11th International Conference on Vibration Problems
Z. Dimitrovová *et al.* (eds.)
Lisbon, Portugal, 9-12 September 2013

A NON-PARAMETRIC APPROACH IN THE UNCERTAINTY QUANTIFICATION OF STRUCTURAL DYNAMIC MODELS

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Keywords: Non-Parametric Approach, Epistemic Uncertainty, Model Uncertainty, Dynamic Models, Uncertainty Quantification.

Abstract. *This paper presents the criteria for application of Non-Parametric Approach in quantifying non-parametric uncertainties - also called epistemic uncertainties or model uncertainties - in structural dynamic models. The goal is to present a procedure for quantifying these uncertainties in such models in order to increase their degree of reliability. The first phase of development is the stochastic modelling which involves the determination of the probability density function for the random matrix of a dynamic linear system with n degrees of freedom. This development includes the application of random matrix theory and maximum entropy method and consists in solving an optimization problem in which occurs the maximization of uncertainty subject to available information. With obtaining of the adequate function one can proceed to the second phase of the approach that is the stochastic simulation through Monte Carlo simulation. An example of a damped linear dynamic system with two degrees of freedom is studied, and in addition to the quantification of uncertainty, analyses about the dispersion parameter are performed as to verify the importance of the correct calculation of this parameter.*

1 INTRODUCTION

Uncertainties have a key hole to establish the reliability of a numerical model and they have been the subject of constant research in many areas of engineering particularly when it comes to complex projects such as aircrafts and automobiles. It is therefore of utmost importance that mathematical models are built to simulate real situations in appropriate softwares to be able to reduce the uncertainty in complex designs systems.

In case of dynamic models, one has in a generalized way two types of uncertainties that must be quantified. They are: parametric and non-parametric uncertainties. The first one is related to system parameters such as modulus of elasticity, density, Poisson's ratio, damping coefficient and geometric parameters. The second one includes the uncertainties arising mainly due to lack of knowledge of the system or due the errors in the estimation of the theoretical models used in the analysis and not depends on their parameters.

However, the focus of this article will be given to the model uncertainties, also called epistemic uncertainties or nonparametric uncertainties.

Finally, the main objectives of this work correspond at first to obtain the appropriate PDF for the propagation of random matrix considered in the analysis, which corresponds to stage of stochastic modelling. After that, one will propagate the uncertainty on stochastic simulation of a simple system so that can be constructed the 95% confidence limits for uncertainty and performed a study concerning the dispersion parameter and its importance in the quantification of uncertainty in dynamic models..

2 STOCHASTIC MODELLING

The appropriated PDF of the dynamic system model will be constructed considered the nonparametric approach that uses the Random Matrix Theory (RMT) and de Maximum Entropy Method (MEM). This theory was proposed by Soize (1998) and Soize (2000), its validation was realizes on Soize (2001), Soize (2003a), Soize (2003b) and Chebli *et al* (2004). An overview about the approach can be found in Soize (2005b). Some details about the nonparametric approach, including RMT and MEM, can also be seen in Justino (2012).

First, one must choose the matrices that will be randomizes. In case of this article the mass, damping and stiffness matrices will be considered in order to make a general procedure to obtain the correct PDF. Thereafter, the sample space must be defined. It identifies the values that can be assumed by the random matrices and corresponds to the construction of the PDF for each of the matrices considered earlier. It should also say that at this stage of modelling, the success of the process depends of use of the appropriate PDF for each of the random matrices, so that the errors of analysis resulting from the use of an incorrect PDF are eliminated.

In addition, the modelling of uncertainty includes vibration problems in the range of low to high frequency.

2.1 Definition of the problem to be solved

For the construction of PDF will be considered the equation of motion of a structural linear dynamic system with n degrees of freedom that can be represented in the frequency domain as follows:

$$-\omega^2 \mathbf{M}\,\ddot{\mathbf{u}}(\omega) + i\omega \mathbf{C}\,\dot{\mathbf{u}}(\omega) + \mathbf{K}\,\mathbf{u}(\omega) = \mathbf{f}(\omega) \tag{1}$$

in which the vectors $\mathbf{u}(\omega)$, $\dot{\mathbf{u}}(\omega)$, $\ddot{\mathbf{u}}(\omega)$ and $\mathbf{f}(\omega)$ respectively represent the vector of displacement, velocity, acceleration of the mass and the vector of the external force applied to the system, all depending on the frequency . $i = \sqrt{-1}$. $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are respectively the $n \times n$ random matrices of mass, damping and stiffness. Such matrices are real, symmetric and posi-

tive-defined, they belong to positive-definite ensemble (PDE) proposed by Soize (2000) and studied in Soize (2003a) and Soize (2005a), whose details can be found in Justino (2012) .
The random matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$, for simplicity will be represented by a $n \times n$ general random matrix $\mathbf{G}$. This may be considered because, according to Adhikari (2007), the random matrices of dynamic system have similar probabilistic characteristics.

2.2 Definition of the mean model

The mean model corresponds to the system modelled in appropriate software. It represents the real model to be studied. The mean model is represented by the following equation in the frequency domain:

$$-\omega^2 \overline{\boldsymbol{M}}\, \ddot{\overline{\boldsymbol{u}}}(\omega) + i\omega \overline{\boldsymbol{C}}\, \dot{\overline{\boldsymbol{u}}}(\omega) + \overline{\boldsymbol{K}}\, \overline{\boldsymbol{u}}(\omega) = \overline{\boldsymbol{f}}(\omega) \tag{2}$$

The bars under the symbols represent mean values. The vectors and the matrices are deterministic, and the matrices are real symmetric and positive-definite.

2.3 Response of the dynamic system and the response of the mean model

The frequency response function (FRF) of the system and of the mean model are respectively:

$$\mathbf{H}(\omega) = (-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K})^{-1}\,, \quad \overline{\boldsymbol{H}}(\omega) = (-\omega^2\overline{\boldsymbol{M}} + i\omega\overline{\boldsymbol{C}} + \overline{\boldsymbol{K}})^{-1} \tag{3}$$

2.4 The optimization problem

According to nonparametric approach and MEM the optimization problem to be solved to obtain the PDF for propagation of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are given by as follows. This optimization problem considers that the entropy is the uncertainty in the system. Thus according to Justino (2012):

$$Maximaze: \quad S(p_{\mathbf{G}}) = -\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, ln[p_{\mathbf{G}}(\mathbf{G})]\, d\mathbf{G} \tag{4}$$

$$Subject\ to: \quad \int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = 1 \tag{5}$$

$$E(\mathbf{G}) = \int_{\mathbf{G}>0} \mathbf{G}\, p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = \overline{\boldsymbol{G}} \tag{6}$$

$$E\big[ln\,[|\mathbf{G}|^{-\nu}]\big] = \int_{\mathbf{G}>0} ln\,[|\mathbf{G}|^{-\nu}] p_{\mathbf{G}}(\mathbf{G}) d\mathbf{G} \tag{7}$$

S is the Shannon entropy. The Eq. (5) is the normalization restriction. In Eq. (6) the mathematical expectation must to be equal to mean model. The last restriction, represented by Eq. (7),

> […] is related to the existence of the inverse moment of random matrix mass, stiffness and damping. Soize (2000) states that it is necessary to introduce a restriction concerning the existence of moments in the inverse random matrix of mass, damping and stiffness ($\mathbf{M}^{-1}, \mathbf{C}^{-1}, \mathbf{K}^{-1}$) in order to obtain a consistent probabilistic model and a convergence property of the stochastic transient response when the size of the matrices approaching infinity. The same author also states that "as the random matrices are almost surely positive-defined, their in-

verses almost certainly exist, but this property does not imply the existence of moments." (Justino, 2012).

For damped systems, the following condition is sufficient to guarantee the existence of the moments of FRF.

$$E\left[\|\mathbf{M}^{-1}\|_F^{\nu_{\mathbf{M}}}\right] < +\infty, \quad E\left[\|\mathbf{C}^{-1}\|_F^{\nu_{\mathbf{C}}}\right] < +\infty, \quad E\left[\|\mathbf{K}^{-1}\|_F^{\nu_{\mathbf{K}}}\right] < +\infty \tag{8}$$

In which $\nu_{\mathbf{M}} \geq 1, \nu_{\mathbf{C}} \geq 1$ and $\nu_{\mathbf{K}} \geq 1$ are positive integers and $\|\cdot\|_F$ is the Frobenius norm.

Considering the Eq. (8) and taking into account that the $n \times n$ matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are real, symmetric and positive-defined, is possible to obtain the Eq. (7). The details of these calculations can be seen in Justino (2012).

2.5 Solution of the optimization problem

The Lagrangian can finally be written as:

$$\mathbb{L}[p_{\mathbf{G}}(\mathbf{G})] = -\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, ln[\mathbf{p}_{\mathbf{G}}(\mathbf{G})]\, d\mathbf{G} - (\mu - 1)\left(\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G}) d\mathbf{G} - 1\right) - \int_{\mathbf{G}>0} ln\,[|\mathbf{G}|^{-\nu}] p_{\mathbf{G}}(\mathbf{G}) d\mathbf{G} - tr\left(\mathbf{\Lambda}\left(\int_{\mathbf{G}>0} \mathbf{G} p_{\mathbf{G}}(\mathbf{G}) d\mathbf{G} - \overline{\boldsymbol{G}}\right)\right) \tag{9}$$

in with $\mathcal{L}(\cdot)$ corresponding to a Lagrangian of $(\cdot)$, λ is a scalar Lagrange multiplier. Was used $(\mu - 1)$ in front of the second restriction only for reasons of simplification on the following steps. $\mathbf{\Lambda}$ also corresponds to a Lagrange multiplier but in this case it is a matrix, ν is the order of the inverse moment constraint, $tr(\cdot)$ stands for trace of $(\cdot)$, $exp(\cdot)$ stands for exponential of $(\cdot)$.

The optimal condition which corresponds to the maximum point is given by the variational calculus that is defined as:

$$\frac{\partial \mathbb{L}[p_{\mathbf{G}}(\mathbf{G})]}{\partial p_{\mathbf{G}}(\mathbf{G})} = 0 \tag{10}$$

Solving the partial integral and after some algebra follows that:

$$p_{\mathbf{G}}(\mathbf{G}) = exp\{-\mu\}\, |\mathbf{G}|^{\nu}\, exp\{tr(-\mathbf{\Lambda G})\} \tag{11}$$

The Eq. (11) corresponds to PDF of the random matrix $\mathbf{G}$. But, to use it, the Lagrange multipliers must be found, what will be done in the next steps.

Substituting Eq. (11) into Eq. (5) and consider that $exp\{-\mu\}$ is a constant, yield:

$$exp\{-\mu\} = \left\{\int_{\mathbf{G}>0} |\mathbf{G}|^{\nu}\, exp\{tr(-\mathbf{\Lambda G})\}\, d\mathbf{G}\right\}^{-1} \tag{12}$$

At this point one should apply the Laplace transform on the right side of the equation above. According to Adhikari (2007), for matrices one have:

$$\mathcal{L}\{F(\mathbf{X})\} = \int_0^{\infty} F(\mathbf{X})\, exp\{tr(-\mathbf{ZX})\}\ d\mathbf{X} \tag{13}$$

in which $\mathcal{L}\{f(\mathbf{X})\}$ is the Laplace transform of the function $f(\mathbf{X})$ that is a function of an $n \times n$ symmetric positive definite matrix. The integral is assumed to be absolutely convergent in the right half plane $\mathbb{R}(\mathbf{Z}) = \mathbf{Z}_r > 0$.

Like in Adhikari (2007), one needs to consider the equations:

$$f(\mathbf{X}) = |\mathbf{X}|^{\nu}, \quad \nu = a - (n+1)/2\ , \quad \mathbf{X} = \mathbf{Z}^{-\frac{1}{2}}\mathbf{Y}\mathbf{Z}^{-\frac{1}{2}}, \quad d\mathbf{X} = |\mathbf{Z}|^{-\frac{1}{2}(n+1)} d\mathbf{Y} \tag{14}$$

Thus, replacing this equations in Eq. (13), considering $tr\left(-\mathbf{Z}\mathbf{Z}^{-\frac{1}{2}}\mathbf{Y}\mathbf{Z}^{-\frac{1}{2}}\right) = tr(-\mathbf{Y})$,

$det(\mathbf{AB}) = |\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$ and after some algebraic manipulations, results:

$$\mathcal{L}\{|\mathbf{X}|^{a-(n+1)/2}\} = |\mathbf{Z}|^{-a} \int_{\mathbf{Y}>0} exp\{tr(-\mathbf{Y})\} \, |\mathbf{Y}|^{a-(n+1)/2} d\mathbf{Y} \quad (15)$$

According to Adhikari (2007), the multivariate gamma function is defined by:

$$\Gamma_n(a) = \int_{\mathbf{X}>0} |\mathbf{X}|^{a-(n+1)/2} \, exp\{\, tr\{-\mathbf{X}\}\} \, d\mathbf{X} \quad (16)$$

Thus, results that:

$$\mathcal{L}\{|\mathbf{X}|^{a-(n+1)/2}\} = |\mathbf{Z}|^{-a} \, \Gamma_n(a) \quad (17)$$

Eq. (17) is the solution of the result of right side of Eq. (12). Then, doing $\mathbf{Z} = \mathbf{\Lambda}$ and substituting the Eq. (17) in Eq. (12) gives:

$$exp\{-\mu\} = |\mathbf{\Lambda}|^{a}\{\, \Gamma_n(a)\}^{-1} \quad (18)$$

The result of Eq. (18) will be used later with the Lagrange multiplier matrix that will be found below. Both will be replaced in Eq. (11).

According to Adhikari (2007), the characteristic function of the random matrix can be obtained as:

$$\varphi_{\mathbf{G}}(\mathbf{\Omega}) = E[exp\,\{tr(i\mathbf{\Omega G})\}\, p_{\mathbf{G}}(\mathbf{G})] = \int_{\mathbf{G}>0} exp\,\{tr(i\mathbf{\Omega G})\}\, p_{\mathbf{G}}(\mathbf{G}) d\mathbf{G} \quad (19)$$

in which $\mathbf{\Omega} \in \mathbb{R}_{n,n}$ is a symmetric matrix.

Substituting Eq. (11) into Eq. (19) and after that substituting the Eq. (18) into Eq. (19) and doing some algebraic manipulations, it results:

$$\varphi_{\mathbf{G}}(\mathbf{\Omega}) = |\mathbf{\Lambda}|^{a}[\Gamma_n(a)]^{-1} \int_{\mathbf{G}>0} exp\{tr(-(\mathbf{\Lambda} - i\mathbf{\Omega})\mathbf{G})\}|\mathbf{G}|^{\nu} \, d\mathbf{G} \quad (20)$$

Taking into account that the integral on the right side of this equation also has as a solution the Eq. (17) and by doing $\mathbf{Z} = (\mathbf{\Lambda} - i\mathbf{\Omega})$ in Eq. (17), one has:

$$\mathcal{L}\{|\mathbf{X}|^{a-(n+1)/2}\} = |\mathbf{\Lambda} - i\mathbf{\Omega}|^{-a} \, \Gamma_n(a) \quad (21)$$

This equation is the final solution of the integral on the right side of Eq. (20). Thus, replacing this equation into Eq. (20) and after some simplifications, one has:

$$\varphi_{\mathbf{G}}(\mathbf{\Omega}) = |\mathbf{I} - i\mathbf{\Omega}\mathbf{\Lambda}^{-1}|^{-a} \quad (22)$$

Considering now, according to Adhikari (2007):

$$ln\, \varphi_{\mathbf{G}}(\mathbf{\Omega}) = ln|\mathbf{I} - i\mathbf{\Omega}\mathbf{\Lambda}^{-1}|^{-a} = a(i\mathbf{\Omega}\mathbf{\Lambda}^{-1} + i\mathbf{\Omega}\mathbf{\Lambda}^{-1} + \cdots) \quad (23)$$

$$E(\mathbf{G}) = \left.\frac{\partial ln\varphi_{\mathbf{G}}(\mathbf{\Omega})}{\partial i\mathbf{\Omega}}\right|_{\mathbf{\Omega}=0} \quad (24)$$

one have:

$$\mathbf{\Lambda} = a\overline{\mathbf{G}}^{-1} \quad (25)$$

This Eq. (25) corresponds to Lagrange multiplier matrix.

Substituting the Eqs. (18) and (25) into Eq. (11), is obtained the final PDF of a random matrix that only depends on its mean and of the order of the random matrix (or degrees of freedom of the system).

$$p_{\mathbf{G}}(\mathbf{G}) = a^{an}|\overline{\boldsymbol{G}}|^{-a}[\, \Gamma_n(a)]^{-1}|\mathbf{G}|^{\nu}\, exp\{tr(-a\overline{\boldsymbol{G}}^{-1}\mathbf{G})\} \quad (26)$$

After some algebraic calculations and considering $a = \nu + (n+1)/2$, one has:

$$p_{\mathbf{G}}(\mathbf{G}) = \left[(2)^{-\frac{n}{2}(2\nu+n+1)}\right]\left[\left|\frac{\bar{\boldsymbol{G}}}{(2\nu+n+1)}\right|^{-\frac{1}{2}(2\nu+n+1)}\right]\left[\left\{\Gamma_n\left(\frac{2\nu+n+1}{2}\right)\right\}^{-1}\right] \times$$
$$\left[|\mathbf{G}|^{\frac{1}{2}[(2\nu+n+1)-(n+1)]}\right]\left[exp\left\{tr\left(-\frac{1}{2}\left(\frac{\bar{\boldsymbol{G}}}{(2\nu+n+1)}\right)^{-1}\mathbf{G}\right)\right\}\right] \quad (27)$$

Now, considering that:

$$(2\nu+n+1) = p\,, \quad \frac{\bar{\boldsymbol{G}}}{p} = \mathbf{\Sigma} \quad (28)$$

Replacing the Eq. (28) into Eq. (27), and after some algebraic manipulations results:

$$p_{\mathbf{G}}(\mathbf{G}) = \left\{(2)^{\frac{1}{2}np}\Gamma_n\left(\frac{1}{2}p\right)|\mathbf{\Sigma}|^{\frac{1}{2}p}\right\}^{-1}|\mathbf{G}|^{\frac{1}{2}(p-n-1)}exp\ tr\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\,\mathbf{G}\right\} \quad (29)$$

It may be noted that the Eq. (29) obtained through the optimization process where occur the maximization of the entropy subject to boundary conditions shown in subsection 2.4 of this paper is the Wishart distribution.

However, were tested and compared four criteria for selecting the more adequate parameters to be used in the propagation of epistemic uncertainty by PDF Wishart. The criteria can be summarized as follows:

Criterion 1 – was proposed by Soize (2000) and Soize (2001): $E[\mathbf{G}] = \bar{\boldsymbol{G}}$ and $\delta_{\mathbf{G}} = \tilde{\delta}_{\mathbf{G}}$.

Criterion 2 – was proposed by Adhikari (2007): $||\bar{\boldsymbol{G}} - E[\mathbf{G}]||_F = minimum\ value$ and $||\bar{\boldsymbol{G}}^{-1} - E[\mathbf{G}^{-1}]||_F = minimum\ value$.

Criterion 3 – was proposed by Adhikari (2008): $E[\mathbf{G}^{-1}] = \bar{\boldsymbol{G}}^{-1}$ and $\delta_{\mathbf{G}} = \tilde{\delta}_{\mathbf{G}}$.

Criterion 4 – was proposed by Adhikari (2008): $E[\mathbf{M}^{-1}] = \bar{\boldsymbol{M}}^{-1}$, $E[\mathbf{K}] = \bar{\mathbf{K}}$, $\delta_{\mathbf{M}} = \tilde{\delta}_{\mathbf{M}}$ and $\delta_{\mathbf{K}} = \tilde{\delta}_{\mathbf{K}}$.

By way of numerical simulation Adhikari (2008) showed that the third criterion produces the best result that is given by:

$$(2\nu+n+1) = p\,, \quad \mathbf{\Sigma} = \frac{\bar{\boldsymbol{G}}}{\theta} \quad (30)$$

Therefore, if the νth order moment of the inverse of a matrix system $\mathbf{G} = \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ exists, and only the mean of $\mathbf{G}$ is available, the not biased distribution of $\mathbf{G}$ follows the Wishart distribution, showed in Eq. (29), with optimal parameters represented by Eq. (30). And this is the final result of stochastic modeling.

3 STOCHASTIC SIMULATION

"Simulation is a process of reproduction of the real world based on a set of hypotheses and models designed reality" (Castanheira (2004) *apud* Ang and Tang (1984)). The simulations are repeated until a convergence criterion is confirmed.

The Monte Carlo method is used to be able to reproduce the behavior of the system under study, perform simulations and obtain statistical response for analysis of results.

According to Soize (2005b), the convergence in accordance with the size of the random matrix and the number of realizations required in Monte Carlo simulation, is given by:

$$conv(n_s, n) = \left\{\frac{1}{n_s}\sum_{k=1}^{n_s}\int_{\omega\in\mathbb{B}}||Q^n(\omega;\theta_k)||^2 d\omega\right\}^{1/2} \quad (31)$$

in which n is the order of random matrices; n_s corresponds to the number of Monte Carlo simulation, ω is the frequency on band $\mathbb{B}$, $Q^n(\omega;\theta_k)$ corresponds to the response of the stochastic system calculated for each simulation k with corresponding result θ_k.

To verify the mean square convergence is determined an acceptable error for the number of simulations and, depending on this error, the result of $conv(n_s, n)$ can be modified.

3.1 Dispersion parameter

The dispersion parameter is the information about the uncertainty on the system. It's very important information for uncertainty quantification.

The dispersion parameter must be calculated within an interval of possible values considering the nonparametric approach. According to Soize (2003a) the equation that must be used for the calculation of this parameter is given by:

$$0 < \delta < \left\{\frac{(n_0+1)}{(n_0+5)}\right\}^{1/2} \tag{32}$$

in which $n_0 > 1$ is an integer that is given and fixed, and in this case, δ is independent of n. It can be stated that, in general, the dimensions of the model in question are large and sometimes above 100. If an example, consider if $n = 10$ then will be $n_0 = 10$ which, when applied in Eq. (32) correspond to a high uncertainty (0.856) which generally is not achieved in applications (Soize (2003a) *apud* Justino (2012)).

3.2 Simulation data

It will be studied now a mass-spring-damper system with two degrees of freedom where the model of the system dynamics is represented by Eq. (1). The mean model is represented by Eq. (2) and the mean deterministic matrices are given by:

$$\overline{\boldsymbol{M}} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \overline{\boldsymbol{C}} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \overline{\boldsymbol{K}} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{33}$$

the vectors $\{\mathbf{u}(t)\}$ and $\{\mathbf{f}(t)\}$ are represented by:

$$\{\mathbf{u}(t)\} = \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}, \quad \{\mathbf{f}(t)\} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix} \tag{34}$$

The data for the simulation were taken from Sampaio *et al* (2007):

$$\begin{gathered} m_1 = 1{,}5\ kg,\ m_2 = 0{,}75\ kg \\ k_1 = 1000\ N/m,\ k_2 = 150\ N/m \\ c_1 = 0{,}5\ N.s/m,\ c_2 = 0{,}05\ N.s/m \\ n = 2,\ \mathbb{B} = [0{,}7]\ \ [Hz],\ f = (1{,}0)^t,\ u = (0, u_2)^t \end{gathered} \tag{35}$$

and the dispersion parameters considered in analyze are:

$$\delta_{1\mathbf{K}} = 0{,}05;\ \delta_{2\mathbf{K}} = 0{,}05;\ \delta_{3\mathbf{K}} = 0{,}05;\ \delta_{4\mathbf{K}} = 0{,}05;\ \delta_{5\mathbf{K}} = 0{,}05 \tag{36}$$

Taking into account Eq. (32) which determines the range of valid values for the dispersion parameter, the maximum value considering $n_0 = n = 2$ is $\delta_{5\mathbf{K}} = 0{,}65$. The other values were arbitrarily considered within the allowable range.

The response of the mean model and the response of the system are given respectively by:

$$\overline{\boldsymbol{H}}(\omega) = \frac{u_2(\omega)}{f_1(\omega)}, \quad \mathbf{H}(\omega) = \frac{u_2(\omega)}{f_1(\omega)} \tag{37}$$

In this simulation will be used the Wishart distribution to propagate the uncertainties in stiffness matrix. Will be performed five simulations each one consider one value for the dispersion parameter showed in Eq. (36).

3.3 Results of simulation

The graphics with the results of the simulation are shown in Figure 1 to Figure 5. They have the response for convergence and the FRF for each dispersion parameter considered.

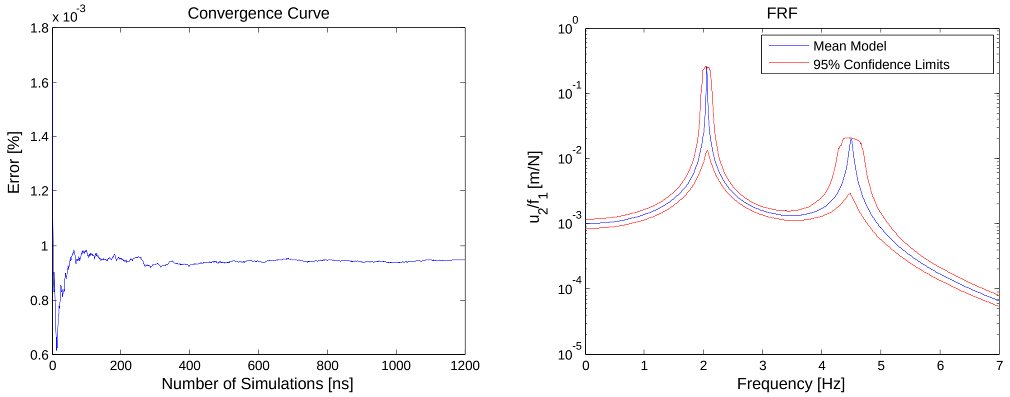

Figure 1: Convergence and 95% confidence limits. $\delta_{\mathbf{K}} = 0{,}05$.

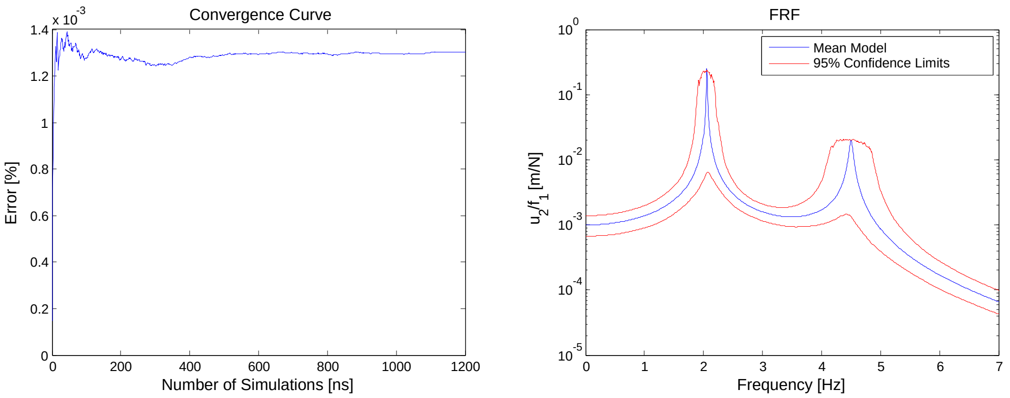

Figure 2: Convergence and 95% confidence limits. $\delta_{\mathbf{K}} = 0{,}1$.

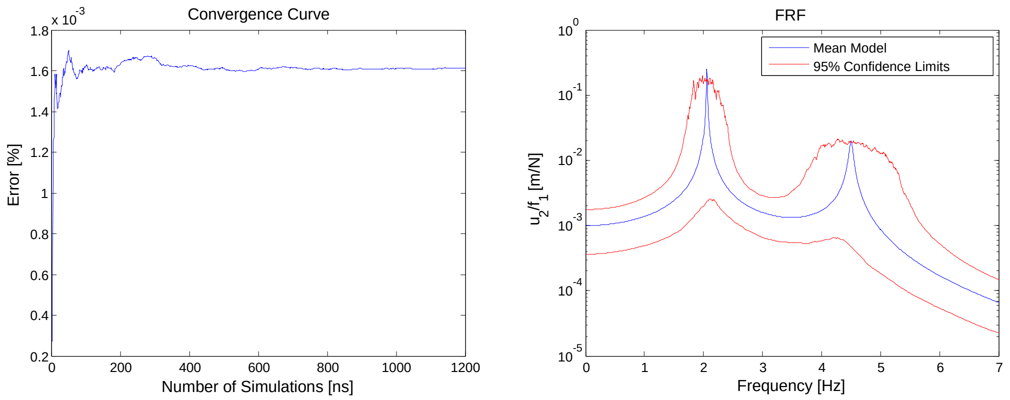

Figure 3: Convergence and 95% confidence limits. $\delta_{\mathbf{K}} = 0{,}2$.

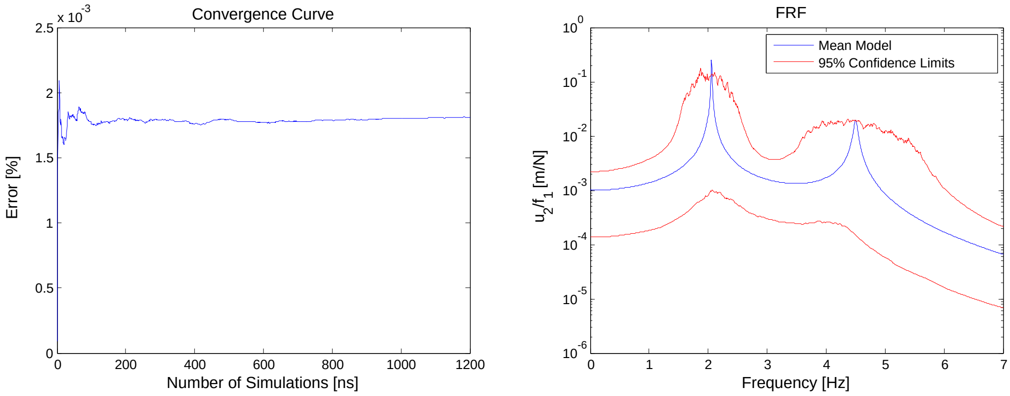

Figure 4: Convergence and 95% confidence limits. $\delta_{\mathbf{K}} = 0{,}3$.

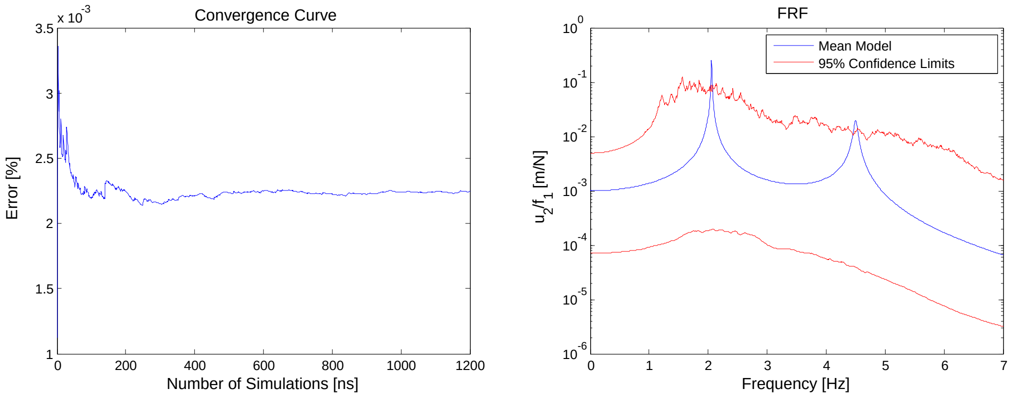

Figure 5: Convergence and 95% confidence limits. $\delta_{\mathbf{K}} = 0{,}65$.

4 CONCLUSIONS

In the present paper is shown how was get the probability density functions of random matrix of mass, damping and stiffness. The stochastic simulation was also performed to an adequate number of simulations and it was made with five different dispersion parameters.

With respect to the convergence of Monte Carlo simulation, it was observed that the value of the number of simulations has been determined to $n_s = 600$ regardless of the dispersion parameter in question, which means that satisfactory results can be obtained for the model when one use this number of simulations in Monte Carlo simulation.

In relation to FRF, the obtained results could show the behavior of the response according the value of dispersion parameter used in the simulation. It was possible to verify that as the frequency increases the width of the curve, by considering the horizontal axis, increases around the peaks of the natural frequencies to achieve a maximum width for a maximum value of the dispersion parameter considered that in this case, worth 0.65. Moreover, as it increases the dispersion parameter, the width of the curve, now with reference to the vertical axis, also increases, and in the same manner as in horizontal reference reaches its maximum width to maximum allowable value of the parameter dispersion. However, the dispersion parameter has to be determined carefully and only then, the uncertainty will be quantified and will correspond to the reality of the problem studied.

Finally, it can be said that the mean model can be used to represent a real structural linear dynamic system with the characteristics described in this article. This is possible because it is inside de 95% confidence limits. Thus, one can proved in this article that is really possible to quantify the model uncertainty to increase the reliability of the system in study, and the nonparametric approach with the Wishart distribution provides very good results.

5 ACKNOWLEDGEMENTS

We would like to thank CAPES (Coordination for Improvement of Higher Level Personnel) and FAPEMIG (Research Support Foundation of Minas Gerais) for the financial support.

6 REFERENCES

[1] S. Adhikari, Matrix Variate Distributions for Probabilistic Structural Dynamics. *AIAA Journal*, 45, 88, 1748-1762, 2007.

[2] S. Adhikari, Wishart Random Matrices in Probabilistic Structural Mechanics. *ASCE Journal of Engineering Mechanics*, 134, 12, 1029-1044, 2008.

[3] A. M. Castanheira, Avaliação da Confiabilidade de Vigas Metálicas Dimensionadas pela NBR 8800, MSc. (Dissertation), Federal University of Ouro Preto – MG, 2004.

[4] H. Chebli and C. Soize, Experimental Validation of a Nonparametric Probabilistic Model of Non Homogeneous Uncertainties for Dynamic Systems, *Journal of the Acoustical Society of America*, 115, 2, 697–705, 2004.

[5] L. B. Justino, Quantification of Non Parametric Uncertainties of Structural Dynamic Models, MSc. (Dissertation), Institute of Mechanical Engineering, Federal University of Itajubá, Itajubá – MG, 2012.

[6] R. Sampaio, T. Ritto and E. Cataldo, Comparison and Evaluation of Two Approaches of Uncertainty Modeling in Dynamical System, *Mecánica Computacional*, XXVI, 3078-3094, 2007.

[7] C. SOIZE, Reduced Models in the Medium Frequency Range for General Dissipative Structural-dynamics Systems, *European Journal Mechanics A/Solids*, 17, 4, 657–685, 1998.

[8] C. SOIZE, A Nonparametric Model of Random Uncertainties for Reduced Matrix Models in Structural Dynamics, *Probabilistic Engineering Mechanics*, 15, 277–294, 2000.

[9] C. SOIZE, Maximum Entropy Approach for Modeling Random Uncertainties in Transient Elastodynamics, *Journal of the Acoustical. Society of America*, 109, 5, 1979–1996, 2001.

[10] C. SOIZE, Random Matrix Theory and Non-Parametric Model of Random Uncertainties in Vibration Analysis, *Journal of Sound and Vibration*, 263, 893–916, 2003a.

[11] C. SOIZE, Uncertainty Dynamic Systems in the Medium Frequency Range, *Journal of Engineering Mechanics*, 129, 9, 1017–1027, 2003b.

[12] C. SOIZE, Random Matrix Theory for Modeling Uncertainties in Computational Mechanics, *Journal of Sound and Vibration*, 194, 1333–1366, 2005a.

[13] C. SOIZE, Comprehensive Overview of a Non-Parametric Probabilistic Approach of Model Uncertainties for Predictive Models in Structural Dynamics, *Journal of Sound and Vibration*, 288, 623–652, 2005b.