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GUITAR STRINGS LOADED WITH LOCALIZED MASSES

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Abstract. *In this work, we have modelled and analysed the unevenness of strings, typically found in fretted instruments. We build a modal method to perform a detailed eigen-analysis, as well as time-domain simulations, and we obtained results considering realistic values for Portuguese guitar strings with point masses. Emphasis is given to the localization analysis. Our results suggest that string-fret interaction can account for the typical loss of brightness of the sound from old guitar strings, due to perturbations on modes with higher frequencies.*

1 Introduction

The sound quality of a stringed instrument is affected not only by its body quality, but also by the strings quality. As the instrument is played, its strings ages, eventually achieving a state at which the instrument playability is highly compromised, commonly called the "*dead strings*". On the other hand, there are guitar players who agree that unused, nearly perfect strings, sound too bright, and therefore prefere to play instrument with strings which are neither *newborn* nor *dead*. Strings properties can be affected by several causes, such as dust (*e.g.* wood dust from a piano), oxidation (*e.g* caused by local air conditions, or fingertips perspiration), or by the repeated pressing of the string against the frets in fretted instruments.

Motivated by our interest in increasing the understanding of string dynamics, in the context of the Portuguese guitar acoustics, we are interested in studying this problem on guitars and alike instruments. And we are specially interested in considering frets as the cause of string deterioration (this is because oxidation is more likely to happen in a larger time scale, and fingertip perspiration depends highly on the guitar player). The acoustics of the guitar have already been addressed [1, 2, 3, 4], as well as the dynamics of musical strings [4, 5, 6, 7]. In this work, we have built a model for localized string deterioration, and searched for the effects of this condition on the eigenfrequencies and the eigenfunctions. Similar models, although not motivated by the context of musical acoustics, can already be found in the literature [10]. To our knowledge, our work goes behind what has already been previously studied or modelled, and it has the advantage of being put in a context with possible applications in the musical instruments industry. A preliminary version of this work was previously presented in [11].

The Portuguese guitar (Cithara Lusitanica) is a twelve metal strings (six courses) pear-shaped instrument, descendant from the renaissance european cittern. This instrument is widely used in Portuguese traditional music, mainly in Fado, and more recently also started to play a considerable role among the modern urban Portuguese musicians. Unlike most common guitars, this guitar has a bent soundboard (arched top) with a movable bridge (*i.e.* a bridge somewhat similar, although smaller in size, to the bridge of a violin), a neck typically with 22 fixed metal frets and it is tuned by a fan-shaped tuning mechanism, consisting in twelve screws, acting as pegs, mounted with small gliding pins where the strings are attached to adjust its tension. It has the typical tuning of the European cittern tradition, and has kept an old plucking technique, described in sixteen century musical books. The first the courses are composes by plain steel strings and tuned in unison, and the remaining are combinations of a plain steel string and an overspun copper on steel string tuned one octave bellow. There are two different models of the Portuguese guitar: the Lisbon guitar and Coimbra guitar, named after two Portuguese cities where the two most important Fado styles (a Portuguese traditional music) emerged. They differ in some details, such as the body measurements, the string length (the Coimbra guitar as a larger arm), and the Lisbon guitar is tuned (each string) a whole tone above the Coimbra guitar. Strings used in these two guitars are therefore different in size and linear mass. More about the Portuguese guitar can be read in [8, 9].

2 String model

Let us start by considering an ideal conservative string with lenght L, under the tension T, and having linear mass μ. Its dynamics are described by the wave equation

$$T\frac{\partial^2 y}{\partial x^2} - \mu\frac{\partial^2 y}{\partial t^2} = F(y,t), \;\; c = \sqrt{\frac{T}{\mu}} \tag{1}$$

with the boundary conditions

$$y(x^{\pm}) = 0,\ x^{-} = 0,\ x^{+} = L, \tag{2}$$

where $F(y,t)$ are any forces which might be acting on the string. Equation (1) admits separation of variables, such that $y(x,t) = \psi(x)q(t)$. Assuming a chromatic wave, the wave function will be the sum of all superposing waves, yielding

$$y(x,t) = \sum_{m_1}^{N} \psi_m(x)q_m(t)\,, \quad N \to \infty\,. \tag{3}$$

Hereafter, we will refer to the spatial part of (3) as modeshapes, which are given by

$$\psi_n(x) = \sin\left(\frac{n\pi x}{L}\right). \tag{4}$$

We can therefore transform (1) in a set of N modal equations

$$m_n\ddot{q}_n(t) + 2m_n\omega_n\zeta_n\dot{q}_n(t) + m_n\omega_n^2 q_n(t) = \mathcal{F}_n(t), \tag{5}$$

where ζ_n are the modal damping coefficients, the eigenfrequencies are $\omega_n = n\,\omega_1$ being $\omega_1 = \frac{\pi c}{L}$, and the modal masses are defined as

$$m_n = \int_0^L \mu\,|\psi_n(x)|^2\,dx = \frac{\mu L}{2}, \tag{6}$$

(notice that this definition of modal masses impose implicitly orthogonality on the equations 5), and, given that in the present work we only consider the string-fret interactions as source for perturbations on the string mass distribution, the generalised forces $\mathcal{F}_n(t)$ are related to the effect of the fret on local mass distributions. The total number of modes truncated with N sufficiently high to cover all the audible spectrum $\rightarrow f \sim 20$ kHz. On each point x_p where string-fret interactions occur, we can define a coupling inertial force of the form

$$\begin{aligned} F_p(x_p,t) &= -M_p\delta(x-x_p)\ddot{y}(x_p,t) \\ &= -M_p\delta(x-x_p)\sum_{m=1}^{N}\psi_m(x_p)\ddot{q}_m(t)\,, \end{aligned} \tag{7}$$

where M_p can be either positive (depositing mass) or negative (removing mass), and represents the variation of the mass at the string point x_p. We can think of $F_p(x_p,t)$ as being the variation of the inertial force felt on a specific string point x_p when the string is modified in this point, or in other words, it's the difference between the inertial force of the unused string and the inertial force of the used string, at x_p. The total effect of the contribution of each x_p point to the string modal dynamics is

$$\mathcal{F}_n(t) = \sum_{p=1}^{P}\int_0^L F_p(x_p,t)\psi_n(x)dx = -\sum_{p=1}^{P} M_p \sum_{m=1}^{N} \ddot{q}_m(t)\psi_m(x_p)\psi_n(x_p),\ \ n = 1,...,N. \tag{8}$$

The result 8 can be recast as a vector of modal excitations, yielding

$$\{\mathcal{F}_n(t)\} = -\sum_{p=1}^{P} M_p\,[\Phi_p]\,\{\ddot{q}_n(t)\}\,, \tag{9}$$

where the matrix $[\Phi_p]$ is given by

$$[\Phi_p] = \begin{bmatrix} \psi_1(x_p) \\ \vdots \\ \psi_n(x_p) \end{bmatrix} . \begin{bmatrix} \psi_1(x_p) & \dots & \psi_n(x_p) \end{bmatrix} . \tag{10}$$

We now can rewrite the dynamical equations 5 in a matrix form, becoming

$$\left([M] + \sum_{p=1}^{P} M_p [\Phi_p] \right) \{\ddot{q}_n(t)\} + [C]\{\dot{q}_n(t)\} + [K]\{q(t)\} = 0 \tag{11}$$

and considering the eigensolutions

$$q_n(t) = \bar{v}_n e^{\bar{\lambda}_n t}, \tag{12}$$

we obtain, by substitution of 12 in 11, the eigenvalue problem

$$\left(\begin{bmatrix} [\emptyset] & [\mathbf{I}] \\ -[\bar{M}]^{-1}[K] & -[\bar{M}]^{-1}[C] \end{bmatrix} - \bar{\lambda}_n [\mathbf{I}] \right) \begin{Bmatrix} \bar{v}_n \\ \bar{\lambda}_n \bar{v}_n \end{Bmatrix} = \{0\}, \tag{13}$$

where $[\bar{M}] = [M] + \sum_{p=1}^{P} M_p [\Phi_p]$. The eigenfrequencies of (13) are given by its relation with the eigenvalues of the system, which are

$$\bar{\lambda}_n = -\bar{\omega}_n \bar{\zeta}_n \pm i \bar{\omega}_n \sqrt{1 - \bar{\zeta}_n^2}, \tag{14}$$

therefore, the damped eigenfrequencies $\bar{\omega}_{d\,n}$ and the damping eigencoefficients $\bar{\zeta}_n$ of the overused string will be

$$\bar{\omega}_{d\,n} = \Im(\bar{\lambda}_n) = |\bar{\lambda}_n| \sqrt{1 - \bar{\zeta}_n^2}\,, \tag{15}$$

$$\bar{\zeta}_n = \frac{-\Re(\bar{\lambda}_n)}{|\bar{\lambda}_n|}\,. \tag{16}$$

Also, the system described by 13 will have N eigenvectors, each of lenght N, and therefore, the constrained modeshapes are given by

$$\bar{\psi}_n(x) = \sum_{m=1}^{N} \bar{v}_m^n \psi_m(x)\,, \tag{17}$$

where $\bar{v}_m^n$ stands for the m^{th} term of the n^{th} eigenvector $\bar{v}$. The constrained modeshapes $\bar{\psi}_n(x)$ may be normalized using the same convention used for the unconstrained modeshapes $\psi_n(x)$ (*i.e.* such that its maximum is equal to unit). To obtain the physical information of the worn string from this modal model, one needs to start by rewriting equation (5) in homogeneous form, in terms of the worn string modes

$$\bar{m}_n \ddot{\bar{q}}_n(t) + 2\bar{m}_n \bar{\omega}_n \bar{\zeta}_n \dot{\bar{q}}_n(t) + \bar{m}_n \bar{\omega}_n^2 \bar{q}_n(t) = 0\,, \tag{18}$$

where the constrained modal masses are obtained from solving

$$\bar{m}_n = \int_0^L m_0 \bar{\psi}_n^2(x) dx + \sum_{p=1}^{P} M_p \bar{\psi}_n^2(x_p)\,, \tag{19}$$

and using numerical techniques to obtain the solutions of $\bar{q}_n(t)$, with initial conditions (plucked string conditions) given by

$$\bar{q}_n(0) = \frac{\int_0^{x_d} m_0\, y_0 \frac{x}{x_d} \bar{\psi}_n(x) dx + \int_{x_d}^{L} m_0\, y_0 \frac{x-L}{x_d-L} \bar{\psi}_n(x) dx + \sum_{p=1}^{P} M_p\, y_0 (\frac{x_p}{x_d} + \frac{x_p - L}{x_d - L}) \bar{\psi}_n(x_p)}{\int_0^L m_0 \bar{\psi}_n^2(x) dx + \sum_{p=1}^{P} M_p \bar{\psi}_n^2(x_p)}\,, \tag{20}$$

where x_d is the position where the string is plucked, and y_0 is the height at this position, apply the following relation:

$$y(x,t) = \sum_{n=1}^{N} \bar{\psi}_n(x) \bar{q}_n(t)\,, \tag{21}$$

and the equivalent relations for the first and second time partial derivatives. Finally, if we neglect the continuation of the string after the bridge, the force applied by the string to the bridge will be

$$\begin{aligned} F_b(t) &= -T \sin\theta_b(t) \sim -T\, \theta_b(t) \sim -T\, \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=L} \\ &= -4\pi m_0 L f_1^2 \sum_{n=1}^{N} \left(\sum_{m=1}^{N} (-1)^m m\, \bar{v}_m^n \right) \bar{q}_n(t), \end{aligned} \tag{22}$$

where T is the string axial tensioning tension, and θ_b is the angle of T with respect to the bridge.

3 Results

Throughout this section, we will consider results for a Si note ($f = 493.88\, Hz$) Lisbon guitar string. The linear mass density and lenght of this string are respectively $\mu = 3.78 \times 10^{-4}\, kg/m$ and $L = 44\, cm$ (total mass of the string $= 166.32\, mg$). In all situations, the string points where we will add/remove mass are coincident with the points where frets are placed in the fretboard. The frets are spaced in the fretboard along intervals given by the equal temperament (as most, if not all, modern luthiers do when building Portuguese guitars, and in fact, when building any other instrument related to the guitar family), which mathematically means, for an ideal string

$$\text{"}position\, of\, the\, n^{th}\, fret\text{"} = L \left(1 - 2^{-\frac{fret\, n^{th}}{12}} \right). \tag{23}$$

We will assume (based on average experimentally identified modal damping results) that the unused string has a constant modal damping $\zeta_n = 0.01\%$, and that the string inharmonicity is negligible, $A \ll 1 \Rightarrow \omega_n = n\, \omega_1$. We will consider a total of $N = 40$ modes, enough to cover the audible spectrum.

We start by analysing how are the frequencies affected when we add/remove a large enough amount of mass to/from the string. Because the guitar player will add/remove tension while tuning the fundamental frequency, we multiply every term of $\bar{\omega}_{d\,n}$ by the factor $\frac{\omega_{d\,1}}{\bar{\omega}_{d\,1}}$, so that the fundamental frequency remain the same. In figure 1 we show plots for the relative frequencies ($= \frac{\bar{\omega}_{d\,n} - \omega_{d\,n}}{\omega_{d\,n}}$) versus mode number in two opposite situations: adding/removing one milligram per fret, along the first ten frets. Although we observe certain (few) similarities in both plots, it is clear that both situations are not symmetrical. As the mode number increases, the effect that removing mass has in the frequencies is much stronger than the effect of adding mass to the string. Furthermore, we stress that we found equivalent results in many simulations with different parameters. When we remove the tuning factor $\frac{\omega_{d\,1}}{\bar{\omega}_{d\,1}}$, the plots shown in figure

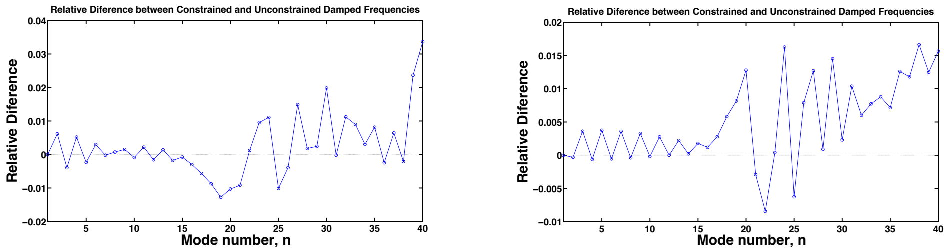

Figure 1: Relative Frequencies vs. Mode Number. In left/right, we show the situation of removing/adding 10 milligrams from/to the string, accordingly to the same mass/fret destribution

1 are linearly translated, becoming always positive for the case of removing mass, and always negative for the case of adding mass. We find a similar evolution for the damping coefficients relative difference, as shown in figure 2. Considering the same mass/fret distributions, we show

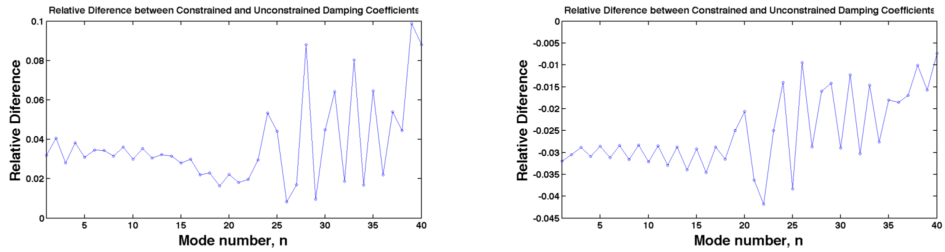

Figure 2: Relative Damping Coefficients vs. Mode Number. In left/right, we show the situation of removing/adding 10 milligrams from/to the string, accordingly to the same mass/fret destribution

in figure 3 the constrained modeshapes for modes 24^{th} and 33^{th}. There is no special reason for the choice of this specific illustrative modes as examples, the choice was in fact somewhat random, but we notice in our results that for the first few modes (in the specific case of Si string, in the first eight modes), the effects of the masses in the modeshape are not much relevant to our opinion, while to any higher mode exhibit similar effects. As was to be expected, it can be seen in the results shown in figure 3 that localized effects will occur in the modeshapes due to the effect of the added/removed masses. this effects correspond to local variations on the modeshape amplitude (which was previously normalized to unit maximum amplitude). In some locations, the amplitude variation is so strong that will even cause a localized phase shift effect.

In order to test the effect of a general distribution of added/removed masses and understand the importance of the localization of the masses, besides the total effect of the total mass added to the string, we show in figure 4 results for a statistical test. In this test, we attribute to each of the nineteen frets of the Portuguese guitar a different random value of extra mass (which can be negative if the mass is removed, or positive if the mass is added) accordingly to a gaussian distribution of variance $\sigma = 1$, and then multiply the result by one milligram times a linear function which decreases from one in the first fret to zero in the nineteenth fret. This procedure is based on an adhoc assumption that the string wear will decrease linearly as one progress from lower to higher frequency frets (most guitar players will rather use the lower frequency frets most of the time). This random distributions of masses/fret will include scenarios in which for the same string, mass is both removed from some positions and added to other positions, obviously giving results for either positive, negative or null total variation of the string mass. We

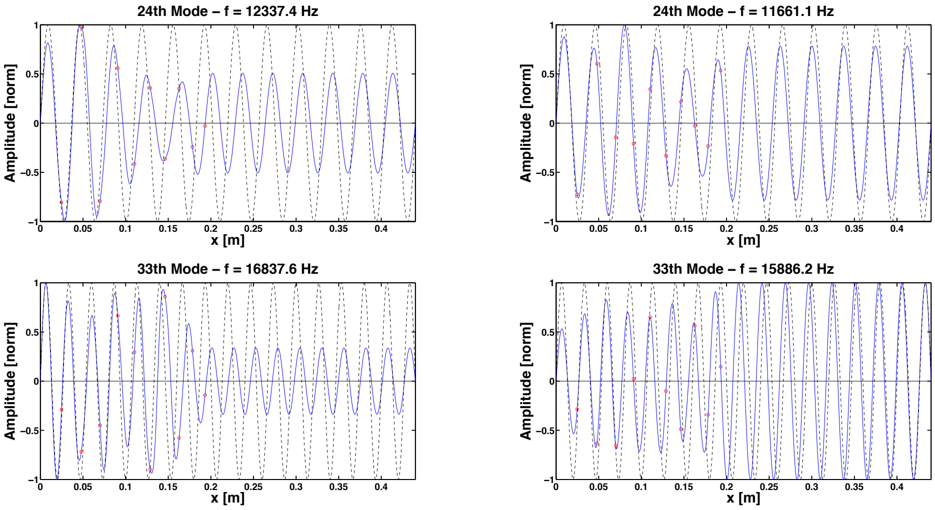

Figure 3: Constrained Modeshapes. In left/right, we show the situation of removing/adding 10 milligrams from/to the string, accordingly to the same mass/fret distribution. Dashed black lines represent the unconstrained modeshapes, red dots represent the positions where string-fret interactions occur.

limit the maximum possible added/removed mass up to 5% of the total string mass (although, we found our simulations to be contained in $-4.47\% \leq \frac{\sum_{p=1}^{P} M_p}{\mu L} \leq 4.14\%$, with an average of $7.5 \times 10^{-3}\%$). To obtain the results shown in figure 4 and 5, we ran results for twenty thousand random distributions. We find in our results that, for the same total mass variation, different distributions will result in different modal frequencies, as well as different modal damping coefficients. Notice that because we did not introduce a specific dissipation connected with the string wear, the changes in modal damping are simply related to the changes in modal frequencies. There seems to be a much clearly defined boundary in the minimum modal frequencies and damping coefficients that any random distribution will generate, when comparing with the maximum modal frequencies and damping coefficients that we obtained. We also notice that, similarly to what was observed in figure 1, when removing mass from the string, rather than adding mass, the effects on the string modal frequencies and modal damping coefficients are much stronger, which leads to non-gaussianities that graphically are much better noticed for the higher modes.

In order to analyze the occurrence of localization phenomenon on the modeshapes of perturbed strings, we define the quantity $\bar{h}_\psi = \frac{\sum_{m=1}^{N} |\bar{\psi}_m(x)|^2}{\sum_{m=1}^{N} |\psi_m(x)|^2}$. Although the modeshapes have been normalized to respect the condition $|\bar{\psi}_n(x)| < 1$, the quantity $\bar{h}_\psi$ does exceed, in the initial frets, the value of the quantity taken for an unperturbed string. We show our results in figure 6 (left plot). We notice that for perturbed strings, $\bar{h}_\psi$ is highly irregular around the positions of the point masses, while it becomes rather regular after the last localized mass.

Also, we show in figure 6 (right plot) the relation $\frac{\bar{m}_n}{m_n}$, which allows us to identify the increase of localization in the higher modes. We notice, in this figure, that for extreme situations, the modeshapes for higher modes can loose up to about 90% of its local amplitude. In lower modes, we find results larger than unity due to the fact that we dont find localization phenomenon for this frequencies, and therefore, the increase in mass, rather than the local loss of amplitude, lead the result of the relation $\frac{\bar{m}_n}{m_n}$.

Finally, we ran time simulations for the case of the plucked unused string and for a string

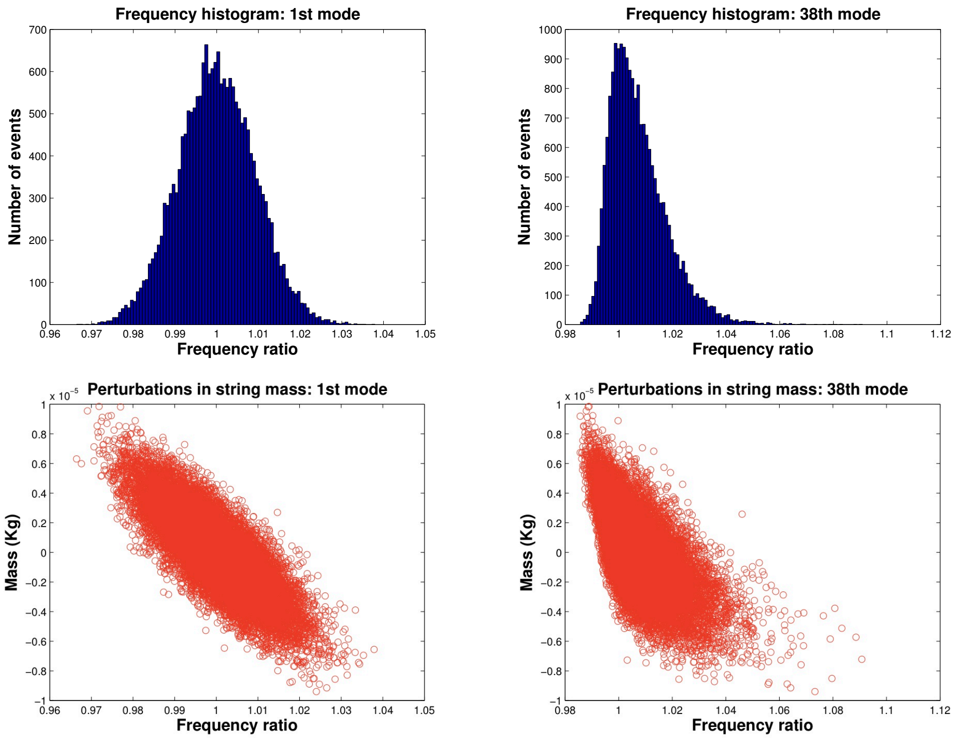

Figure 4: Analysis of Random Distributions. The label Frequency ratio stands for $\frac{\bar{\omega}_{d\,n}}{\omega_{d\,n}}$. In the bottom plots, the label Mass stands for the total variation of the string mass.

with one milligram added per fret along the first ten frets. The string in plucked at the point $x_d = \frac{6.5}{8}L$, at the height $y_0 = 2mm$. We show in figure 7 spectrograms for the force exerted on the bridge. We notice that while some of the high frequency modes vanish when we add mass to the string, some of the surviving high frequency modes have an higher amplitude comparing with the unused string. When listening to sound simulations of these vibrations, we notice that both situations generate sounds which are audibly different. We have performed a study of the time evolution of the string energy, where we find that, because there are no significant changes in the damping coefficients, the decay time will be very similar in both situations, being that the used string has a slightly larger sustain.

4 Conclusions and Perspectives

We have built and studied a model for localized perturbations of the string mass. We find in our results that these perturbations will affect mainly the higher modes of vibration of the string, which explain why players feel that the guitar sounds less bright after some usage, comparing to its sound when with new strings. Also, as should be expected, we find that the change on the modal frequencies of the string depends not only on the amount of mass added to/removed from the string, but also on the positions where these perturbations are located: the same amount of mass will produce considerably different results when distributed along somewhat different positions. We have shown the existence of localization phenomenon in the perturbed mode-shapes. Based on what we listen from our response simulations, we expect that, if the mass added to/removed from the string is enough to considerably change the frequencies of the partials, than, string sound will be significantly affected, and the string might be considered a dead string by the guitar player. Although we considered results for a Portuguese guitar string, we see

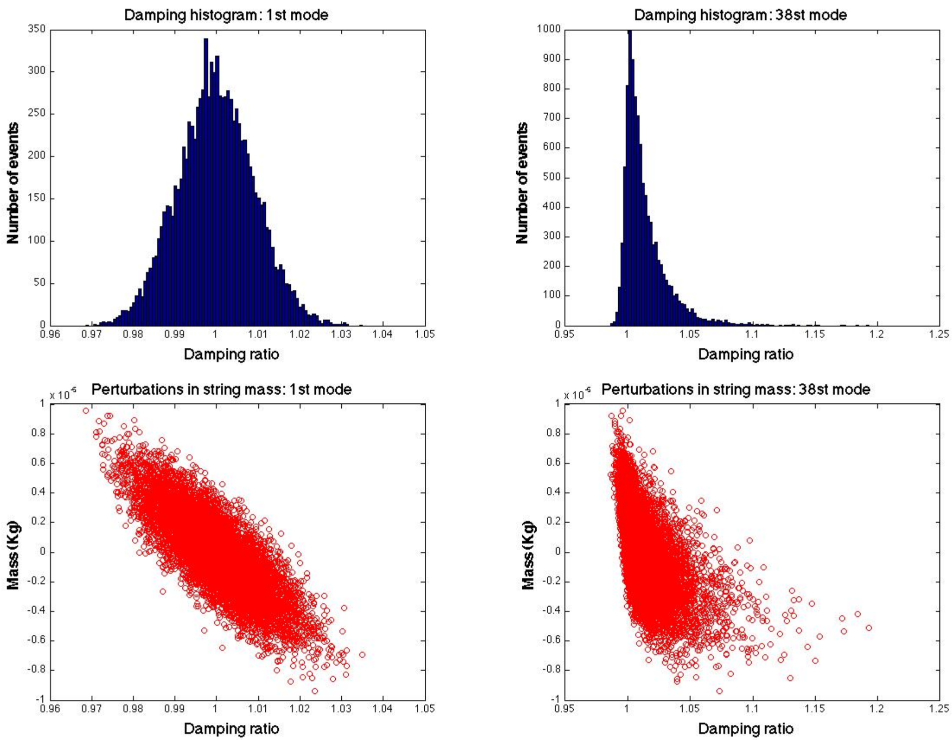

Figure 5: Analysis of Random Distributions. The label Frequency ratio stands for $\frac{\bar{\omega}_{dn}}{\omega_{dn}}$. In the bottom plots, the label Mass stands for the total variation of the string mass.

no reasons for these results to be qualitatively different from those one would find considering a steel string for any other type of guitar or alike instrument.

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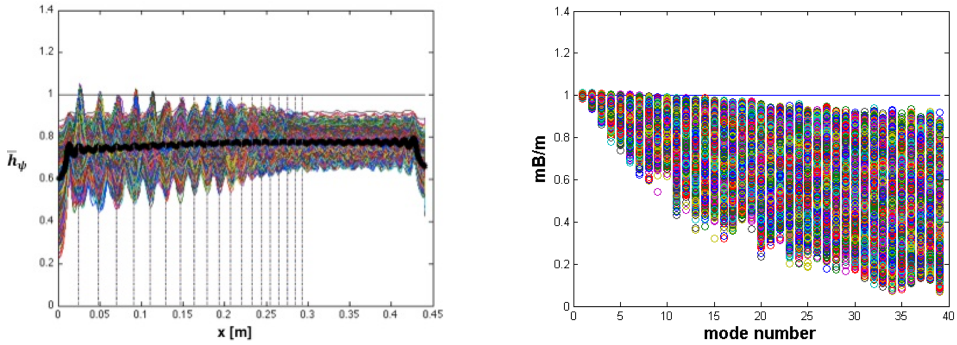

Figure 6: Statistical analysis assuming gaussian distribution. In the left, the black large line represents the average of $\bar{h}_\psi$; the vertical dashed lines represent the affected locations. In the right, we represent the quantity $\frac{\bar{m}_n}{m_n}$

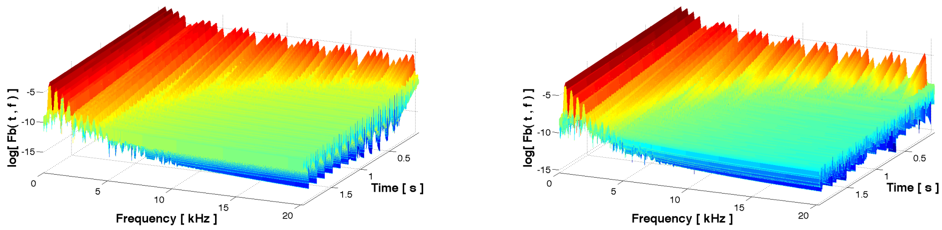

Figure 7: Spectrogram of the force exerted on the bridge. Left plot represents an unused string, right plot represents a string which adds one milligram per fret in the first ten frets.

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