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TRANSIENT HEAT CONDUCTION IN FUNCTIONALLY GRADED MATERIAL HOLLOW CYLINDERS BY NUMERICAL METHOD

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Abstract. *This study presents the temperature distribution of a hollow cylinder for transient heat conduction. The solution is employed using a numerical method called Durbin's Inverse Method. The material under consideration is the functionally graded material (FGM), made from two different material constituents, in which the material properties change from one surface to another surface continuously or gradually. In this study, the material properties such as elasticity modulus, density, and thermal conductivity are considered as a function of radius only. The effect of in-homogeneity parameter, which is a non-dimensional quantity design point, is determined for its positive values for all material properties to determine temperature distribution for various time.*

1 INTRODUCTION

The functionally graded materials (FGMs) were first used in Japan for some applications in space, aviation, and nuclear systems in order to obtain more efficient thermal barrier materials and insulations for structures, etc. [1-5]. Since these materials are made of a mixture with arbitrary composition of the constituents (e.g. metallic and ceramic) the properties of the whole material can be changed continuously and gradually from one surface to the other. Piezoelectric, sensors, dental and orthopaedic implants, and thermo generators are a few examples of FGM applications [6,7]. Because of its characteristics, thermal stresses, residual stresses, and stress concentration factors can be reduced controlling for the material properties such as elasticity modulus, density, thermal conductivity, linear expansion coefficient, and so on [8]. They can be defined a function as power-law, exponential, linear, and quadratic, etc.

Here through the literature can be given some studies employed both analytically and numerically about FGM behaviours under different conditions and different dimensional sizes. Jabbari et al. [9] used a direct method of solution of the Navier equation on a thick hollow cylinder of FGM under a one-dimensional steady-state temperature distribution with general types of thermal and mechanical boundary conditions. Another analytical solution was studied by Hosseini et. al for transient temperature distribution in functionally graded thick hollow cylinders [10]. Zhao [11] solved a two-dimensional (2D) thermoelastic problem where the cylinder is assumed to be composed of a number of fictitious layers in the radial direction. Guo and Noda [12] studied a problem for an FGM cylinder which is reduced to a plane problem. They used a perturbation method to investigate thermal shock of a FGM shell in 2D. Asgari and Akhlaghi used the classical theory of thermoelasticity to investigate the transient thermal stresses in 2D functionally graded thick-hollow cylinder with finite length [13]. Darabseh et. al [14] considered an orthotropic cylinder under the hyperbolic heat conduction model to determine the transient thermal stresses. Different studies in 3D can be reviewed in detail. See Cheng and Batra for elliptic plates [15], Vel and Batra for rectangular plates [16], and Shao et. al for hollow cylinders [17].

The numerical method considered is called Durbin’s inverse method. It gives a simple solution for problems are difficult to solve in Laplace domain. Ref [18] provides the theoretical background for this method. It has been applied to such problems as hyperbolic heat conduction of FGM cylinders [19], spheres [20], pressure vessels [21], and beams [22]. These references show that the method mentioned above can be used to obtain a numerical approach for problems which are difficult to solve analytically.

The purpose of this study is to determine numerically the transient temperature distribution of a thick-walled FGM hollow cylinder due to varying material properties in radial direction. The material properties such as elasticity modulus, density, and thermal conductivity are considered as a function of radius only. The governing equation is generated based on power-law. All numerical solutions are employed using Mathematica. The effect of in-homogeneity parameter β, which is a non-dimensional quantity design point, is determined for its positive values for all material properties to determine temperature distribution for various time.

2 GOVERNING EQUATIONS

The transient heat conduction equation for a thick-walled FGM hollow cylinder to find the temperature distribution in radial direction is given the following form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(k(r)r\frac{\partial T}{\partial r}\right) = \rho(r)c(r)\frac{\partial T}{\partial t} \tag{1}$$

Where k is the thermal conductivity, ρ is the density, and c is the specific heat of the material. All of them are a function of radial coordinate r.They are assumed as a power function of R which is called nondimensionless radial coordinate.

$$k(R) = k_0 R^{\beta_1}, \qquad \rho(R) = \rho_0 R^{\beta_2}, \qquad c(R) = c_0 R^{\beta_3} \tag{2}$$

Where k_0, ρ_0 and c_0 are material constants, and β_1, β_2 and β_3 are inhomogeneity parameters. Let's introduce the other nondimensional variables which used to obtain nondimensional equation.

$$\theta = \frac{T - T_0}{T_{w0} - T_0}, \qquad R = \frac{r}{r_0}, \qquad r_\gamma = \frac{r_i}{r_0}, \qquad K_0 = \frac{k_0}{c_0 \rho_0}, \qquad \tau = \frac{K_0 t}{{r_0}^2} \tag{3}$$

Here r_γ gives the relative thickness of the cylinder i.e. it is the rate of inner radius with respect to outer radius. T indicates the temperature, T_0 is symbolized for ambient temperature, and T_{w0} gives the absolute temperature of the outer surface of the cylinder. Furthermore τ is used to show nondimensional time.

For convenience let's take $\beta_1 = \beta_2 = \beta_3 = \beta$. After some algebraic arrangement the transient heat conduction as a second-order differential equation can be written into nondimensional form as:

$$\frac{\partial^2 \theta}{\partial R^2} + (\beta + 1) \frac{1}{R} \frac{\partial \theta}{\partial R} = R^\beta \frac{\partial \theta}{\partial \tau} \tag{4}$$

To find the temperature distribution now we apply Laplace transform to Eq. (4).

$$\tilde{\theta}_{RR} + (\beta + 1) R^{-1} \tilde{\theta}_R - R^\beta s \tilde{\theta} = 0 \tag{5}$$

Where symbol "~" indicates that Laplace transform of the function, and s is the Laplace parameter. The solution of the Eq. (5) is based on Bessel's equation

$$\tilde{\theta}(R, s) = \frac{\left(R^{1+\beta}\right)^A \left[\left(I_{-A,B}\right)\left(I_{A,C}\right) - \left(I_{-A,C}\right)\left(I_{A,B}\right)\right]}{s\left[\left(I_{A,B}\right)\left(I_{A,D}\right) - \left(I_{-A,D}\right)\left(I_{A,B}\right)\right]} \tag{6}$$

Where

$$A = \frac{\beta}{2(1+\beta)}, \qquad B = \frac{2\sqrt{s}(1+\beta)\left({r_\gamma}^{1+\beta}\right)^{\frac{2+\beta}{2(1+\beta)}}}{(1+\beta)(2+\beta)},$$
$$C = \frac{2\sqrt{s}(1+\beta)\left(R^{1+\beta}\right)^{\frac{2+\beta}{2(1+\beta)}}}{(1+\beta)(2+\beta)}, \qquad D = \frac{2\sqrt{s}(1+\beta)}{(1+\beta)(2+\beta)} \tag{7}$$

And finally the boundary conditions are given below

$$\theta = 0 \text{ at } R = 0.6$$
$$\theta = 1 \text{ at } R = 1 \tag{8}$$

3 DURBIN'S INVERSE METHOD

The problem is solved analytically in the Laplace domain, and the results obtained are transformed to the real-time space using the modified Durbin's numerical inversion method [18]. It is based on Fast Fourier Transform (FFT) algorithm. The formulation of this method can be summarized as below:

The function $f(t)$ at time t_n is given by

$$f(t_n) \cong \frac{2Exp[an\Delta t]}{T}\left[-\frac{1}{2}Re\{\bar{F}(a)\} + Re\left\{\sum_{k=0}^{N-1}(\bar{F}(p_k)L_k)Exp\left[i\left(\frac{2\pi}{N}\right)n_k\right]\right\}\right] = 0 \quad (n = 0,1,2,\dots,N-1) \tag{9}$$

where a is an arbitrary number and it is recommended to choose its value between 5 and 10 [19]. In this study $a = 7.5$. $\bar{F}(p_k)$ indicates Laplace transform of $f(t)$, and p_k gives the kth Laplace parameter. Δt is called the time increment, and $T = N\Delta t$ is the solution interval where N is a number between 50 and 5000. The choosing of constrains is important for minimizing both discretization and truncation errors in the subroutine. Let's introduce Lancosz factor, L_k to modified the results where $L_k = \sin(k\pi/N)/(k\pi/N)$ [23]. It is noticed that when $k = 0$, $L_k = 1$.

4 RESULTS

The solution gives transient heat conduction problem in one-dimensional problem where a thick-walled FGM hollow cylinder. In order to test the effect of in-homogeneity parameter β, the algorithm was applied for three positive β values (0.1, 1.5, and 5) in three τ values (0.01, 0.03, and 0.5). All material properties considered are calculated based on power-law function type, and all results are obtained numerically using a Mathematica program. Two figures are depicted here to see temperature distribution along radial coordinate for $\tau = 0.01$ (Fig. 1), and for $\tau = 0.5$ (Fig. 2).

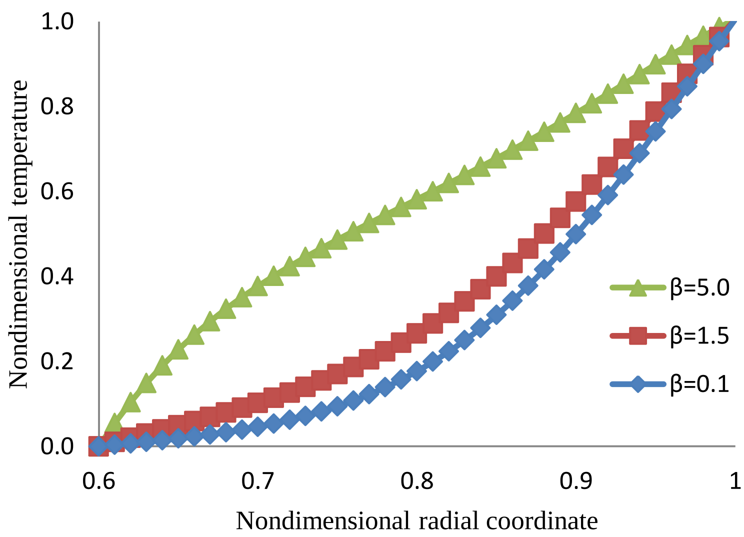

Figure 1: Temperature distribution along radial coordinate when $\tau = 0.01$.

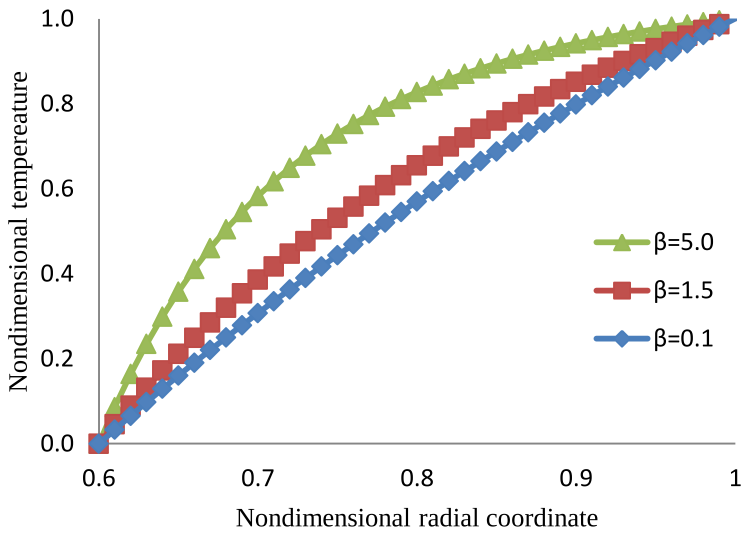

Figure 2: Temperature distribution along radial coordinate when $\tau = 0.5$.

Figure 3 is plotted to see clearly the effect of highest β for different time values. A combination among β_1, β_2, and β_3 for the lowest value as 0.1 and for the highest value as 5 are given in Table 1.

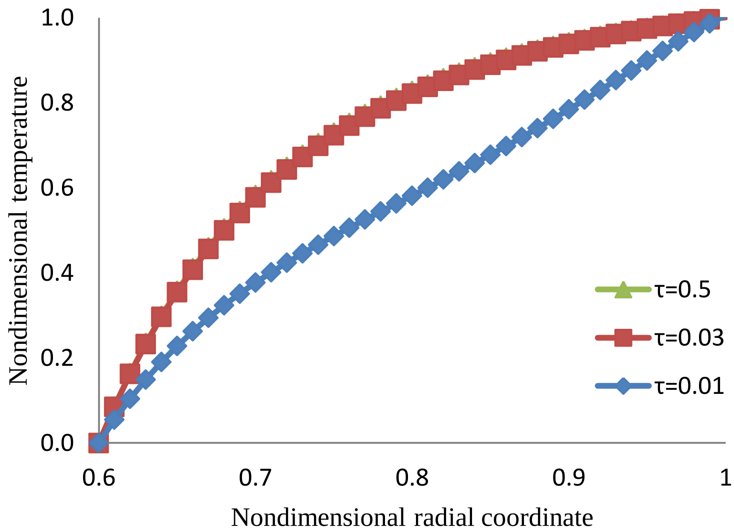

Figure 3: Temperature distribution along radial coordinate when $\beta = 5$.

$\beta_1 = 0.1,\ \beta_2 = \beta_3 = 5$	$\beta_2 = 0.1,\ \beta_1 = \beta_3 = 5$	$\beta_3 = 0.1,\ \beta_1 = \beta_2 = 5$
0.569763	0.827461	0.827461

Table 1: Comparison of β values when $\tau = 0.5$.

5 CONCLUSIONS

- When β values increase the value of nondimensional temperature increases.
- For higher β values the nondimensional temperature reaches steady state more quickly.
- The changing of β_1 is less effective than the other parameters β_2 and β_3 on temperature distribution.
- Since we have the same result for β_2 and β_3 they can be considered as only one parameter.
- The results show that β is a useful parameter to determine temperature distribution of FGM.
- These data can be used for calculation of stress distribution for similar applications.

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