

DYNAMIC RESPONSE OF A BRIDGE DUE TO MOVING LOADS

Jozef Melcer*¹

¹University of Zilina, Faculty of Civil Engineering, Zilina, Slovak Republic
jozef.melcer@fstav.uniza.sk

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Abstract. *The problem of vehicle-bridge interaction can be followed in the literature from the year 1849. It was induced by the collapse of the Chester Rail Bridge in England in the year 1847. At the early stage the analytical methods were applied. The development of computers brings the change in the approach to the used methods of solution. The Finite Element Methods and the Component Element Methods represent the revolution and qualitative jump in the development. The dynamic deflections and the values of dynamic coefficients are remarkable from the point of bridge designers. The analysis of dynamic coefficients of a bridge and dependence of dynamic coefficients on the speed of vehicle motion are presented in this contribution. For the solution of the problem the plane discrete computing models of a lorry and of a bus were adopted. The computing model of a bridge is adopted as simply supported Euler beam with continuously distributed mass. The equations of motion are derived in the form of differential equations and they are solved numerically. Numerical calculations were realised in the environment of the program system MATLAB.*

1 INTRODUCTION

The problem of vehicle-bridge interaction can be followed in the literature from the year 1849. It was induced by the collapse of the Chester Rail Bridge in England in the year 1847, [1]. While the problem of dynamic of railway bridges was followed from the year 1847 the problems of dynamic of highway bridges start to be followed in the 20th century only. The 1st important report on this topic was published by the American Society of Civil Engineers [2]. Total review about results of solution to the year 1975 was published by Tseng Huang in [3]. Also in the Czech and Slovak Republic important works arose in this field. In the area of railway bridges they were published by Frýba, L. [4] and in the area of highway bridges they were published by Melcer, J. [5]. Numerical modeling of the vehicle motion along bridge structure requires paying attention minimally to these facts: creation of computing models of vehicles, creation of computing models of bridges, creation of computing programs for the solution of the equations of motion and displaying of obtained results.

2 COMPUTING MODELS OF VEHICLES

The discrete computing models of vehicles can be created on three qualitative different levels: 1D – quarter model, 2D – plane model and 3D – space model. Every model has its advantages and disadvantages and under certain assumptions it can be used for the solution of real practical problems. Possible plane computing models of a lorry and a bus are shown in the Figure 1. The relation between the components of displacements $\{\mathbf{r}(t)\}$, corresponding to individual degrees of freedom, and deformations of jointing members $\{\mathbf{d}(t)\}$ gets the transpose static matrix $[\mathbf{A}]^T$

$$\{\mathbf{d}(t)\} = [\mathbf{A}]^T \cdot \{\mathbf{r}(t)\}, \quad (1)$$

Dependence between elastic forces in jointing members (in the sense of action of mass objects on jointing members) and its deformations is described by the equation

$$\{\mathbf{F}_{re}(t)\} = [\mathbf{k}] \cdot \{\mathbf{d}(t)\}, \quad (2)$$

where $[\mathbf{k}]$ is the stiffness matrix of jointing members. Dependence of damping forces on the velocity of deformations $\{\dot{\mathbf{d}}(t)\}$ is described by the equation

$$\{\mathbf{F}_d(t)\} = [\mathbf{b}] \cdot \{\dot{\mathbf{d}}(t)\}, \quad (3)$$

where $[\mathbf{b}]$ is damping matrix. By dot is denoted derivation with respect of time t . Friction forces are considered as

$$\begin{aligned} \text{if } \{\dot{\mathbf{d}}(t)\} &\geq + \dot{d}_c & \{\mathbf{F}_f(t)\} &= + \{\mathbf{f}\}, \\ \text{if } \{\dot{\mathbf{d}}(t)\} &\leq - \dot{d}_c & \{\mathbf{F}_f(t)\} &= - \{\mathbf{f}\}, \\ \text{if } \text{abs } \{\dot{\mathbf{d}}(t)\} &< \dot{d}_c & \{\mathbf{F}_f(t)\} &= + \{\mathbf{f}\}^T \cdot \{\dot{\mathbf{d}}(t)\} / \dot{d}_c. \end{aligned} \quad (4)$$

Resulting forces in jointing members in action on mass objects are

$$\{\mathbf{F}_{sc}(t)\} = - \{\mathbf{F}_{re}(t)\} - \{\mathbf{F}_d(t)\} - \{\mathbf{F}_f(t)\}. \quad (5)$$

Sign (-) is due to the principle of action and reaction. From the forces in jointing members $\{\mathbf{F}_{sc}(t)\}$ the static equivalents corresponding to individual degrees of freedom $\{\mathbf{F}_{sv}(t)\}$ are calculated

$$\{\mathbf{F}_{sv}(t)\} = [\mathbf{A}] \cdot \{\mathbf{F}_{sc}(t)\}. \quad (6)$$

To the forces corresponding to individual degrees of freedom $\{\mathbf{F}_{sv}(t)\}$ the gravity forces $\{\mathbf{F}_G\}$ and reactions in support $\{\mathbf{F}_{rv}(t)\}$ must be added. In this manner we obtain the complete vector of forces $\{\mathbf{F}_v(t)\}$ acting on the computing model of vehicle

$$\{\mathbf{F}_v(t)\} = \{\mathbf{F}_{sv}(t)\} + \{\mathbf{F}_G\} + \{\mathbf{F}_{rv}(t)\} . \quad (7)$$

The system of equations of motion describing the vibration of the computing model of vehicle is then expressed by the relation

$$[\mathbf{m}] \cdot \{\ddot{\mathbf{r}}(t)\} = \{\mathbf{F}_v(t)\} , \quad (8)$$

where $[\mathbf{m}]$ is mass matrix.

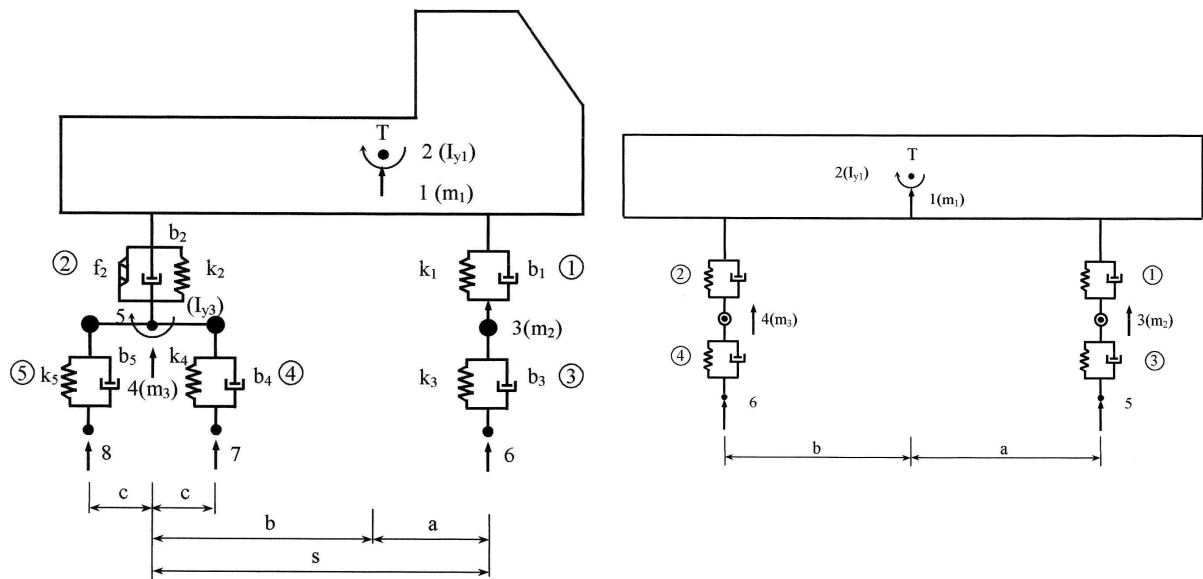


Figure 1: Plane computing model of a lorry and a bus.

3 COMPUTING MODEL OF A BRIDGE

For the description of bridge vibration the simplified computing model in the form of simply supported Euler beam excited by moving forces is adopted, Figure 2.

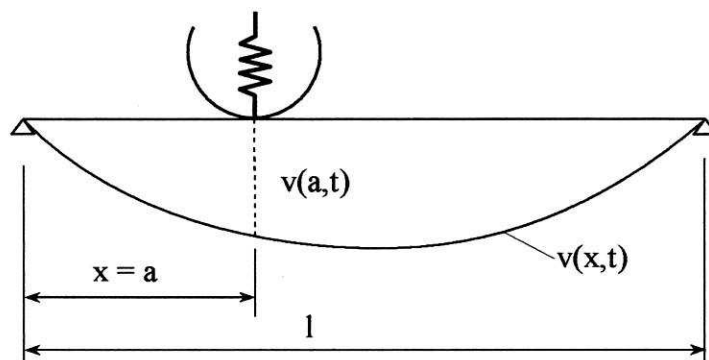


Figure 2: Computing model of a bridge as a simply supported Euler beam.

Equation of motion can be written as

$$E \cdot I \cdot \frac{\partial^4 y(x,t)}{\partial x^4} + \mu \cdot \frac{\partial^2 y(x,t)}{\partial t^2} + 2 \cdot \mu \cdot \omega_b \cdot \frac{\partial y(x,t)}{\partial t} = p(x,t). \quad (9)$$

In the next the following identification will be used:

$h(x)$ respectively $h(t)$ –function defining the road unevenness

$y(x,t)$ - dynamic deflection curve of the beam axis

$v(x,t)$ - profile of the runway defined by the term (10)

$$v(x,t) = y(x,t) + h(x). \quad (10)$$

The assumption about the shape of dynamic deflection curve is adopted in the form

$$y(x,t) = f(t) \cdot \psi(x,t), \quad (11)$$

where $f(t)$ is a coefficient of proportionality dependent on the time t and $\psi(x,t)$ is the static deflection curve induced by static effect of vehicle.

$$\psi(x,t) = \sum_{n=1}^{\infty} \eta_n(t) \cdot \sin \frac{n \cdot \pi \cdot x}{l}, \quad (12)$$

where

$$\eta_n(t) = \frac{2}{E \cdot I} \cdot \left(\frac{l}{\pi} \right)^4 \cdot \frac{1}{l} \cdot \left(\sum_{j=1}^m \varepsilon_j \cdot \frac{G_j}{n^4} \cdot \sin \frac{n \cdot \pi \cdot x_j}{l} \right). \quad (13)$$

l is span of the beam, G_j is weight j -th vehicle axis. Coefficient $\varepsilon_j = 1$, if the corresponding axis is on the bridge and $\varepsilon_j = 0$, if the corresponding axis is outside the bridge. For the practical use only the 1st member of infinite series is used for the calculation. Than

$$\psi(x,t) = \eta_1(t) \cdot \sin \frac{\pi \cdot x}{l} \quad (14)$$

and in the case of plane computing model of vehicle

$$\eta_1(t) = \frac{2}{E \cdot I} \cdot \left(\frac{l}{\pi} \right)^4 \cdot \frac{1}{l} \cdot \left(\sum_{j=1}^m \varepsilon_j \cdot G_j \cdot \sin \frac{\pi \cdot x_j}{l} \right). \quad (15)$$

Expression for $y(x,t)$ than can be written as

$$y(x,t) = f(t) \cdot \psi(x,t) = f(t) \cdot \eta_1(t) \cdot \sin \frac{\pi \cdot x}{l} = q(t) \cdot \sin \frac{\pi \cdot x}{l} \quad (16)$$

and expression for the profile of the runway is

$$v(x,t) = y(x,t) + h(x) = q(t) \cdot \sin \frac{\pi \cdot x}{l} + h(x). \quad (17)$$

$q(t)$ in equations (16) a (17) is generalized Lagrange coordinate. Assume that $y(x,t) = q(t) \cdot \sin(\pi \cdot x/l)$, than

$$\frac{\partial^4 y(x,t)}{\partial x^4} = q(t) \cdot \frac{\pi^4}{l^4} \cdot \sin \frac{\pi \cdot x}{l}, \quad (18)$$

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \ddot{q}(t) \cdot \sin \frac{\pi \cdot x}{l}, \quad (19)$$

$$\frac{\partial y(x,t)}{\partial t} = \dot{q}(t) \cdot \sin \frac{\pi \cdot x}{l}. \quad (20)$$

After substitute (18), (19), (20) into equation (9) we obtain

$$E \cdot I \cdot q(t) \cdot \frac{\pi^4}{l^4} \cdot \sin \frac{\pi \cdot x}{l} + \mu \cdot \ddot{q}(t) \cdot \sin \frac{\pi \cdot x}{l} + 2 \cdot \mu \cdot \omega_b \cdot \dot{q}(t) \cdot \sin \frac{\pi \cdot x}{l} = p(x,t) \quad (21)$$

respectively

$$\left\{ \ddot{q}(t) \cdot \mu + \dot{q}(t) \cdot 2 \cdot \mu \cdot \omega_b + q(t) \cdot E \cdot I \cdot \frac{\pi^4}{l^4} \right\} \cdot \sin \frac{\pi \cdot x}{l} = p(x,t). \quad (22)$$

The moving forces $F_{\text{int},j}$ can be transform by the use of Dirac δ function on the continual load

$$p(x,t) = \sum_j \varepsilon_j \cdot \delta(x - x_j) \cdot F_{\text{int},j}(t). \quad (23)$$

For the plane computing model we can write

$$p(x,t) = \sum_j \varepsilon_j \cdot \delta(x - x_j) \cdot F_{\text{int},j}(t) = \sum_j \sum_{n=1}^{\infty} p_{n,j}(t) \cdot \sin \frac{n \cdot \pi \cdot x}{l}, \quad (24)$$

where

$$p_{n,j}(t) = \frac{2}{l} \int_0^l p_j(x,t) \cdot \sin \frac{n \cdot \pi \cdot x}{l} \cdot dx = \frac{2}{l} \cdot \varepsilon_j \cdot F_{\text{int},j}(t) \cdot \sin \frac{n \cdot \pi \cdot x_j}{l} \quad (25)$$

Than

$$\begin{aligned} p(x,t) &= \sum_j \sum_{n=1}^{\infty} p_{n,j}(t) \cdot \sin \frac{n \cdot \pi \cdot x}{l} = \sum_j \sum_{n=1}^{\infty} \frac{2}{l} \cdot \varepsilon_j \cdot F_{\text{int},j}(t) \cdot \sin \frac{n \cdot \pi \cdot x_j}{l} \cdot \sin \frac{n \cdot \pi \cdot x}{l} = \\ &= \sum_j \sum_{n=1}^{\infty} \frac{2}{l} \cdot \sin \frac{n \cdot \pi \cdot x}{l} \cdot \varepsilon_j \cdot F_{\text{int},j}(t) \cdot \sin \frac{n \cdot \pi \cdot x_j}{l}. \end{aligned} \quad (26)$$

When we will use only the 1st member of the series than the expression for $p(x,t)$ can be simplified

$$p(x,t) = \frac{2}{l} \cdot \sin \frac{\pi \cdot x}{l} \cdot \sum_j \varepsilon_j \cdot F_{\text{int},j}(t) \cdot \sin \frac{\pi \cdot x_j}{l}. \quad (27)$$

4 RESULTS OF NUMERICAL CALCULATIONS

Numerical calculations were realised in the environment of the program system MATLAB. For the purpose of numerical calculations the parameters of the prestressed concrete bridge with the span $l = 29,0$ m were used. Moment of inertia of the cross section $I = 1,60622 \text{ m}^4$, modulus of elasticity $E = 3,85 \text{e}10 \text{ N.m}^{-2}$, intensity of the mass $\mu = 19680 \text{ kg.m}^{-1}$, circular frequency of damping $\omega_b = 0,1 \text{ rad.s}^{-1}$. From the aspect of bridge designers the values of dynamic coefficients versus speed of vehicle motion are interested. Therefore in the next the values of dynamic coefficients δ versus speed of vehicle motion in interval of $V = 0 - 120$ km/h are presented, Figure 3, 4. As the moving vehicles the lorry TATRA T815 and the bus KAROSA with following parameters are applied.

TATRA T815:

$$m_1 = 22\,950 \text{ kg}, m_2 = 910 \text{ kg}, m_3 = 2\,140 \text{ kg}, I_{y1} = 62\,298 \text{ kg.m}^2, I_{y3} = 932 \text{ kg.m}^2,$$

$k_1 = 287\,433\text{ N/m}$, $k_2 = 1\,522\,512\text{ N/m}$, $k_3 = 2\,550\,600\text{ N/m}$, $k_4 = k_5 = 5\,022\,720\text{ N/m}$,
 $b_1 = 19\,228\text{ kg/s}$, $b_2 = 260\,197\text{ kg/s}$, $b_3 = 2\,746\text{ kg/s}$, $b_4 = b_5 = 5\,494\text{ kg/s}$.

Initial conditions are assumed as:

$r_1(0) = -0,02\text{ m}$, $\dot{r}_1(0) = 0,0\text{ m/s}$, $r_2(0) = 0,0\text{ m}$, $\dot{r}_2(0) = 0,0\text{ m/s}$,
 $r_3(0) = -0,002\text{ m}$, $\dot{r}_3(0) = 0,0\text{ m/s}$, $r_4(0) = -0,003\text{ m}$, $\dot{r}_4(0) = 0,0\text{ m/s}$,
 $r_5(0) = 0,0\text{ m}$, $\dot{r}_5(0) = 0,0\text{ m/s}$.

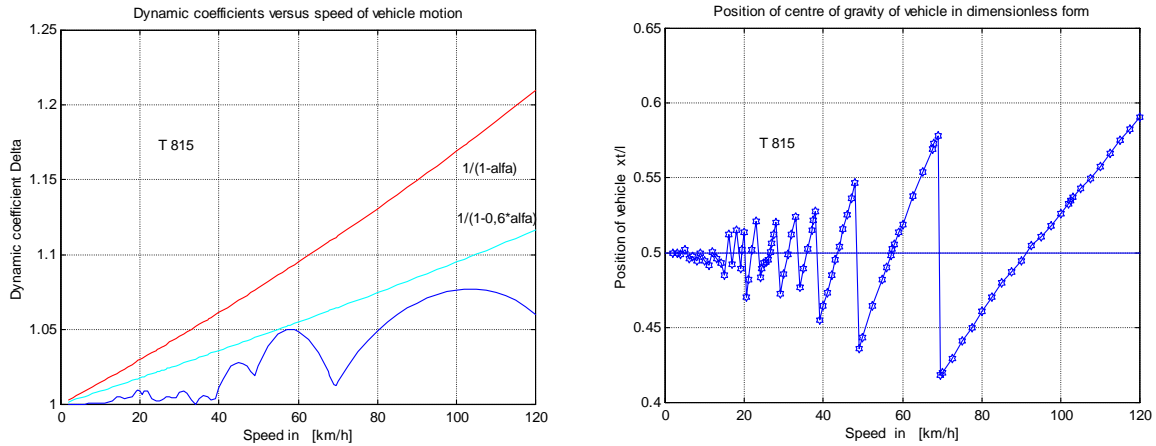


Figure 3: Dynamic coefficients and position of vehicle versus speed of vehicle motion, T815

KAROSA:

$m = 18\,150\text{ kg}$, $m_1 = 600\text{ kg}$, $m_2 = 1\,250\text{ kg}$, $I_y = 330\,420\text{ kg.m}^2$,
 $k_1 = 850\,000\text{ N/m}$, $k_2 = 1\,500\,000\text{ N/m}$, $k_3 = 1\,700\,000\text{ N/m}$, $k_4 = 3\,400\,000\text{ N/m}$,
 $b_1 = 80\,000\text{ kg/s}$, $b_2 = 160\,000\text{ kg/s}$, $b_3 = 4\,000\text{ kg/s}$, $b_4 = 8\,000\text{ kg/s}$.

Initial conditions are assumed as:

$r_1(0) = -0,02\text{ m}$, $\dot{r}_1(0) = 0,0\text{ m/s}$, $r_2(0) = 0,0\text{ m}$, $\dot{r}_2(0) = 0,0\text{ m/s}$,
 $r_3(0) = -0,01\text{ m}$, $\dot{r}_3(0) = 0,0\text{ m/s}$, $r_4(0) = -0,009\text{ m}$, $\dot{r}_4(0) = 0,0\text{ m/s}$.

The function $\delta(V)$ is not smooth curve. It has many local maxima and spikes. Its character is connected with discontinuities in the function $x_t(V)$ indicating the position of vehicle on the bridge at the moment of arising of maximal dynamical deflection in the mid span of the bridge. The position of spikes in the function $\delta(V)$ corresponds to the position of discontinuities in the function $x_t(V)$, Figure 3, 4.

With respect to the character of the curve $\delta(V)$ it would be convenient to approximate the maximal values of dynamic coefficients by some envelope curve, for example in the shape

$$ok = 1 / (1 - \alpha) \quad (28)$$

or in the shape

$$ok = 1 / (1 - 0,6.\alpha). \quad (29)$$

The dimensionless coefficient α indicates the influence of the speed of vehicle motion. It is defined as

$$\alpha = T_{(1)} / 2.T_P. \quad (30)$$

$T_{(1)}$ is the period of bridge vibration in the 1st natural mode and T_P is the transit time of one axle along the bridge. Every type of vehicle has its own suitable envelope curve. But envelope curve (28) covers all cases.

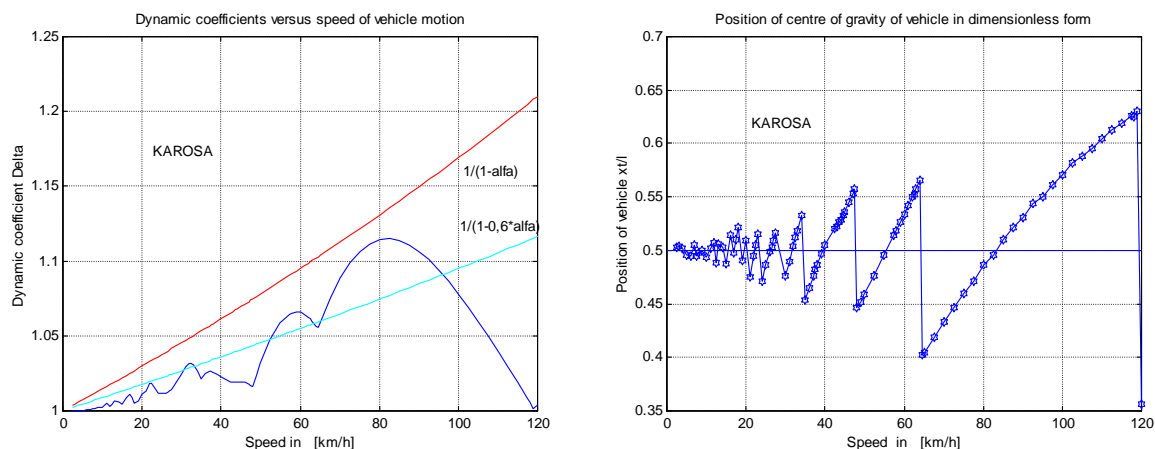


Figure 4: Computing model of a bridge as a simply supported Euler beam.

5 CONCLUSIONS

- The moving load effect on bridges can be followed by theoretical or by experimental way but the combination of both advances is the most effective way.
- The numerical solution requires the creation of vehicle and bridge computing models and numerical solutions of equations of motion.
- The values of bridge dynamic coefficients are interested for the bridge designers.
- The function describing the values of dynamic coefficient in dependence on the speed of vehicle motion is not smooth curve. It has many local maxima and spikes. Its character is connected with discontinuities in the function indicating the position of vehicle on the bridge at the moment of arising of maximal dynamical deflection in the mid span of the bridge.
- It would be convenient to approximate the maximal values of dynamic coefficients by some envelope curve.
- Every type of vehicle has its own suitable envelope curve. But the envelope curve covering all cases can be found.

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