11th International Conference on Vibration Problems
Z. Dimitrovová *et al.* (eds.)
Lisbon, Portugal, 9-12 September 2013

THE RELIABILITY-BASED OPTIMIZATION AND SENSITIVITY ANALYSIS OF SANDWICH PLATES

E. Kormaníková[1], K. Kotrasová*[1]

[1]Technical University of Košice, Civil Engineering Faculty, Slovakia
eva.kormanikova@tuke.sk, kamila.kotrasova@tuke.sk

Keywords: Optimization, Sensitivity analysis, Sandwich plate, Tsai-Wu criterion, Buckling analysis, Free vibration analysis

Abstract. *This paper presents optimization and sensitivity analysis of sandwich plates with laminate facings subjects the Tsai-Wu criterion. A symmetric sandwich plate has the objective function of maximizing the natural frequencies or maximizing the buckling load. The design variables are the fiber orientations of individual layers and are computed by using the Sequential linear programming method. Within this method the Modified feasible direction method was used. The sensitivity analysis is similar in principle to the design optimization. In the sensitivity analysis the design variables are changes between their lower and upper bounds in a specified number of steps. The optimization of a composite plate with the sensitivity is important analysis for design of structures ranging from aircrafts to civil structures.*

1 INTRODUCTION

Material that has two or more different material components or phases is composed material, but only in that case as the phases that make it up have very different physical properties and also properties of composite material are clearly distinct from its components. If these characteristics are fulfilled we are talking about composite material, which sandwich composites are too.

The outer layers are made of a material that has high strength (metals and fiber reinforced laminates), which can transfer axial forces and bending moments, while the core is made of lightweight materials such as foam, alder wood etc. The material used in sandwich core must be resistant to compression and capable of transmitting shear [1, 5].

2 FREE VIBRATION OF SANDWICH PLATE

The equations to determine the natural frequencies of symmetric sandwich panel are used:

$$\begin{aligned} &D_{11}\frac{\partial^2\bar{\alpha}}{\partial x^2}+D_{66}\frac{\partial^2\bar{\alpha}}{\partial y^2}+(D_{12}+D_{66})\frac{\partial^2\bar{\beta}}{\partial x\partial y}- \\ &-k^s A_{55}\left(\bar{\alpha}+\frac{\partial w}{\partial x}\right)-I\frac{\partial^2\bar{\alpha}}{\partial t^2}=0, \end{aligned} \tag{1}$$

$$\begin{aligned} &(D_{12}+D_{66})\frac{\partial^2\bar{\alpha}}{\partial x\partial y}+D_{66}\frac{\partial^2\bar{\beta}}{\partial x^2}+D_{22}\frac{\partial^2\bar{\beta}}{\partial y^2}- \\ &-k^s A_{44}\left(\bar{\beta}+\frac{\partial w}{\partial y}\right)-I\frac{\partial^2\bar{\beta}}{\partial t^2}=0, \end{aligned} \tag{2}$$

$$\begin{aligned} &k^s A_{55}\left(\frac{\partial\bar{\alpha}}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+k^s A_{44}\left(\frac{\partial\bar{\beta}}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)- \\ &-\rho_m h\frac{\partial^2 w}{\partial t^2}=0, \end{aligned} \tag{3}$$

$$\begin{aligned} &\rho_m=\frac{1}{h}\sum_{k=1}^{N}\rho_k(z^{(k)}-z^{(k-1)}), \\ &I=\frac{\rho_m h^3}{12}\frac{1}{3}\sum_{k=1}^{N}\rho_k(z^{(k)})^3-(z^{(k-1)})^3, \end{aligned} \tag{4}$$

where:
k^s is the transverse shear deformation factor given by value 5/6,
ρ_k is the mass density of the k^{th} layer.
For the simply supported plate let:

$$\begin{aligned} w(x,y,t)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C'_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{i\omega_{mn}t}, \\ \bar{\alpha}(x,y,t)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A'_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{i\omega_{mn}t}, \\ \bar{\beta}(x,y,t)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B'_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{i\omega_{mn}t}, \end{aligned} \tag{5}$$

where:
m, *n* – are integers only,
a, *b* – are the panel dimensions in *x*, *y* axis direction respectively,

ω_{mn}– is natural angular velocity.
Substituting term (5) into equations (1), (2) and (3) results in a set of homogeneous equations that are used to solve the natural frequencies of vibration:

$$\begin{pmatrix} L'_{11} & L_{12} & L_{13} \\ L_{12} & L'_{22} & L_{23} \\ L_{13} & L_{23} & L'_{33} \end{pmatrix} \begin{pmatrix} A'_{mn} \\ B'_{mn} \\ C'_{mn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6}$$

Matrix elements are given by the formulas:

$$L'_{11} = L_{11} - \frac{\rho_m h^3}{12}\omega_{mn}^2,\ L'_{22} = L_{22} - \frac{\rho_m h^3}{12\omega_{mn}^2},$$
$$L'_{33} = L_{33} - \rho_m h \omega_{mn}^2, \tag{7}$$

where:

$$\begin{aligned} L_{11} &= D_{11}\lambda_m^2 + D_{66}\lambda_n^2 + k^s A_{55}, \\ L_{12} &= (D_{12} + D_{66})\lambda_m \lambda_n, \\ L_{13} &= k^s A_{55}\lambda_m, \\ L_{22} &= D_{66}\lambda_m^2 + D_{22}\lambda_n^2 + k^s A_{44}, \\ L_{23} &= k^s A_{44}\lambda_n, \quad L_{33} = k^s A_{55}\lambda_m^2 + \lambda_n^2, \end{aligned} \tag{8}$$

$$\lambda_m = \frac{m\pi}{a}, \quad \lambda_n = \frac{n\pi}{b}. \tag{9}$$

If the rotatory inertia terms are neglected then $L'_{11} = L_{11}$, $L'_{22} = L_{22}$, and we get:

$$\omega_{mn}^2 = \frac{(QL_{33} + 2L_{12}L_{23}L_{13} - L_{22}L_{13}^2 - L_{11}L_{23}^2)}{\rho_m h Q}, \tag{10}$$
$$Q = L_{11}L_{22} - L_{12}{}^2.$$

Also applies:

$$\begin{aligned} A'_{mn} &= \frac{L_{12}L_{23} - L_{22}L_{13}}{Q} C'_{mn}, \\ B'_{mn} &= \frac{L_{12}L_{13} - L_{11}L_{23}}{Q} C'_{mn}. \end{aligned} \tag{11}$$

In a similar way the governing equations for buckling problems can be derived. In the matrix equations (6) only the differential operator L'_{33} is substituted by [2, 4]:

$$L_{33} - \left(N_1 \frac{\partial^2}{\partial x^2} + 2N_6 \frac{\partial^2}{\partial x \partial y} + N_2 \frac{\partial^2}{\partial y^2} \right) \tag{12}$$

3 OPTIMIZATION PROCESS

The design problem consists of determining the optimal fiber orientation ${}^k\theta$ where k = 1,2,...,N, with N denoting the number of layers so as to satisfy the following objectives:

1. Maximization of the natural frequencies when the plate undergoes free vibrations
2. Maximization of the buckling load N when the plate is subjected to an in-plane load N.

Moreover, the design variables should satisfy the constraints:

$$g_{kj} = \left(P_j^{(k)}\varepsilon_{1k} + Q_j^{(k)}\varepsilon_{2k} + R_j^{(k)}\gamma_{12k}\right) - 1 \le 0 \tag{13}$$

for $k = 1,\dots,N,$ $j = 1,\dots,J,$

where ρ_k and t_k are the density and the thickness, respectively, of the *k*-th layer, $P_j^{(k)}$, $Q_j^{(k)}$, $R_j^{(k)}$, are coefficients that define the j-th boundary of a failure envelope for each layer in the strain space, and the $\varepsilon_{1k}, \varepsilon_{2k}, \gamma_{12k}$ are the strains in the principal material direction in the *k*-th layer. For a Tsai-Wu criterion, which puts bounds on the values of the strains in the principal material directions, the failure envelope is ellipsoid. *P* and *Q* defined as an inverse of the normal failure strains in the longitudinal and transverse directions to the fibers, once in tension and once in compression. The coefficient *R* is the inverse of the shear failure strain for positive shear and for negative shear [3].

The optimization process is applied to the approximate problem represented by the polynomial approximation. The coefficients of the polynomial function are determined by the least squares regression.

For regression analysis the singular value decomposition is used. When the objective function and constraints are approximated and their gradients with respect to the design variables are calculated based on chosen approximation, it is possible to solve the approximate optimization problem. The algorithm SLP (Sequential Linear Programming) method was used for solving the approximate optimization problem. The iterative process of SLP within each optimization loop is shown below:

1. p=0, X^p=X^m
2. p=p+1
3. Linearize the problem at X^{p-1} by creating a first order Taylor Series expansion of the objective function and retained constraints:
   $F(X) = F(X^{p-1}) + \nabla F(X^{p-1})(X - X^{p-1})$
   $g(X) = g(X^{p-1}) + \nabla g(X^{p-1})(X - X^{p-1})$
4. Use this approximation of optimization instead of the original nonlinear functions:
   Maximize: $F(X)$
   Subject to: $g(X) \le 0$ and $\overline{X}_i^L \le X_i \le \overline{X}_i^U$
5. Find an improved design X^p (using the Modified Feasible Direction method)
6. Check feasibility and convergence. If both of them are satisfying, go to 7. Otherwise, go to step 2.
7. $X^{m+1} = X^p$

Using the SLP method the solving process is iterated until convergence is achieved. Convergence or termination checks are performed at the end of each optimization loop in general optimization. The optimization process continues until either convergence or termination occurs.

4 TSAI-WU CRITERION

We can distinguish the failure between fiber failure (FF) and inter fiber failure (IFF). In the case of plane stress, the IFF criteria discriminates three different modes. The IFF mode A is when perpendicular transversal cracks appear in the lamina under transverse tensile stress with or without in-plane shear stress. The IFF mode B denotes perpendicular transversal cracks, but in this case they appear under in-plane shear stress with small transverse compression stress. The IFF mode C indicates the onset of oblique cracks when the material is under significant transversal compression [2].

Strength of a composite layer in any other direction is evaluated based on various failure criteria. The basic premise in predicting the failure of fiber-reinforced layers using maximum stress and maximum strain criteria is the same as for isotropic material. Failure is predicted when the maximum stress along the fiber or transverse to the fiber directions exceed the strength of the tension or compression .

Tsai-Wu criterion is the general form of the failure criterion for orthotropic materials under plane stress. The assumption is expressed as:

$$F_{01}\sigma_1 + F_{11}\sigma_1^2 + 2F_{12}\sigma_1\sigma_2 + F_{02}\sigma_2 + F_{22}\sigma_2^2 + F_{44}\tau_{12}^2 < 1, \tag{14}$$

where:

$$F_{01} = \frac{1}{X_t} - \frac{1}{X_c}, \qquad F_{11} = \frac{1}{X_t X_c},$$
$$F_{02} = \frac{1}{Y_t} - \frac{1}{Y_c}, \qquad F_{22} = \frac{1}{Y_t Y_c},$$
$$F_{12} = -\frac{1}{2}\frac{1}{\sqrt{X_t X_c Y_t Y_c}}, \qquad F_{44} = \frac{1}{S^2}. \tag{15}$$

The failure criterion for orthotropic material under strain assumption is expressed as:

$$G_{01}\varepsilon_1 + G_{11}\varepsilon_1^2 + G_{12}\varepsilon_1\varepsilon_2 + G_{02}\varepsilon_2 + G_{22}\varepsilon_2^2 + G_{44}\gamma_{12}^2 < 1, \tag{16}$$

where:

$$G_{01}=F_{01}E_{11}+ F_{02}E_{12} \qquad G_{02}=F_{02}E_{22}+ F_{01}E_{12}$$

$$G_{11} = F_{11}E_{11}^2 + F_{22}E_{12}^2 + F_{12}E_{11}E_{12} \qquad G_{22} = F_{22}E_{22}^2 + F_{11}E_{12}^2 + F_{12}E_{22}E_{12}$$

$$G_{12} = 2E_{12}(F_{11}E_{11} + F_{22}E_{22}) + 2F_1(E_{12}^2 + E_{11}E_{22}) \qquad G_{44} = F_{44}E_{44}^2. \tag{17}$$

When $F_{12} = \frac{-1}{2X_t^2}$, the Tsai-Wu criterion is reduced to Tsai-Hill criterion, and when $F_{12} = \frac{-1}{2X_t X_c}$ the Tsai-Wu criterion is reduced to Hoffman criterion [3].

These failure criteria are used to calculate a failure index (F.I.) from the computed stresses and user-supplied material strengths. A failure index denotes the onset of failure, and a value less than 1 denotes no failure. The failure index according to this theory is computed using the following equation [2]:

$$I_F = F_{01}\sigma_1 + F_{11}\sigma_1^2 + 2F_{12}\sigma_1\sigma_2 + F_{02}\sigma_2 + F_{22}\sigma_2^2 + F_{44}\tau_{12}^2 \tag{18}$$

Failure load factor is inverse value to the failure index.

5 SOLUTION AND RESULTS

Solve the optimization and sensitivity of sandwich plate consists of a 6-layer Boron-Epoxy laminate facings $[\theta/\theta-60/\theta+60]_s$ and polystyrene core. The thickness h of the laminate is 0.001m. The material properties for laminate layers are given as:

E_1 = 194GPa, E_2 = 8.7GPa, G_{12} = 3.2GPa, ν_{12} = 0.33, $\rho = 2100\,\text{kg/m}^3$

X_t = 1300MPa, X_c = 2000MPa, Y_t = 140MPa, Y_c = 300MPa, S = 90MPa.

The material properties for sandwich core are given as:

E = 42MPa, ν = 0.3, $\sigma_u = 1\,\text{MPa}$, $\rho = 150\,\text{kg/m}^3$.

The plate is simply supported at all boundaries and loaded by a uniaxial uniform load see Fig. 1. Thickness h is for the facings and 8*h is for the core (Fig. 2).

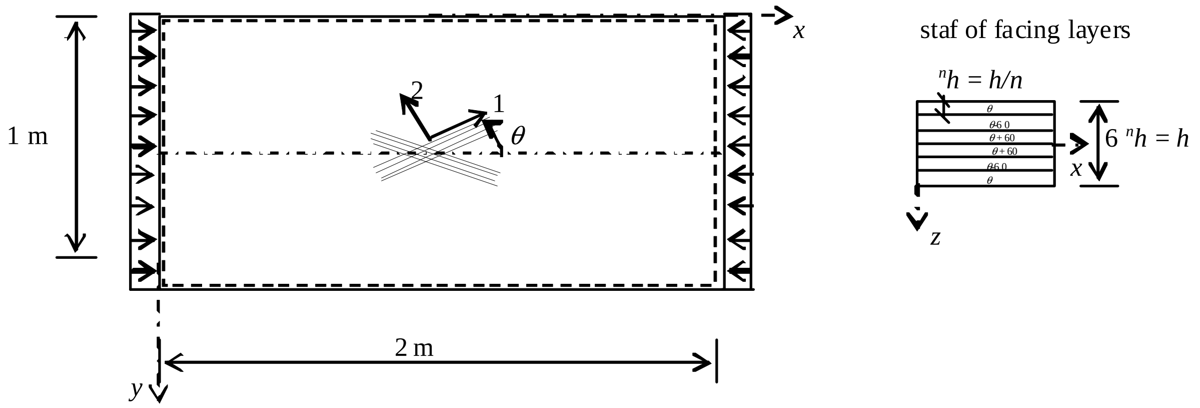

Figure 1: Geometry of the sandwich plate

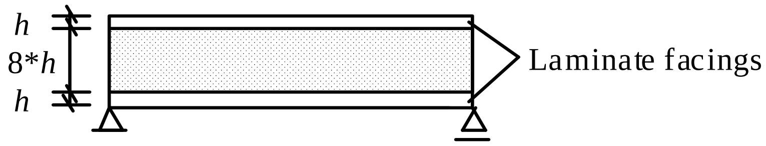

Figure 2: Cross-section of sandwich plate

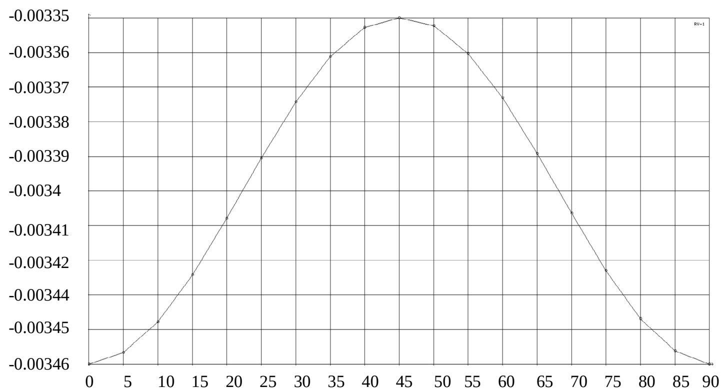

Figure 3: Maximum F.I for changed angle orientation $0^\circ - 90^\circ$

Eigen Value#	Buckling Load Factor
1	1.279965e+001
2	1.544049e+001
3	1.973424e+001
4	2.125276e+001
5	2.979095e+001
6	4.138276e+001
7	5.431924e+001
8	5.670496e+001
9	5.796148e+001
10	6.183806e+001

Table 1: First 10 buckling load factors

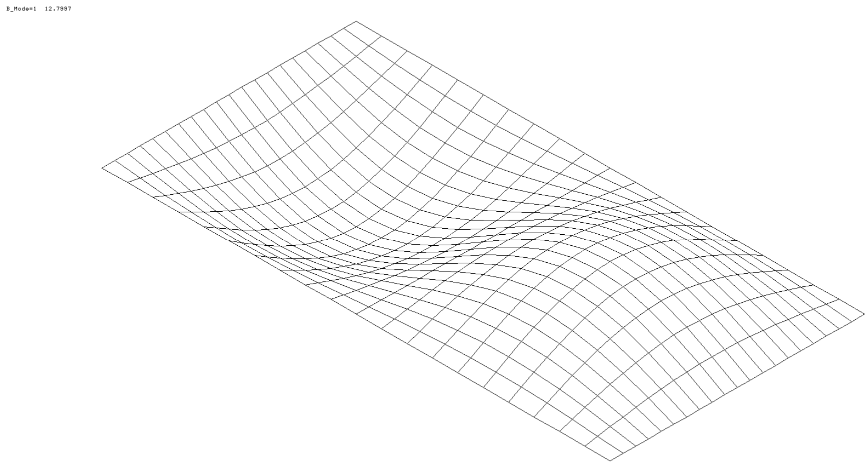

Figure 4: Eigen mode for the first eigen value in buckling analysis after optimization

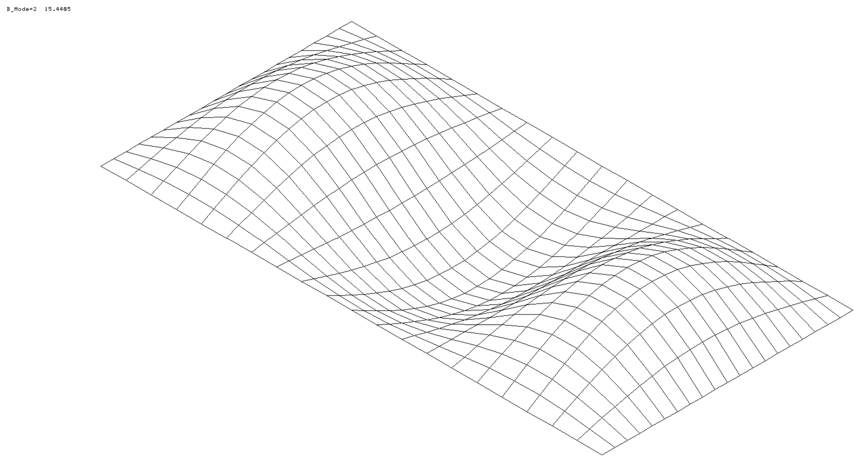

Figure 5: Eigen mode for the second eigen value in buckling analysis after optimization

Frequency#	Frequency (Rad/sec)	Frequency (cycles/sec)	Period (seconds)
1	3.78204e+002	6.01931e+001	1.66132e-002
2	4.53367e+002	7.21556e+001	1.38589e-002
3	5.88496e+002	9.36620e+001	1.06767e-002
4	6.20171e+002	9.87033e+001	1.01314e-002
5	8.62661e+002	1.37297e+002	7.28349e-003
6	1.18672e+003	1.88873e+002	5.29456e-003
7	1.54041e+003	2.45163e+002	4.07892e-003
8	1.60620e+003	2.55635e+002	3.91183e-003
9	1.65680e+003	2.63688e+002	3.79237e-003
10	1.72507e+003	2.74553e+002	3.64229e-003

Table 2: First 10 frequencies in buckling analysis

Frequency#	Frequency (Rad/sec)	Frequency (cycles/sec)	Period (seconds)
1	3.01247e+002	4.79450e+001	2.08573e-002
2	4.84689e+002	7.71406e+001	1.29633e-002
3	7.92671e+002	1.26157e+002	7.92660e-003
4	1.02714e+003	1.63474e+002	6.11719e-003
5	1.10744e+003	1.76255e+002	5.67361e-003
6	1.22031e+003	1.94218e+002	5.14886e-003
7	1.22929e+003	1.95647e+002	5.11124e-003
8	1.54335e+003	2.45632e+002	4.07113e-003
9	1.80077e+003	2.86601e+002	3.48917e-003
10	2.00034e+003	3.18363e+002	3.14106e-003

Table 3: First 10 eigen frequencies in free vibration analysis

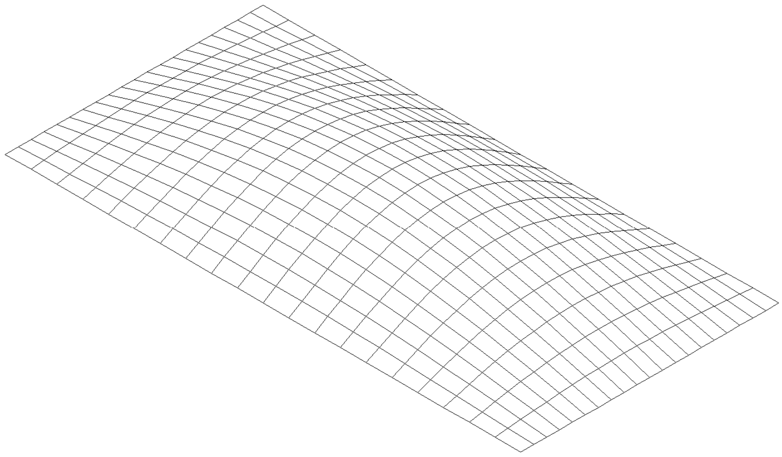

Figure 6: First eigen mode in free vibration analysis after optimization

6 CONCLUSIONS

From sensitivity analysis (Fig.3) one can see, that angle orientation has minor influence on the maximum failure index. The reason is the quasi-isotropic character of the laminate facings. Tsai-Wu criterion is violated, than failure load factor is 289. The results for the buckling factors are shown in Table 1. The first buckling load factor is 22.5 times minor than maximum failure load factor. Optimal design variable is $\theta = 45^\circ$. It means that fibre angle near 45° leads to the highest buckling load for a quadratic plate. Eigen modes in buckling analysis depend on fibre angle orientation [4] and have another shapes then isotropic homogeneous plates. The buckling modes are symmetric to the symmetric axis in loading direction (Figs. 5, 6). The first 10 frequencies in buckling and frequency analysis you can see in Table 2 and 3, respectively. The first ten frequencies in buckling analysis are higher than in free vibration analysis. Eigen mode for the first eigen value in buckling analysis is different than first eigen mode in free vibration analysis. Very similar first eigen modes in both analysis would be, if the fibre angle would be less than 30°.

Preparation of the paper was supported by the Scientific Grant Agency of the Ministry of Education of Slovak Republic and the Slovak Academy of Sciences under Project 1/0201/11.

REFERENCES

[1] H. Altenbach, J. Altenbach, W. Kissing, *Structural analysis of laminate and sandwich beams and plates,* Lublin, 2001.

[2] E. J. Barbero, *Finite element analysis of composite materials*, CRC Press, USA, ISBN-13: 978-1-4200-5433-0, 2007.

[3] Z. Gürdal, R.T. Haftka, P. Hajela, *Design and Optimization of Laminated Composite Materials*, J. Wiley & Sons, 1999.

[4] E. Kormaníková, I. Mamuzic, Buckling analysis of a laminate plate. *Metalurgija*. Vol. 47, no. 2 (2008), p. 129-132. - ISSN 0543-5846.

[5] M. Mihaliková [et al.], Influence of strain rate on automotive steel sheet breaking. *Chemické listy*. Vol. 105, no. 17 (2011), p. 836-837. - ISSN 0009-2770.