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# REDUCTION OF VIBRATIONS TRANSMITTED THROUGH THE SOIL BY MULTIPLE BURIED INCLUSIONS – NUMERICAL ANALYSIS

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**Abstract.** *The displacements wave-field in an elastic soil induced by dynamic loads can be significantly modified by the presence of heterogeneities. This concept can be exploited in order to provide some protection against ground-borne vibrations transmitted through the soil. For this purpose, some authors have studied the use of trench solutions, open or filled with a different material than the hosting medium. In the present work, the authors analyse the possibility of adopting a set of buried solid inclusions to reduce the vibrations that propagate through an elastic half-space. The proposed methodology is based on the implementation of a numerical model, developed from a meshless method, the Method of Fundamental Solutions (MFS). This is a Trefftz-type method, based on the knowledge of fundamental solutions for infinite and semi-infinite media, in which the solution of the propagation problem is obtained from the linear combination of those solutions. The implemented model accounts for the presence of multiple inclusions, filled by an elastic material different from the elastic soil where they are embedded. The authors analyse numerically the influence, on the attenuation provided by the heterogeneities, of different parameters, such as, the number and diameter of the solid inclusions, the depth and geometric configuration used on its installation, or the characteristics of the elastic materials filling the inclusions and observed on the hosting media. Some of the obtained results seem to indicate that the presence of the buried inclusions may lead to changes on the transmitted vibrations, reducing some of its effects in some system configurations.*

## 1 INTRODUCTION

The displacements wave-field in an elastic soil induced by dynamic loads, such as those resulting from the activity of surface and underground transportation systems, can be significantly modified by the presence of heterogeneities. This concept can be exploited in order to provide some protection against ground-borne noise and vibrations transmitted through the soil. The problem of vibration mitigation for soils generated by different sources has been an interesting research topic since the middle of last century [1]. Some authors have studied the use of trench solutions, open or in-filled with a different material than the hosting medium [1-2]. According to the works referred, the use of trenches to provide attenuation of ground-borne vibrations is very dependent on the frequency range of vibrations and should be more effective in the range of moderate to high frequencies.

For the analysis of wave propagation and vibration transmission problems in elastic media, different numerical methods have been applied, mainly based on the well-established techniques, such as the Finite Difference Method (FDM), Finite Element Method (FEM), Boundary Element Method (BEM), and some combinations of these approaches [1].

An important strategy, mainly when dealing with boundary element methods, has been the adoption of specific Green´s functions that account for part of the boundaries of the analysis domain (e.g. on half-spaces or layered propagation media), since they may allow for smaller discretization schemes, leading to accuracy and lower computational effort. One such set of functions has been proposed by Tadeu et al. [3], for the cases of half-spaces and solid layers, subjected to 2.5D loads. The functions proposed by these authors are defined as summations of the effects of plane waves with different inclinations with respect to the horizontal axis; these functions have, in fact, been extensively used in subsequent works.

Recently, a new class of numerical methods, the so-called meshless methods, since they don't use domain or boundary discretizations, has emerged and being developed. Several meshless methods are described in the literature, but in the present work we focus our attention on the Method of Fundamental Solutions (MFS), a Trefftz-type method [4].

In the present work, a formulation of the MFS is presented, based on the fundamental solutions proposed by Tadeu et al. [3] and Tadeu and Kausel [5], for the analysis of wave propagation in the presence of a set of multiple solid inclusions, located in an elastic half-space medium. To establish those solutions, the wavenumber integral is conveniently transformed in a discrete summation, assuming the presence of an infinite number of virtual sources, equally spaced along the horizontal direction. The model thus defined is verified by comparing its results with those directly obtained by the Green's functions, when no inclusions are present. Finally, a numerical application illustrates the applicability of the method to model a system with different sets of buried inclusions, in order to assess their capabilities to provide attenuation of ground-borne vibrations.

## 2 MATHEMATICAL FORMULATION

### 2.1 Method of Fundamental Solutions (MFS)

Consider an elastic half-space $\Omega_1$ with density $\rho_1$, allowing a shear wave velocity $\beta_1$ and a compressional wave velocity $\alpha_1$, containing an elastic inclusion $\Omega_2$, with density $\rho_2$ and where P and S waves propagate with velocities $\alpha_2$ and $\beta_2$, respectively, as depicted in Figure 1. This system is excited by a dynamic load, oscillating with a frequency $\omega$, and located in the host medium at position $(x_0, y_0)$.

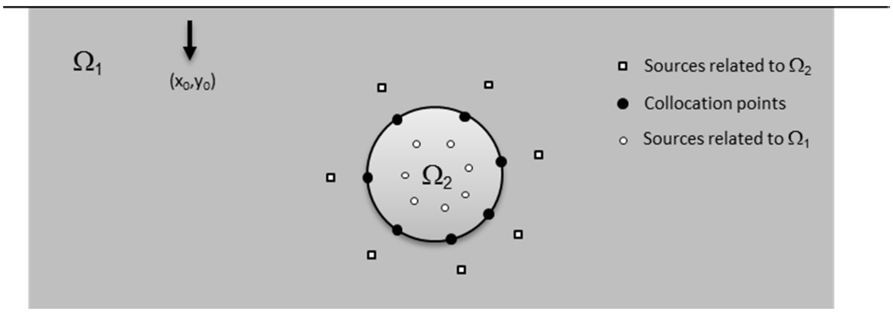


Figure 1: Schematic representation of the MFS model.

The system described above is analysed by means of the MFS, in which two sets of NS virtual loads are distributed along the interface between the two media, at distances *D* from that boundary. At each position, loads acting along the *x* and *y* directions must be considered. Sources inside the inclusion have unknown amplitudes $P_{nj}$, while those placed in the host medium have unknown amplitudes $Q_{nj}$; *n* is the load order number, and *j* represents the direction in which the load acts. The reflected displacement field inside each domain is then defined as a summation of the contribution of these virtual sources.

In the outer elastic medium, the displacements are given by:

$$u_i^{(1)}(x,y)=\sum_{n=1}^{NS}\sum_{j=1}^{3}P_{nj}G_{ij}^{(1)}(x_n^{(1)},y_n^{(1)},x,y,\rho_1,\alpha_1,\beta_1)\,, \tag{1}$$

while inside the inclusion, they are

$$u_i^{(2)}(x,y)=\sum_{n=1}^{NS}\sum_{j=1}^{3}Q_{nj}G_{ij}^{(2)}(x_n^{(2)},y_n^{(2)},x,y,\rho_2,\alpha_2,\beta_2)\,, \tag{2}$$

where $G_{ij}^{(m)}(x_n^{(m)},y_n^{(m)},x,y,\rho_m,\alpha_m,\beta_m)$ is the displacement at point $(x,y)$, in medium *m*, along direction *i*, generated by a load acting along *j* at position $(x_n^{(m)},y_n^{(m)})$, computed using the adequate fundamental solutions.

Here, two sets of fundamental solutions are used, namely those for an elastic half-space to simulate the host medium, and those for an elastic full-space, to simulate the field within the inclusion. In the next section some details are given concerning these solutions. To determine the unknown load amplitudes, it is necessary to impose the boundary conditions of continuity of tangential and normal displacements and stresses along the interface, at NS collocation points, establishing a linear equation system with $4\times\mathrm{NS}$ equations and $4\times\mathrm{NS}$ unknowns. After solving these equations, the displacements at any point $(x,y)$ in the propagation domain can be determined.

## 2.2 Fundamental solutions

To avoid the discretization of the horizontal surface of the half-space, fundamental solutions that take its presence into account are used, following the works of Tadeu et al. [3]. For a half-space medium, the total wavefield can be expressed by taking into account the incident field generated by the source (source terms), and the terms generated at the surface (surface terms). The source terms can be written making use of the equations proposed by Tadeu and Kausel [5] for 2.5D loads, while the surface terms can be represented by a set of one dilatational and two shear potentials, with unknown amplitude values. Both the source and the surface terms are expressed as continuous integrals of the effects of plane waves. These integrals can then be discretized into summations of discrete terms, assuming the existence of an infinite number of virtual sources placed along the $x$ direction at equal intervals, $L_x$. The distance separating them is large enough to prevent the virtual loads from contaminating the response. For loads acting along the *x* and *y* directions, and considering the contribution of 2N+1 terms, the potentials that define the surface terms are:

*Load acting along the x direction:*

$$\phi_0^x = E_a \sum_{n=-N}^{n=+N} \left( \frac{k_n}{\nu_n} E_{b0} A_n^x \right) E_d \; ; \; \psi_0^x = -E_a \sum_{n=-N}^{n=+N} \left( E_{c0} C_n^x \right) E_d \tag{3}$$

*Load acting along the y direction:*

$$\phi_0^y = E_a \sum_{n=-N}^{n=+N} \left( E_{b0} A_n^y \right) E_d \; ; \; \psi_0^y = E_a \sum_{n=-N}^{n=+N} \left( \frac{k_n}{\gamma_n} E_{c0} B_n^y \right) E_d \tag{4}$$

with $E_a = \dfrac{1}{2\rho_1 \omega^2 L_x}$ , $E_{b0} = \mathrm{e}^{-\mathrm{i}\nu_n y}$ , $E_{c0} = \mathrm{e}^{-\mathrm{i}\gamma_n y}$ , $E_d = \mathrm{e}^{-\mathrm{i}k_n(x-x_0)}$ , $\nu_n = \sqrt{k_{p1}{}^2 - k_n{}^2}$ ( $\mathrm{Im}(\nu_n) \le 0$ ), $\gamma_n = \sqrt{k_{s1}{}^2 - k_n{}^2}$ ($\mathrm{Im}(\gamma_n) \le 0$), $k_{p1} = \omega/\alpha_1$, $k_{s1} = \omega/\beta_1$ , $\mathrm{i} = \sqrt{-1}$, and with $k_n = 2\pi n/L_x$ being the horizontal wavenumber along *x*. Using this methodology, and imposing the necessary boundary conditions (null tangential and normal stresses at the half-space free surface), systems of equations can be established for each value of $k_n$ that allow the unknown amplitude factors to be calculated. As an example, taking the specific case of a load acting along *x*, after solving the relevant equation systems for a full sequence of values of $k_n$, in the range [$-\frac{2\pi}{L_x}\mathrm{N}$ ; $+\frac{2\pi}{L_x}\mathrm{N}$ ], the final displacements can be computed as

$$\begin{aligned} G_{xx}^{(1)}(x_0, y_0, x, y, \rho_1, \alpha_1, \beta_1) &= G_{xx}^{full}(x_0, y_0, x, y, \rho_1, \alpha_1, \beta_1) + \\ &+ E_a \sum_{n=-N}^{n=+N} \left( A_n^x \frac{-\mathrm{i}k_n{}^2}{\nu_n} E_{b0} - \mathrm{i}\gamma_n E_{c0} C_n^x \right) E_d \end{aligned} \tag{5}$$

$$\begin{aligned} G_{yx}^{(1)}(x_0, y_0, x, y, \rho_1, \alpha_1, \beta_1) &= G_{yx}^{full}(x_0, y_0, x, y, \rho_1, \alpha_1, \beta_1) + \\ &+ E_a \sum_{n=-N}^{n=+N} \left( -\mathrm{i}k_n A_n^x E_{b0} + \mathrm{i}k_n C_n^x E_{c0} \right) E_d \end{aligned} \tag{6}$$

where $G_{ij}^{full}(...)$ represents the displacement generated along direction *i*, due to a load acting along *j*, by a dynamic source in an unbounded elastic medium. The displacements originated by loads acting along different directions can be determined in a similar way.

Within the inclusion, the required fundamental solution for the displacement field can be described just by the full-space Green's function (see [5]), such that

$$G_{xx}^{(2)}(x_0, y_0, x, y, \rho_2, \alpha_2, \beta_2) = G_{xx}^{full}(x_0, y_0, x, y, \rho_2, \alpha_2, \beta_2) \tag{7}$$

$$G_{yx}^{(2)}(x_0, y_0, x, y, \rho_2, \alpha_2, \beta_2) = G_{yx}^{full}(x_0, y_0, x, y, \rho_2, \alpha_2, \beta_2) . \tag{8}$$

## 3 VERIFICATION AND ANALYSIS OF THE NUMERICAL MODEL

### 3.1 Numerical verification

The proposed MFS formulation was verified by comparing the computed results with those obtained directly by means of the appropriate Green's functions for an elastic homogenous half-space medium. For this purpose, consider a circular solid inclusion of unit radius centred at point $(x_c = 0\ \mathrm{m}, y_c = 3\ \mathrm{m})$, embedded in an elastic half-space that is excited by a dynamic load acting at point $(x_0 = -10\ \mathrm{m}, y_0 = 2\ \mathrm{m})$, along the vertical direction (see Figure 2). For this system, if the same elastic properties are ascribed both to the host medium, $\Omega_1$, and to the inclusion, $\Omega_2$, the solution of the problem at any point inside the inclusion can be obtained analytically using Green's functions derived in a similar way to equations (5) and (6). Assuming the following elastic properties of both the unbounded half-space and the inclusion materials, density $\rho_1 = \rho_2 = 2500\ \mathrm{kg/m^3}$, Young's modulus $E_1 = E_2 = 2\ \mathrm{GPa}$ and Poisson's coefficient $\nu_1 = \nu_2 = 0.2$, the response of the system is computed for a specific frequency of $50\ \mathrm{Hz}$, over a set of receivers placed along a circular line with radius of $0.5\ \mathrm{m}$, within the inclusion.

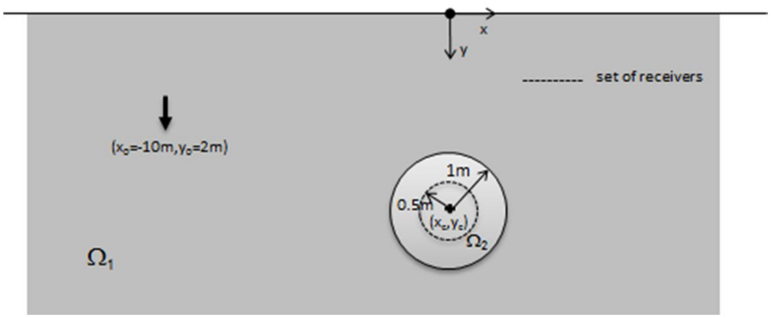


Figure 2: Schematic representation of the system used for model verification, prescribing identical elastic properties to media $\Omega_1$ and $\Omega_2$.

Figure 3 presents the modified positions of the circular line of receivers, with the displacements in both directions being amplified by a factor of $1*10^{10}$, for representation purposes of the Real and Imaginary parts of the response. For the MFS model, a total number of 28 collocation points is adopted, with the two sets of virtual sources being placed at distances $D = 0.4\ \mathrm{m}$ from the media interface, towards the interior of the inclusion and the host medium. The responses computed by the MFS model are represented by the continuous lines and the circular marks correspond to the analytical response determined directly with the Green's functions. For the analysed frequency, the responses evaluated by the two methodologies match almost perfectly, revealing that the proposed formulation is accurate.

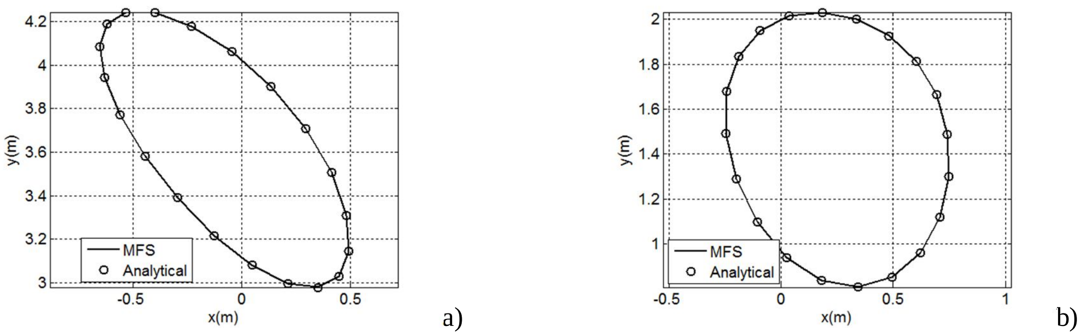


Figure 3: Numerical model verification – modified positions of receivers, f=50Hz: a) and b) respectively, represent the Real and Imaginary parts of the response, amplified by a factor of $1*10^{10}$.

### 3.2 Analysis of the behaviour of the numerical model

In order to better understand the behaviour of the MFS model, an additional numerical study is performed concerning the variability of its results with the number of collocation points and with the position of the virtual sources. For this purpose, consider now the system represented in Figure 4, in which one elastic circular inclusion is buried within an elastic half-space ( $\rho_1 = 1700\ \text{kg/m}^3$ , $E_1 = 116\ \text{MPa}$ and $\nu_1 = 0.33$ ). The radius of the inclusion is $0.4\ \text{m}$ , its central point is located at $(x_c = -15\ \text{m}, y_c = 0.9\ \text{m})$ , and it is perfectly filled with solid material with elastic properties of $\rho_2 = 2700\ \text{kg/m}^3$ , $E_2 = 4416\ \text{MPa}$ and $\nu_2 = 0.2$ . This system is excited by a dynamic load, acting near the surface at $(x_0 = 0\ \text{m}, y_0 = 0.1\ \text{m})$ , along the vertical direction. The response, in terms of *y* displacement component (both Real and Imaginary parts), is calculated for frequencies of $12.5\ \text{Hz}$ and $50\ \text{Hz}$ , at an exterior receiver R placed at $(x_R = 10\ \text{m}, y_R = 0.1\ \text{m})$ .

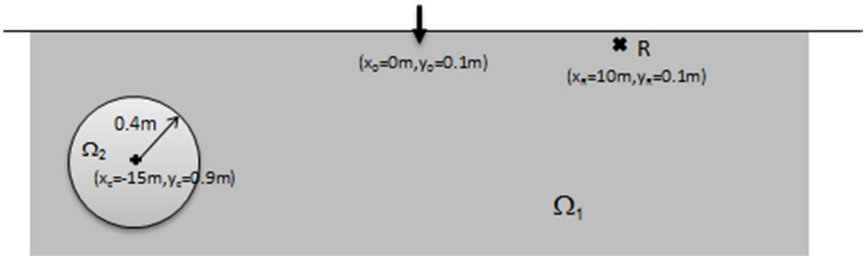


Figure 4: Schematic representation of the system used for studying the behaviour of the model.

In Figure 5a, the *y* displacement at receiver R can be observed, calculated for a fixed number of 38 collocation points, when the distance between the virtual sources and the interface ( $D$ ) assumes different values. In those figures, the relation $D/R$ is used to define the distance as a function of the radius of the elastic inclusion ( $R$ ). As can be observed, the responses are stable as long as the virtual sources are not very close to the interface. For that case, a singularity of the fundamental solution occurs very close to the boundary, degrading the quality of the results. When $D/R$ is 0.3 or larger, the responses are very similar, indicating a good behaviour of the MFS model. Additionally, vertical displacements at R are presented in Figure 5b, computed in the case of a fixed relation $D/R = 0.6$ , when different numbers of collocation points are chosen. Here, the responses can be seen to stabilize above 30 collocation points.

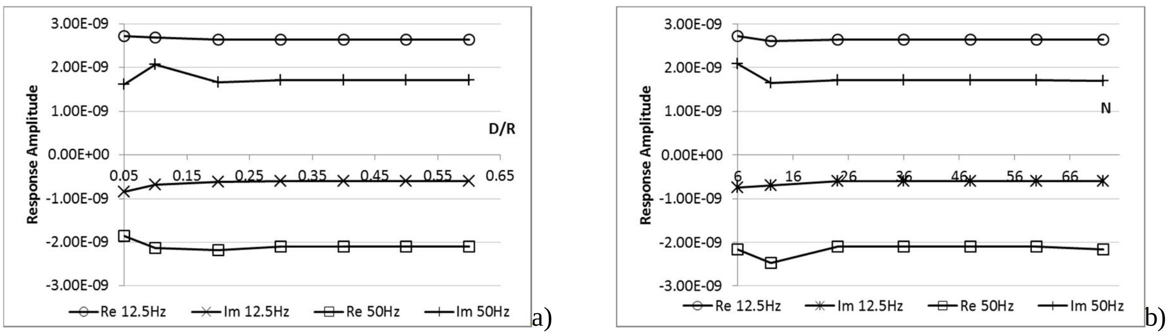


Figure 5: Responses at receiver R, vertical (*y*) component of displacement responses, for model analysis: a) placement of virtual sources and b) number of virtual sources.

# 4 NUMERICAL APPLICATION

## 4.1 General description of the analysed systems

In the examples presented, a set of $n*m$ solid circular inclusions (with radii equal to $0.4\,\text{m}$) are buried in an elastic half-space, excited by a load acting near the surface at $(x_0 = 0\text{ m}, y_0 = 0.1\text{ m})$ along the vertical direction, with the generic geometry of the system being represented schematically in Figure 6. All the $n*m$ solid inclusions are assumed to be buried at a certain depth to the surface and made by a homogenous material with elastic properties of $\rho_2 = 2700\text{ kg/m}^3$, $E_2 = 4416\text{ MPa}$ and $\nu_2 = 0.2$, while the hosting soil is characterized by $\rho_1 = 1700\text{ kg/m}^3$, $E_1 = 116\text{ MPa}$ and $\nu_1 = 0.33$. In the present analysis, the circular inclusions are placed orthogonally, aligned along both *x* and *y* directions, with the distance between their centers being kept constant and equal to twice the inclusions' diameter. Also, in those $n*m$ configurations, the center of the central inclusions along the *x* direction is kept at $x_c = -15\text{ m}$.

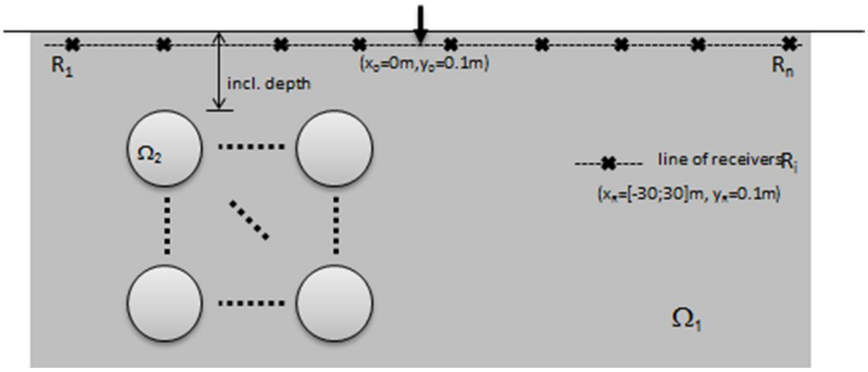


Figure 6: Model generic geometry defined for the numerical application with a set of $n*m$ buried inclusions.

The response, in terms of horizontal (*x*) and vertical (*y*) displacements, is computed along a line receivers $R_i$, placed next to the surface from $x_{R1} = -30\text{ m}$ to $x_{Rn} = 30\text{ m}$, equally spaced $0.1\text{ m}$ apart, and placed $0.1\text{ m}$ below the surface. The computations are performed using the MFS model, with 28 collocation points distributed along the boundary of each inclusion, and with two sets of 28 virtual sources placed at either side of that interface, at a distance of $0.24\text{ m}$. Computations are performed for two specific frequencies of $12.5\text{ Hz}$ and $50\text{ Hz}$, in the range of frequencies of interest of the addressed problem.

In order to evaluate the effect of the presence of the circular inclusions on the vibrations registered at the receivers, a reduction of vibration levels, in dB, is evaluated. This reduction is computed in terms of an insertion loss, *IL*, between displacement vibration levels obtained, respectively, in the presence of the set of inclusions and in the same system but without inclusions, and which can be given by

$$IL_i = -20\log\frac{\left|u_i^{(1)}\right|}{\left|G_{ij}^{(1)}\right|}, \tag{9}$$

where $u_i^{(1)}$ denotes the displacement field, along direction *i*, calculated at receivers on the outer elastic medium when the set of inclusions is present, and $G_{ij}^{(1)}$ represents the displacements (along direction *i* and generated by a load acting along *j*) determined from the analysis of the system excluding the circular inclusions. According to this equation, positive values of the insertion loss are achieved when the presence of the set of elastic inclusions reduces the displacement vibration levels, corresponding to a screening effect and to the mitigation of vibration transmission through the soil. On the other hand, negative values of the insertion loss are observed when the presence of the buried inclusions leads to higher vibration levels and less efficient protective solutions.

### 4.2 Influence of the set of inclusions' depth

In this first case, consider a set of 5*3 inclusions, that is placed at two different depths from the soil surface to the top of the upper inclusion, namely $1\,\mathrm{m}$ and $3\,\mathrm{m}$. Figure 7 illustrates the insertion loss evolution, in dB, along the line of selected receivers near the surface, for *x* and *y* components of the displacements and for frequencies of $12.5\,\mathrm{Hz}$ and $50\,\mathrm{Hz}$.

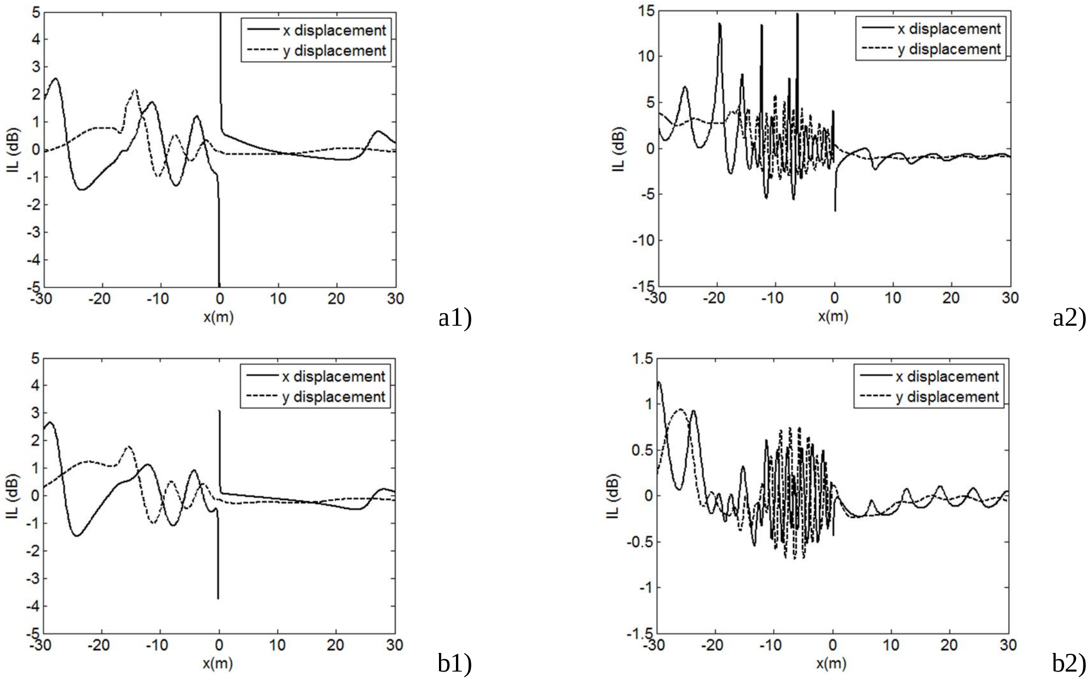


Figure 7: Influence of the depth of the set of 5*3 inclusions: a) depth from the top of the upper inclusion 1m and b) 3m; 1 and 2, respectively, frequencies f=12.5Hz and 50Hz.

From the observation of the presented results, the influence of the depth of the inclusions is not very perceptible at a low frequency of $12.5\,\mathrm{Hz}$. However, at a higher frequency of $50\,\mathrm{Hz}$, the effect of this geometric parameter reveals to be determinant in the effect provided by the presence of the inclusions. Although at receivers placed between the inclusions and the load position some amplification of the vibration levels are observed (negative *IL* values), at receivers placed "downward" from the set of inclusions, considerable attenuation values (positive *IL* values) are identified, for the lowest depth values.

### 4.3 Influence of the number of inclusions

In order to identify the effect of a varying number of inclusions, consider now the presence of a line with 3 inclusions and the case when this configuration is repeated 5 times (e.g. sets of 1*3 and 5*3 inclusions, regarding the same depth of $0.5\,\mathrm{m}$ is observed, from the top of the upper inclusions to the surface). For these systems, the computed results in terms of insertion loss, in dB, at the same line of receivers are presented in Figure 8.

For the selected configurations of the inclusions and their position in the elastic medium, the number of inclusions has not shown great influence in the obtained insertion loss levels, mainly for a frequency of 50 Hz. For the lower frequency, the response is more dependent on the number of inclusions, with higher *IL* levels observed for a larger number of inclusions.

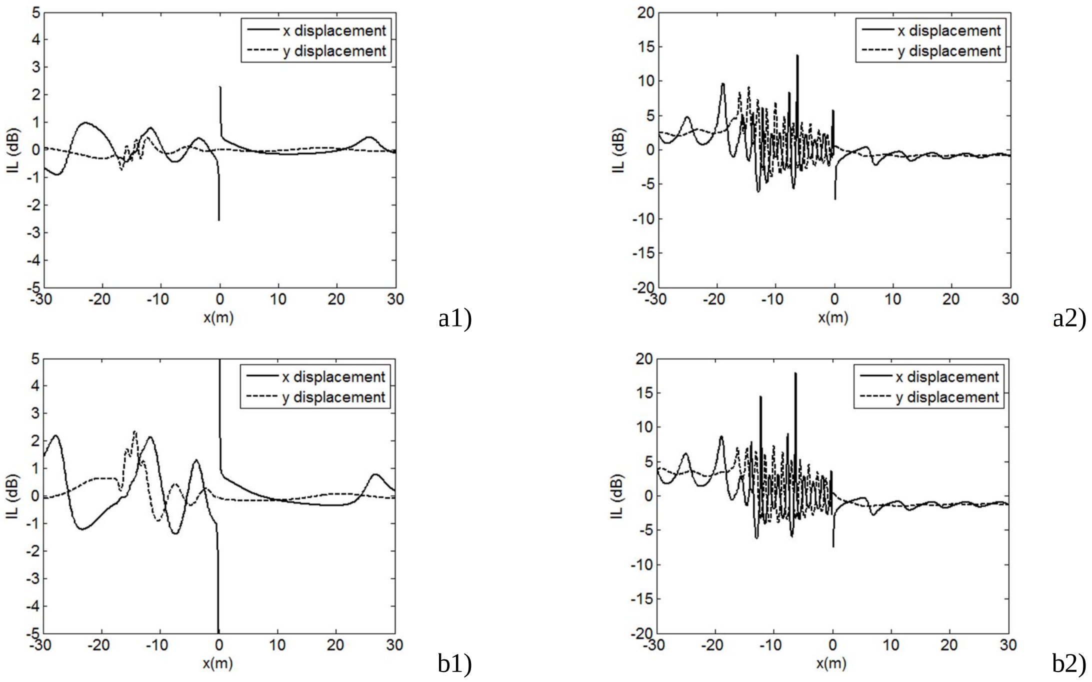


Figure 8: Influence of the number of inclusions with spatial arrangements of: a) 1*3 and b) 5*3; 1 and 2, respectively, frequencies f=12.5Hz and 50Hz.

### 4.4 Influence of the alignment of a set of inclusions'

In the third situation presented in this work, two sets of 5 inclusions are embedded in the half-space: the first in a horizontal alignment of one row with 1*5 inclusions, and the second in a vertical alignment of one column with 5*1 inclusions. In both cases, the depth from the top of the upper inclusion to the surface is kept constant at 0.5 m. In Figure 9, the results obtained with the MFS model for insertion loss, at the line of receivers, can be observed.

In this situation, there are evident differences between system responses, for both analysed frequencies, when installing the line of inclusions along a horizontal or vertical configuration. However, as before, higher values of attenuation are provided for the higher frequency of 50 Hz. Comparing the responses for horizontal and vertical alignments, more intense fluctuations and higher values of insertion losses are observed when a horizontal line of 1*5 inclusions is adopted, reaching interesting reduction levels of almost 10 dB.

## 5 FINAL REMARKS

In this work, the analysis of the reduction of ground-borne and transmitted vibrations by multiple buried inclusions has been addressed, using a 2D formulation of the Method of Fundamental Solutions. The method presented is based on fundamental solutions for wave propagation in unbounded and semi-infinite elastic media, formulated in the frequency domain. It was found to be both applicable and accurate for the analysis of the specific problems addressed here, and could thus be a valuable and efficient tool for analysing wave and vibration propagation problems in the presence of buried elastic heterogeneities.

As a numerical application, the authors analysed the case of a set of solid circular inclusions embedded in a homogeneous half space, ascribing different properties to the two media and geometric configurations. The computed results, in the frequency domain, enabled the observation of interesting signal features of the modified displacement field observed in the elastic half-space as a consequence of the presence of the buried inclusions. In terms of the reduction of (horizontal and vertical) displacement vibration levels, insertion loss positive and negative values were identified, indicating that further analyses should be performed in order to optimize its configuration as a protective system/device.

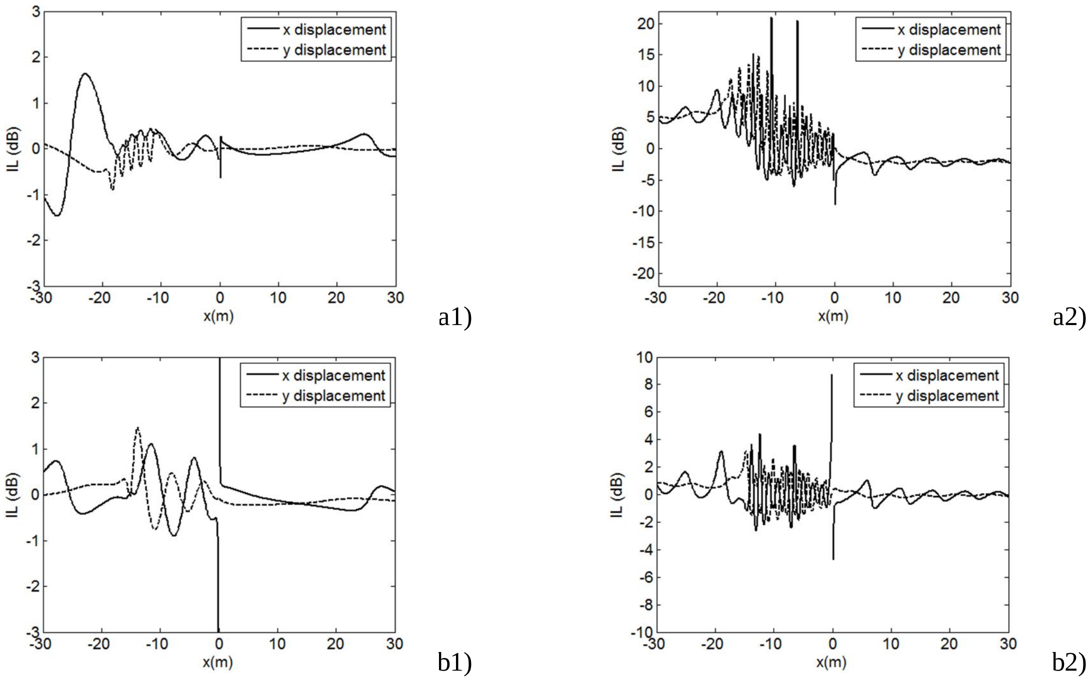


Figure 9: Influence of the alignment of a set of inclusions': a) horizontal alignment of a set of 5 inclusions and b) vertical alignment of a set of 5 inclusions; 1 and 2, respectively, frequencies f=12.5Hz and 50Hz.

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