ [-1, ..1] & : \dot{x} = 0 \end{cases} \quad (1)$$

is the convex closure of the sign-function.

## 2 DIVERGENCE INSTABILITY – BUCKLING

In statics equilibria may lose stability and become unstable due to divergence. Such types of instability often occur in buckling of structures for instance. The influence of Coulomb-type damping on simple buckling problems is discussed for the simple text-book example shown in figure 2 a): it consists of a rigid rod (length  $\ell$ , mass moment of inertia w.r.t. point A) which is elastically hinged at point A (torsional spring stiffness  $c_T$ ). At the tip of the rod a constant load

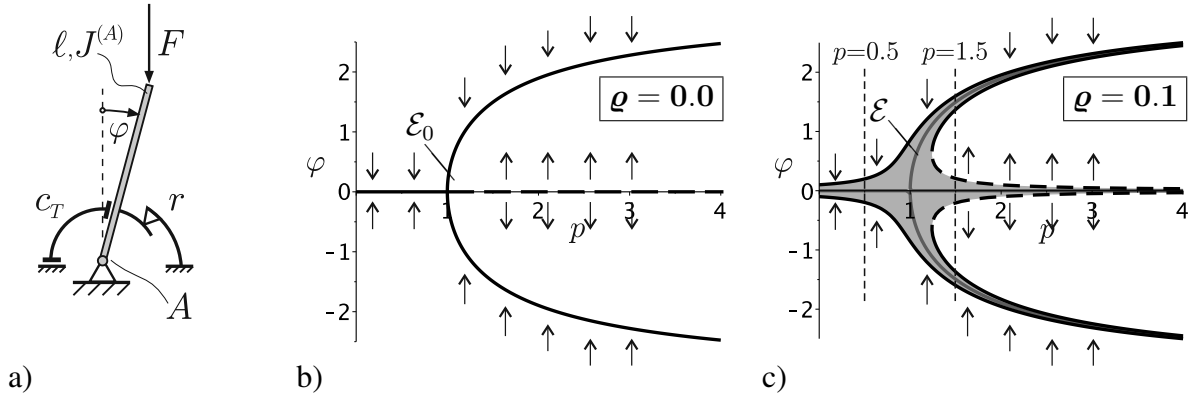


Figure 2: a) Simple buckling model. b),c) Pitchfork - bifurcations (buckling) for  $\varrho = 0$  and  $\varrho > 0$ .

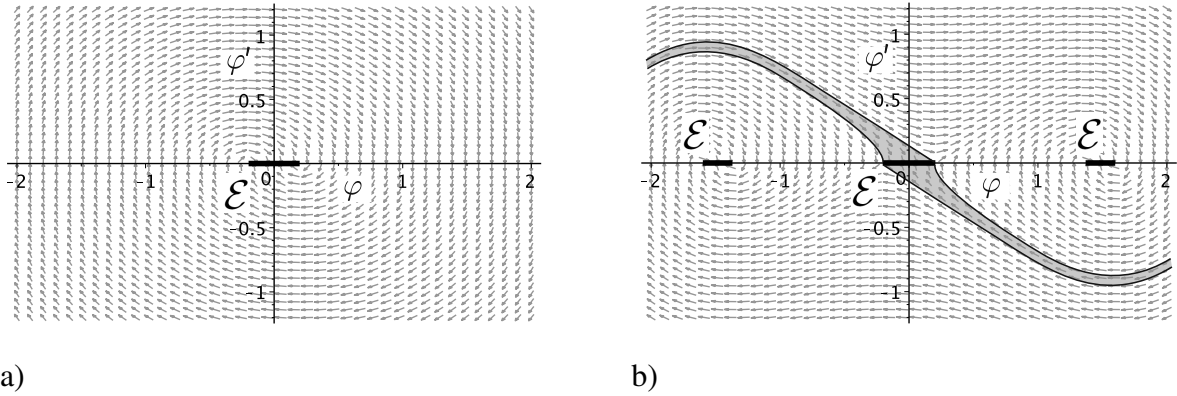


Figure 3: Phasefield and equilibria: a)  $p = 0.5$ . b)  $p = 1.5$  (trajectories in the shaded area converge to  $\mathcal{E}$ ).

$F$  applies. This classical setup is extended by a frictional torque  $M_R = -r\text{Sign}(\dot{\varphi})$  at  $A$ . Using the time scaling  $\tau = \omega_0 t$  (where  $\omega_0 = \sqrt{c_T/J^{(A)}}$ ) the equation of motion reads

$$\varphi'' + \varphi - p \sin \varphi \in -\varrho \text{Sign}(\varphi') \quad (2)$$

where  $p = F\ell/c_T$ ,  $\varrho = r/c_T$  and  $()' = d()/d\tau$ . Existence of  $C^0$  continuous solutions in forward time follow from compactness and upper semi-continuity of the right hand side. Uniqueness is assured since trajectories cross the switching surface  $\varphi = 0$  in terms of transversal crossings or attractive sliding (cf. fig. 3) [15]. The set of equilibria is given by

$$\mathcal{E} = \{(\varphi_0, \varphi' = 0) \mid \varphi_0 - p \sin \varphi_0 \in -\varrho[-1, 1]\}. \quad (3)$$

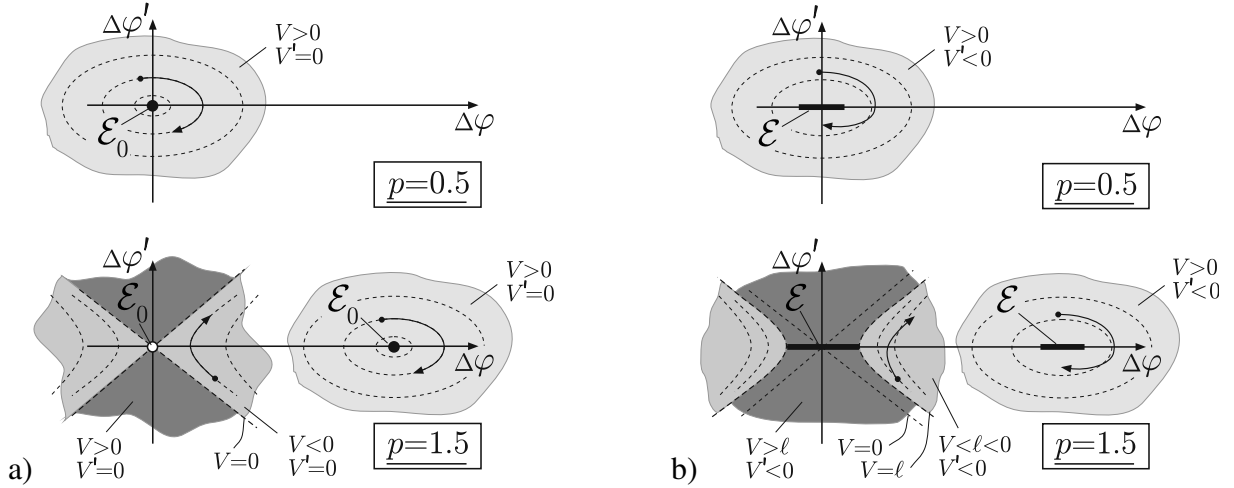
For vanishing Coulomb damping the equilibrium set  $\mathcal{E}_0 = \mathcal{E}|_{\varrho=0}$  of the corresponding smooth problem is obtained. These sets are outlined in figure 2 b) and c). Stability and attractivity can easily be investigated using Lyapunov's direct method and LaSalle's Invariance Principle.

For an equilibrium solution  $\varphi_0$ , one may introduce the coordinate shift  $\varphi = \varphi_0 + \Delta\varphi$ , yielding

$$\Delta\varphi'' + \varphi_0 + \Delta\varphi - p \sin(\varphi_0 + \Delta\varphi) = -\varrho \text{Sign}(\Delta\varphi'). \quad (4)$$

Taylor expansion about  $\varphi_0$  and assuming that  $\varphi_0 \in \mathcal{E}_0$  is an equilibrium of the smooth system ( $\varrho = 0$ ) yields

$$\Delta\varphi'' + (1 - p \cos \varphi_0)\Delta\varphi + (\dots \text{h.o.t.} \dots) \in -\varphi \text{Sign}(\Delta\varphi'). \quad (5)$$


 Figure 4: Equilibria, trajectories and iso-levels of  $V$ .

As for the smooth problem, a Lyapunov function may be constructed according to

$$V(\varphi_0) = \frac{1}{2}\Delta\varphi'^2 + \frac{1}{2}(1 - p \cos \varphi_0)\Delta\varphi^2 + \int (\dots \text{h.o.t.} \dots) d\Delta\varphi \quad (6)$$

whose time derivative along solutions  $(\Delta\varphi, \Delta\varphi')$  reads

$$V' = -\varrho|\Delta\varphi'| \leq 0. \quad (7)$$

From this the following results on stability or attractivity of the equilibria may be given: For the smooth problem  $\varrho = 0$  (cf. fig. 4 a)  $V' = 0$  for all parameter values. For  $p < 1$  only one equilibrium  $\varphi_{01} = 0$  exists. The corresponding Lyapunov-function  $V(\varphi_{01} = 0) > 0$  is positive definite and thus this equilibrium is stable (Theorem of Lyapunov). For  $p > 1$  three equilibria  $\varphi_{01} = 0$  and  $\varphi_{02} = -\varphi_{03} > 0$  exist. The Lyapunov-function  $V(\varphi_0 = 0)$  of the center equilibrium now is no longer positive definite and this equilibrium is unstable (Theorem of Chetayev). The Lyapunov functions  $V(\varphi_{02/03}) > 0$  are positive definite within a non-empty region about  $\varphi_{02/03}$  and therefore these solutions are stable.

For the non-smooth problem  $\varrho > 0$  (cf. fig. 4 b)  $V' = -\varrho|\Delta\varphi'| \leq 0$  is negative semi-definite for all values of  $p$ . For  $p < 1$  only one set of equilibria exists, which is centered around  $\varphi_{01} = 0$ . Since the corresponding function  $V(\varphi_{01} = 0) > 0$  is positive definite this equilibrium is attractive (LaSalle's Invariance Principle, App. A). For  $p > 1$  three sets of equilibria exist, which are centered around the equilibria  $\varphi_{01} = 0$  and  $\varphi_{02} = -\varphi_{03} > 0$  of the corresponding smooth system. The function  $V(\varphi_{01} = 0) \leq 0$  is no longer p.d. and instability of this center equilibrium set follows from an extension of Chetayevs theorem (App. B). Analogously, attractivity of the equilibrium sets around  $\varphi_{02} = -\varphi_{03}$  can be shown.

### 3 OSCILLATORY INSTABILITY

In oscillatory systems, steady states or equilibria may loose their stability in terms of oscillatory instabilities, which are often related to negative damping or non-conservative coupling. The effect of non-smooth damping on such types of problems has been discussed in [22], [20],[7] or [9] for instance.

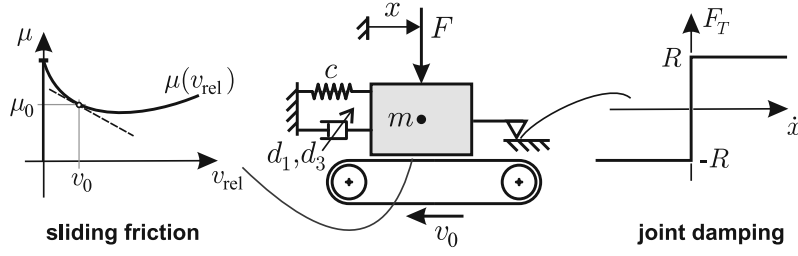


Figure 5: Engineering example for negative damping: prototype model for self-excitation due to sliding friction.

### 3.1 Negative Damping

As a typical example from engineering application, negative damping may occur in systems with self-excitation due to sliding friction when the sliding friction force has a negative gradient with respect to the relative velocity. Figure 5 displays a typical prototype model, consisting of a mass  $m$ , which is pressed onto a moving belt (belt velocity  $v_0$ ) by a vertical force  $F$ . The position of the mass is  $x$ . The mass is connected to the environment by a linear elasticity (stiffness  $c$ ) and a nonlinear damper (linear damping  $d_1$ , cubic damping  $d_3$ ). The sliding friction between the mass and the belt is described by the coefficient of sliding friction  $\mu = \mu(v_{rel})$ , which depends on the relative velocity  $v_{rel} = v_0 - \dot{x}$ . Here, this classical setup is extended by an additional Coulomb friction element as a model for joint damping. Introducing the dimensionless time  $\tau = \omega_0 t$  where  $\omega_0 = \sqrt{c/m}$ , centering the displacement about the steady state displacement  $x_0 = F\mu_0/c$  of the smooth problem and scaling it by a reference length  $L$  according to  $q = (x - x_0)/L$  yields the equation of motion

$$q'' + 2D_1 q' + D_3 q'^3 + q - \frac{F}{cL} (\mu(V_0 - q') - \mu(V_0)) \in -\varrho \text{Sign}(q') \quad (8)$$

where  $()' = d/d\tau$ ,  $V_0 = v_0/\omega_0$ ,  $2D_1 = d_1/\omega_0$ ,  $D_3 = d_3\omega_0^2$ ,  $\varrho = \frac{R}{cL}$ . The equilibrium set reads

$$\mathcal{E} = \{(q, q') \mid q \in -R\text{Sign}(0), q' = 0\}. \quad (9)$$

The equilibrium of the corresponding smooth problem simply reads  $\mathcal{E}_0 = \mathcal{E}|_{\varrho=0} = (0, 0)$ .

In the vicinity of this equilibrium set, the (smooth) sliding friction characteristic may be expanded into a Taylor series according to  $\mu(v_{rel}) = \mu_0 + \sum_k \frac{1}{k!} \frac{\partial^k \mu}{\partial (v_{rel})^k} \Big|_{v_{rel}=v_0} \dot{x}$ . It is assumed that between belt and mass stiction will not occur. Assuming that the nonlinear damping is stronger than the higher order terms stemming from this Taylor expansion, equation (8) simplifies to

$$q'' + \underbrace{\left[ 2D_1 + \frac{F}{cL} \frac{\partial \mu}{\partial v_{rel}} \Big|_{v_{rel}=v_0} \right]}_{=: 2D} q' + D_3 q'^3 + q \in -\varrho \text{Sign}(q') \quad (10)$$

where  $\varrho = R/(cL)$ .

Motivated by the smooth problem, the Lyapunov - function  $V = \frac{1}{2}q'^2 + \frac{1}{2}q^2 > 0$  may be introduced. Along solutions  $\mathbf{z}(t) = (q(t), q'(t))^\top$  the evolution of  $V$  reads

$$V' = -2Dq'^2 - D_3q'^4 - \varrho|q'|. \quad (11)$$

First, stability and attractiveness of the equilibria  $\mathcal{E}_0, \mathcal{E}$  will be assessed. For that purpose, one may focus on the immediate vicinity of the equilibria, where (11) may be simplified to

$$V' \approx 2Dq'^2 - \varrho|q'| \quad (q'^2 \ll 2D/D_3). \quad (12)$$

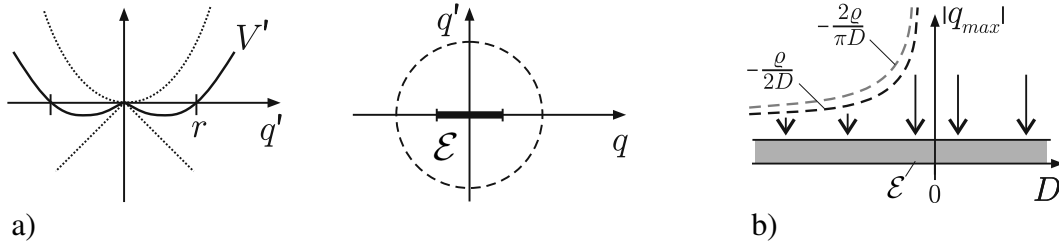
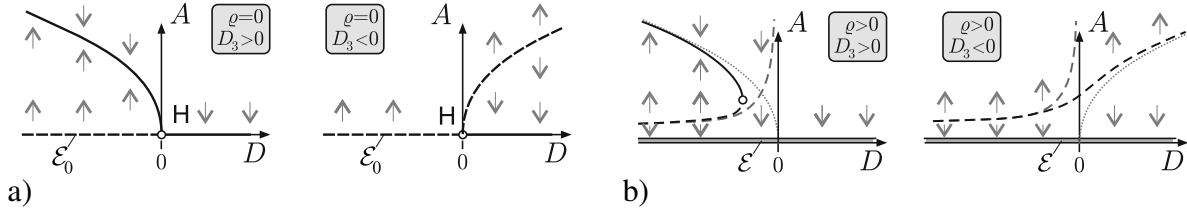

 Figure 6: Lyapunov function and corresponding estimate of the basin of attraction of  $\mathcal{E}$ .


Figure 7: Equilibria and stationary amplitudes.

For the smooth problem  $\varrho = 0$  in the vicinity of  $\mathcal{E}_0$  the stability is entirely determined by  $D$ . For  $D > 0$  follows  $V' > 0$  and thus the steady state will be asymptotically stable while it is unstable for  $D < 0$  due to  $V' < 0$ . Since the problem is oscillatory for both stable and unstable regimes, the steady state loses its stability in terms of a Hopf-bifurcation.

In the non-smooth problem  $\varrho > 0$  there exists always a non-empty region about the origin where  $V' < 0$ : according to the chosen Lyapunov function this region is (at least) a circle of radius  $r = -\frac{\varrho}{2D}$  (cf. fig. 6 a). Attractivity of  $\mathcal{E}$  follows from LaSalle's invariance principle provided that  $\mathcal{E}$  is contained within the circle of radius  $r$  which holds for  $D > -1/2$ . Moreover, solutions are bounded. Hence, for  $D > -1/2$  the set  $\mathcal{E}$  is always (locally) attractive with a finite basin of attraction and motions are bounded.

Assuming weak Coulomb damping  $|\varrho| \sim \mathcal{O}(\varepsilon) \ll 1$ , further insight into the bifurcation behaviour may be obtained by means of non-smooth averaging [5], [18]. Using the van-der-Pol - transformation  $q = A \cos \varphi$ ,  $q' = -A \sin \varphi$  equation (10) is transformed to standard form  $\mathbf{z}' \in \varepsilon \mathbf{f}(\mathbf{z})$  where  $\mathbf{z} = (A, \varphi)$  and the right-hand side is periodic. Decomposing  $\mathbf{z}$  into a large, slowly varying part and small fast oscillations according to  $\mathbf{z} = \bar{\mathbf{z}} + \varepsilon \tilde{\mathbf{z}}$  and averaging over one period yields the first order approximation

$$A' \approx -4\frac{r}{\pi} - 2DA - \frac{3}{4}D_3A^3 \quad , \quad \varphi' \approx 1. \quad (13)$$

Stationary amplitudes are given by the implicit equation  $A' \stackrel{!}{=} 0$ . For the smooth case  $\varrho = 0$  this equation may readily be solved, yielding the trivial solution (i.e. the stationary set  $\mathcal{E}_0$ ) as well as  $A_1 = \sqrt{-\frac{8D}{3D_3}}$ . Depending on  $D_3$  the bifurcation may be super- or subcritical. For the non-smooth case  $\varrho = 0$  explicit solutions may not be stated – however, a lower and an upper asymptote are easily calculated

$$A_L = -\frac{2\varrho}{\pi D} \quad , \quad A_U = A_1. \quad (14)$$

Additionally, the stationary sets  $\mathcal{E}_0$  or  $\mathcal{E}$  exist. Figure 7 outlines these stationary solutions. The bifurcation behaviour may be summarized as follows:

For the smooth problem ( $\varrho = 0$ ) the trivial solution  $\mathcal{E}_0$  loses its stability at  $D = 0$  where the system undergoes a Hopf-bifurcation (i.e. pitchfork bifurcation of the amplitudes). The type of bifurcation is determined by  $D_3$ : for  $D_3 > 0$  the bifurcation is supercritical, for  $D_3 < 0$  it is subcritical.

For the non-smooth problem ( $\varrho > 0$ ) the equilibrium set  $\mathcal{E}$  is attractive (at least) for all values of  $D > -1/2$ . Depending on the cubic damping term the stationary amplitudes behave similarly to the super- / subcritical scenario found for the smooth problem. However, the non-trivial stationary amplitudes do no longer originate from a bifurcation point but are created in terms of a global bifurcation. For the subcritical-like behavior  $\mathcal{E}$  is attractive with a finite basin of attraction. For the supercritical-like behavior,  $\mathcal{E}$  is globally attractive for  $D > D_F = -\frac{1}{2} \sqrt[3]{\left(\frac{9}{\pi}\right)^2 \varrho^2 D_3}$  and locally attractive for  $D < D_F$ .

If stiction between mass and belt is considered, also stick-slick limit-cycles may be observed for both cases.

### 3.2 Non-conservative Coupling

Self-excitation due to non-conservative coupling between coordinates of a system occurs in many technical systems like friction brakes [12], [21] or rotor systems [3]. This mechanism is sometimes also referred to as mode-coupling or modal coupling. Figure 8 shows a

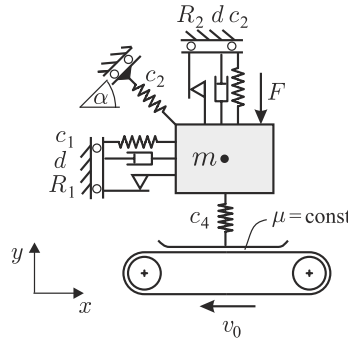


Figure 8: Abstract 2DoF-model.

simple 2-DoF-model problem which is often used to study self-excitation due frictional self-excitation [10] and which has been modified by adding joint damping elements. The model consists of a mass  $m$ , that is attached to the environment by spring-damper elements (stiffnesses  $c_1, c_2$ , viscous damping  $d$ ) which have been extended by additional Coulomb friction elements (intensities  $R_1, R_2$ ). The mass is pressed onto a moving conveyer belt (speed  $v_0$ ): the contact between mass and belt has the stiffness  $c_4$  and it is assumed that only sliding friction occurs (coefficient of friction  $\mu = \text{const}$ ). The mass is subjected to a static preload  $F = \text{const}$  in negative  $y$ -direction. Introducing the abbreviations  $c = \frac{1}{2}(c_1 + c_2 + c_3 + c_4)$ ,  $\Delta c = \frac{1}{2}((c_2 + c_4) - c_1 + c_3(\sin^2 \alpha - \cos^2 \alpha))$ , subtracting the steady state displacements  $x_0, y_0$  of the corresponding smooth problem and introducing the dimensionless variables

$$\Omega^2 = \frac{c}{m}, \quad \tau = \Omega t, \quad q_1 = \frac{x - x_0}{L}, \quad q_2 = \frac{y - y_0}{L}, \quad \varrho_i = \frac{R_i}{cL}, \quad \kappa = \frac{\Delta c}{c}, \quad \bar{\mu} = \mu \frac{c_4}{c}, \quad (15)$$

$$p = \frac{\Delta c}{c}, \quad D = \frac{d}{2\sqrt{cm}} \quad (16)$$

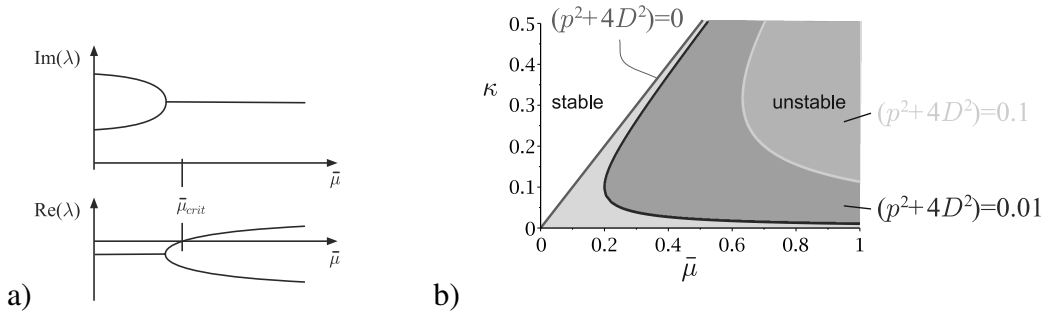


Figure 9: a) Schematic behaviour of the eigenvalues  $\lambda_i$ . b) Stability chart for different values of  $(p^2 + 4D^2)$ .

where  $L$  is a characteristic length. With  $\mathbf{q} = (q_1, q_2)^\top$  and  $(\cdot)' = d/d\tau$  the equation of motion reads

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{q}'' + \begin{bmatrix} 2D & 0 \\ 0 & 2D \end{bmatrix} \mathbf{q}' + \underbrace{\begin{bmatrix} 1-p & -\kappa + \bar{\mu} \\ -\kappa & 1+p \end{bmatrix}}_{=: \mathbf{Q}} \mathbf{q} \in - \underbrace{\begin{bmatrix} \varrho_1 & 0 \\ 0 & \varrho_2 \end{bmatrix}}_{=: \mathbf{R}} \text{Sign}(\mathbf{q}'). \quad (17)$$

For vanishing joint damping  $\varrho_1 = \varrho_2 = 0$  equation (17) is a smooth linear differential equation. The steady state is the equilibrium  $\mathcal{E}_0 = \mathbf{0}$  and its stability is readily evaluated by means of an eigenvalue analysis. Introducing  $\mathbf{q} = \text{re}^{\lambda\tau}$  yields the eigenvalues  $\lambda_i = -D \pm j\sqrt{(1-D^2) \pm \sqrt{p^2 + \kappa^2 - \bar{\mu}\kappa}}$ ,  $i = 1, \dots, 4$ . For small values of  $p$  and  $D$  divergence will not occur. The stability border  $\bar{\mu} = \bar{\mu}_{crit}$  for flutter instability is given by [7], [8]

$$\kappa\bar{\mu}_{crit} - p^2 - (\kappa^2 + 4D^2) = 0. \quad (18)$$

Figure 9 a) outlines qualitatively the behaviour of the eigenvalues as  $\bar{\mu}$  varies, figure 9 b) displays the corresponding stability chart.

For small joint damping  $\varrho_1, \varrho_2 > 0$  the equilibrium set  $\mathcal{E}$  reads

$$\mathcal{E} = \{(\mathbf{q}, \mathbf{q}') \mid \mathbf{q} \in -\mathbf{Q}^{-1}\mathbf{R}\text{Sign}(\mathbf{0}), \mathbf{q}' = \mathbf{0}\}. \quad (19)$$

It can be shown that  $\mathcal{E}$  is locally attractive for all values of  $\bar{\mu}$ : the corresponding basin of attraction is either infinite or finite, where the latter case implies the possibility of self-excitation if the system state is driven off this attracting region. Assuming  $\varrho_1, \varrho_2, D, \bar{\mu} \sim \mathcal{O}(\varepsilon)$ ,  $\varepsilon \ll 1$  averaging can be employed again in order to approximate stationary limit cycles bordering the basin of attraction. Introducing the van-der-Pol - transformations  $q_1 = A_1 \sin \varphi$ ,  $q_1' = A_1 \cos \varphi$ ,  $q_2 = A_2 \sin(\varphi + \alpha)$ ,  $q_2' = A_2 \cos(\varphi + \alpha)$ , separating small, fast from large, slow motions and integrating over one fast oscillation cycle yields evolution equations for the slowly varying part

$$\bar{A}_1' = f_1(\bar{A}_1, \bar{A}_2, \bar{\alpha}) \quad , \quad \bar{A}_2' = f_2(\bar{A}_1, \bar{A}_2, \bar{\alpha}). \quad (20)$$

Stationary amplitudes are given by  $\bar{A}_i' = 0$ : Figure 10 a) outlines an example of stationary amplitudes – characterising the basin of attraction of  $\mathcal{E}$  – as well as the equilibrium set  $\mathcal{E}$ . Figure 10 b) displays Poincaré -sections through the basins of attraction at  $\mathbf{q}' = \mathbf{0}$ . First, it is observed that for  $\bar{\mu} < \bar{\mu}_{crit}$  the equilibrium set  $\mathcal{E}$  is globally attractive. For  $\bar{\mu} > \bar{\mu}_{crit}$  the equilibrium set  $\mathcal{E}$  is still attractive but now the basin of attraction is finite: its extent is proportional to  $\varrho_i$  and roughly scales inversely proportional to  $(\bar{\mu} - \bar{\mu}_{crit})$ .



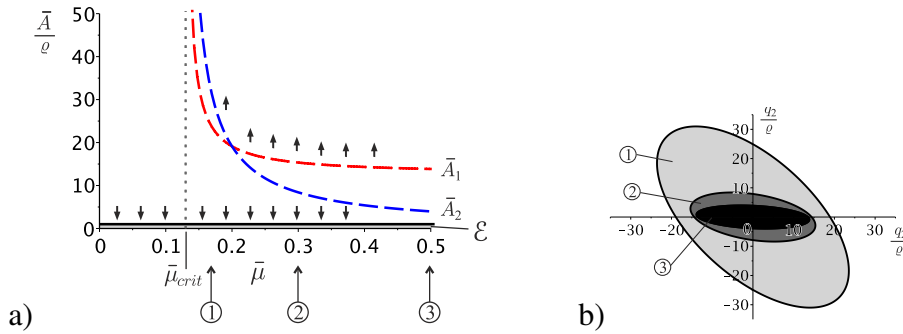


Figure 10: a) Stationary amplitudes characterizing the stability behavior and basin of attraction of  $\varepsilon$  for  $D = 0.01$ ,  $\kappa = 0.1$ ,  $p = 0.05$ . b) Poincaré-sections ( $\mathbf{q}' = \mathbf{0}$ ) of the basin of attraction for different values of  $\bar{\mu}$ .

## 4 CONCLUSIONS

Within this contribution the effect of frictional damping on three classical stability problems has been discussed. For all problems, the corresponding differential equations turn into differential inclusions. While for smooth systems Lyapunov - stability may be investigated by means of eigenvalue analyses, this is no longer possible for the non-smooth problems: instead, one may use Lyapunov's direct method in conjunction with Lasalle's invariance principle in order to proof attractivity of equilibrium sets and boundedness of nearby trajectories. Both properties eventually guarantee some kind of practical stability. Further results have been derived using non-smooth averaging. As a common result it is found that adding Coulomb damping changes equilibrium points into sets of equilibria which contain the original equilibrium points of the smooth systems. For static problems of divergence type, the stability results for the equilibrium points of the smooth problem may immediately transferred to attractivity properties of the equilibrium sets for the non-smooth case. For the oscillatory problems discussed within this contribution, the impact of Coulomb - damping is more fundamental: by adding Coulomb damping unstable equilibria changed into locally attractive equilibrium sets with a finite basin of attraction, stable equilibria turned into attractive sets of equilibria. If the intensity of the Coulomb damping is decreased further and further, the smooth problem is recovered as the extent of the basin of attraction goes to zero. Hence, for the oscillatory problems the notion of stability is replaced by the question whether the basin of attraction is finite or infinite. In the context of applications this implies that the question for practical stability may only be decided with respect to a specific level of ambient perturbations.

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### A LaSalle's Invariance Principle [11], [14], [15]

Suppose having an autonomous dynamical system in state space form,  $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z})$  and positive trajectories  $\varphi = \{\mathbf{z}(\mathbf{z}_0, t_0) | t \geq t_0\}$  starting in  $\mathbf{z}_0$  at  $t_0$ , which are unique. Suppose having a strictly increasing Lyapunov function  $V(\mathbf{z}) \in C^1$  and a subset  $\Omega_\ell$  of the state space with

$$\Omega_\ell = \{\mathbf{z} | V(\mathbf{z}) \leq \ell\} \quad , \quad V(\mathbf{z}) > 0 \quad \forall \mathbf{z} \in \Omega_\ell \setminus \{\mathbf{0}\} \quad , \quad \dot{V}(\mathbf{z}) \leq 0 \quad \forall \mathbf{z} \in \Omega_\ell.$$

Moreover,  $\mathcal{Z}$  is a set on which  $V$  is stationary, i.e.  $\mathcal{Z} = \{\mathbf{z} | \dot{V}(\mathbf{z}) = 0\}$  and  $\mathcal{M} \subset \mathcal{Z} \subset \Omega_\ell$  is the largest (positively) invariant set within  $\mathcal{Z}$  and  $\Omega_\ell$ . Then, every trajectory starting within  $\Omega_\ell$  approaches  $\mathcal{M}$  for  $t \rightarrow \infty$ . Thus, the set  $\mathcal{M}$  is attractive.

Please note that in contrast to problems with a single values equilibrium the stability of the steady state may not be concluded from LaSalle's Principle.

### B Extension of Chetayev's Theorem to set-valued equilibria [11], [15]

Suppose having an autonomous dynamical system in state space form,  $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z})$  and positive trajectories  $\gamma^+ = \{\mathbf{z}(\mathbf{z}_0, t_0) | t \geq t_0\}$  starting in  $\mathbf{z}_0$  at  $t_0$ , which are unique. Suppose having a Lyapunov function  $V(\mathbf{z}) \in C^1$  and the subsets of the state space where

$$\mathcal{E} = \{\mathbf{z} | \mathbf{z}' = 0\} \quad (\text{equilibrium}) \quad , \quad (21)$$

$$\mathcal{Z} = \{\mathbf{z} | V'(\mathbf{z}) = 0\} \quad , \quad \Omega_\ell = \{\mathbf{z} | V(\mathbf{z}) \leq \ell\} \quad (22)$$

and  $\ell$  is chosen in order to have the cone  $\Omega_\ell$  touch  $\mathcal{E}$ , i.e.  $\min \text{dist}(\Omega_\ell, \mathcal{E}) = 0, \mathcal{E} \cap \Omega_\ell = \emptyset$  For  $V' \leq 0$  the equilibrium set is unstable.