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DYNAMICAL CHARACTERIZATION OF FRACTAL OBJECTS: DETERMINATION OF THE FINE FRACTAL TOPOLOGY USING THE ENERGY COVERING

L. Bevilacqua*[1], **M. M. Barros**[2]

[1]COPPE/UFRJ, Brazil
bevilacqua@coc.ufrj.br

[2]LNCC/MCT, Brazil
barros@lncc.br

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Abstract. *The foundation of the theory presented here has already been proved to be effective for the case of curves belonging to the Koch family. The present paper extends the investigation to more complex curves, namely randomly generated curves and the Weierstrass-Mandelbrot curve. The analysis is focused on numerical experiments. The results obtained with the numerical analysis allow advancing some interesting proposition concerning the fine fractal structure of plane curves. The method uses the dynamical response of appropriate harmonic oscillators built up according to the geometry of the fractal set. For the plane motion three fundamental periods are determined. Each period plotted against the characteristic length of the corresponding curve in a log-log scale estimates the fractal dimension. It is shown that each degree of freedom corresponds to a distinct energy covering associate to a certain topological property. Using this technique it is possible to distinguish the fine fractal structure of plane curves. The dynamic technique is proved to work also for the identification problem, that is, to determine the fractal characteristics embedded in some sample of a given fractal set. It is proposed the classification of fractal structures into at least two categories, perfectly fractal and partially fractal. The affine similarity can be precisely detected using this method.*

INTRODUCTION

The dynamical characterization of fractal objects has already been proved to work satisfactorily for family of curves that follow strictly the Hausdorff criterion [1]. The dynamical characterization is more precise in the sense that it requires three different tests for objects on a plane. It is possible to distinguish the three tests associated to three degrees of freedom characterizing simple harmonic oscillators built up according to the geometry of the object under consideration. The three distinct degrees of freedom generate three fundamental periods which may indicate a unique dynamical dimension provided that the fractal object obeys the Hausdorff criterion [2].

As shown in Figure 1a we may build three harmonic oscillators with the same geometry of the quadric Koch generator. For each different excitation there exists a well determined natural period that may be written under a normalized form as: (T_M/T_{0M}), (T_H/T_{0H}) and (T_V/T_{0V}). The "spring" is assembled with a wire folded according to the geometry of the quadric generator with arbitrary mechanical – Young modulus E – and cross section characteristics – moment of inertia I.

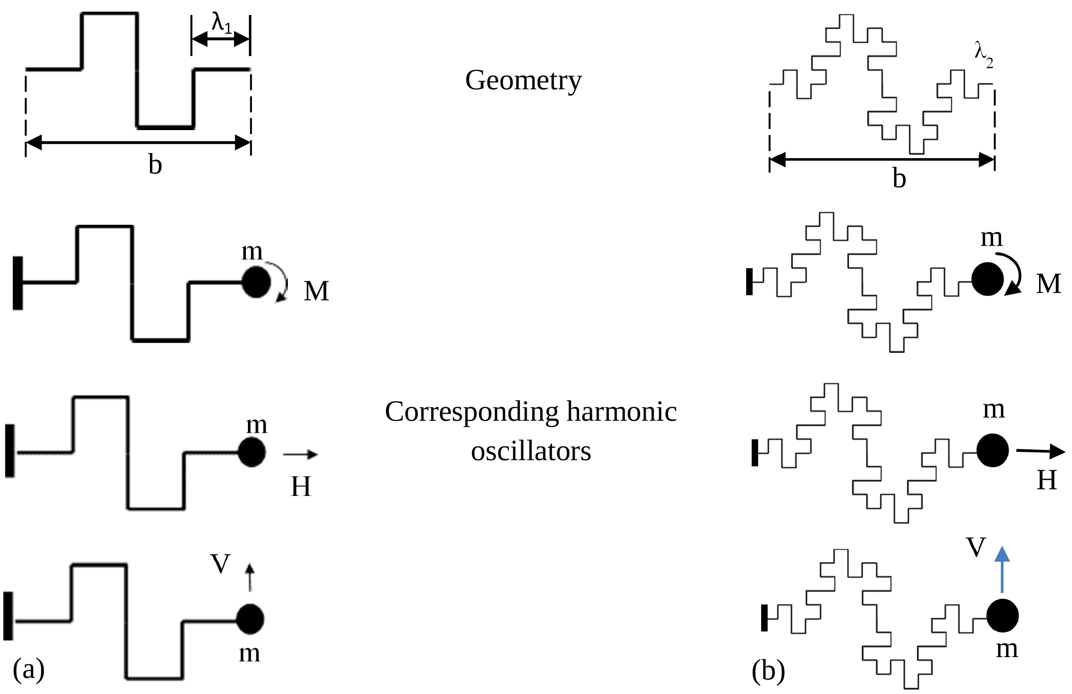


Fig. 1. The Koch quadric curve and the corresponding oscillators with the three distinct excitations characterizing there fundamental frequencies

Now if we repeat the same process for the next terms of the fractal sequence (Figure 1b) new sets of normalized periods are determined. The elastic and geometric characteristics E and I are kept the same for the subsequent oscillators. The sequences of the normalized periods for the three cases as function of the shortest element λ_k represented in a log-log scale approach asymptotically a straight line with a well determined slope that determines the

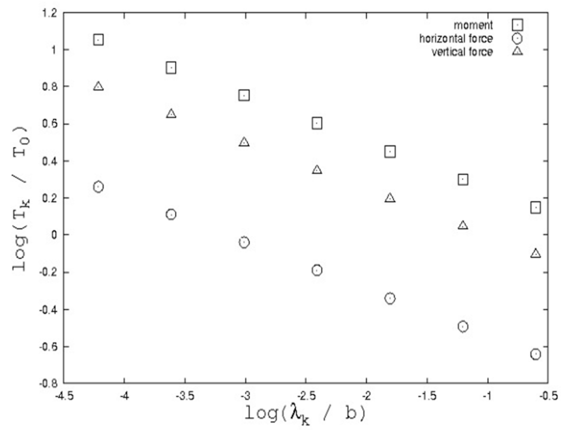


Fig. 2. Koch quadric curve $\log(T_k/T_0)$ x $\log(\lambda_k/b)$ for the three excitations M, H, V. $D_M = 1.5$, $D_H = 1.5$ $D_V = 1.5$

dynamical fractal sequence (Figure 2). All three cases provide the same dimension which is related to the Hausdorff geometric dimension. That is given a certain term n of a sequence, for sufficiently large n, the dynamical approach provides an accurate value for the geometric fractal dimension of the sequence. We call the method described above as “direct method”.

Consider now the more complex case of identification. It has been shown [1] [3] that it is also possible to determine the fractal dimension of the set by cutting successive samples of a given element. The normalized periods of each sample related to its projection on the horizontal axis represented in a log-log scale provide concrete information about the fractal dimension of the entire set. Certainly some spurious perturbations appear for this case. However the deviations do not induce critical inaccuracies in the determination of the fractal dimension. As shown in the Figure 3, the strongest noise appears in the sequence corresponding to the periods related to the initial condition induced by the horizontal force. The adjusted straight line however gives the same fractal dimension as the other two cases. The maximum absolute error is of the order of 0.013%.

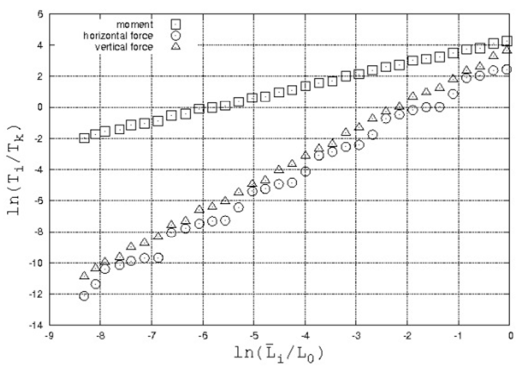


Fig. 3. Koch quadric curve $\log(T_k/T_0)$ x $\log(\lambda_k/b)$ for the three excitations M, H, V. $D_M = 1.498505$, $D_H = 1.497955$, $D_V = 1.497938$ Scale of the horizontal projections $L_i/L_0=(1/3)^i$

This preliminary presentation has the purpose to show that a curve that is fractal in the sense of Hausdorff may be characterized as fractal with the dynamical technique. A complete analysis of the dynamical dimension of classical fractal curves was developed in previous papers. In the next section it will be discussed some consequences of the fractal characterization using the dynamical response of simple oscillators.

THE ENERGY COVERING

The dynamical determination of the fractal dimension is sustained by three different excitations corresponding to three natural periods. The question that comes out naturally is in what sense are these three sequences different. Do each excitation reveals a different aspect of the fractal characteristic of the curve under examination?

To answer this question let us first recall that the periods are associated to the potential energy stored in the oscillators. Note that the periods are determined using the bending energy only. Therefore if each initial perturbation, M, H or V has a distinct characteristic we may say that each one determines a particular aspect of the "fractality" of the given curve. This is indeed the case. The energy induced by the initial displacement corresponding to a moment M is equally distributed over all the elementary segments λ_k for the k^{th} element. Therefore the fractal characterization generated by this initial perturbation corresponds to the total length of the k^{th} element. Indeed the bending energy stored in this element is:

$$W_k = \frac{1}{2}\frac{1}{EI}\sum_1^k \lambda_k = \frac{1}{2}\frac{1}{EI} L_k$$

where $L_k = k\lambda_k$ is the total length of the fractal element. The normalized period is easily obtained:

$$\left(\frac{T_k}{T_0}\right)^2 = \frac{L_k}{L_0} \qquad \text{or} \qquad \log\left(\frac{\mathrm{T_k}}{T_0}\right) = \frac{1}{2}\log\left(\frac{L_k}{L_0}\right)$$

If the curve is fractal with respect to the total length the plot $\log(T_k/T_0)\, x \log(\lambda_k/L_0)$ will show this characteristic.

Now for the other two cases the bending energy distribution over the elements depends also on the spatial distribution of the elementary segments. The question can now be raised if it is possible, for the dynamical dimension, that a curve presents different dimensions for different excitations, or even if it is possible for a curve to have a fractal characterization for some particular excitation and no fractal characterization for another excitation. In other words is it possible that the dynamical dimension depends on the energy covering? This is indeed the case. It was shown [1] that the random orientation of the reduced generator to assemble a Koch triadic sequence (Figure 4) can be detected by the dynamical response of the terms in the sequence. The energy cover generated by the moment doesn't distinguish the self-similar sequence from the random sequence. The energy cover induced by the horizontal force however can clearly show that the assemblage was performed with a random orientation of the reduced generator. Clearly the energy cover for the case of the moment is independent of the orientation of the reduced elements in the curve while the orientation of the elements is an important data to evaluate the total energy induced by the horizontal force. For this case due to the strong characterization of the fractal property of the Koch triadic the energy cover derived from the horizontal force reveals fractal characteristics that are distributed in the neighborhood of the self-similar assemblage. The

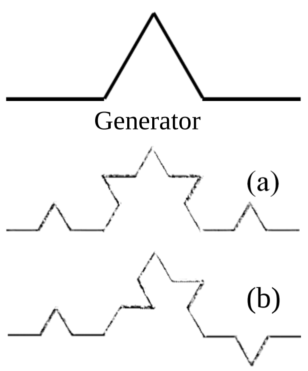


Fig.4. Self-similar grouping (a) and random grouping (b) for the Koch triadic

rms value calculated from the values given by the different grouping converge to the triadic dimension D=1.26186 .

An useful test to verify the characterization of the three different dynamical excitations is to build harmonic oscillators considering the vertical bars rigid and the horizontal bars flexible as shown in Figure 5. Taking a sufficiently number of terms in the sequence – as a rule the set of the first 12 terms is enough – it is possible to obtain the fractal dimension by plotting the curves:

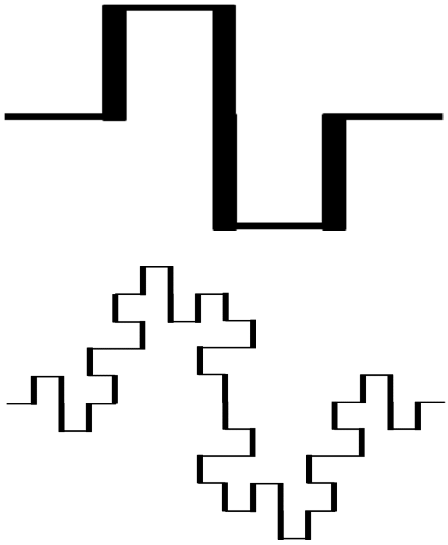


Fig.5. Generator and the first term of the quadric Koch curve. Vertical bars rigid and horizontal flexible

$$\log\left(\frac{\mathrm{T_k}}{T_0}\right)_{M,H,V} = \frac{1}{2}\log\left(f\left(\frac{\lambda_k}{L_0}\right)\right)$$

for the three different excitations M,H,V. If the curves come out to be straight lines the slope of these lines will determine the dynamical fractal dimension. For the Koch quadric with rigid vertical bars we obtain:

$D_M = 1.558880$ $D_H = 1.552680$ $D_V = 1.550688$

Given that the geometric fractal dimension is D=1.5 for the flexible configuration the deviations due to introduction of rigid components with the dynamical procedure remain less than 0,4%. The topological identification obtained with the three energy covers clearly reveals a single fractal dimension very close to the case of the flexible configuration. If we invert the flexibility characteristics of the bars, taking the horizontal rigid and the vertical flexible, the result remains unaltered. Therefore we may say that the dynamical method used to evaluate the fractal characteristic of a plane curve uses three different energy covers associated to three distinct topologies. The above results suggest the introduction of the following postulations:

i) *The energy cover due to the moment excitation is related to the curve length topology.*
ii) *The energy cover due to the horizontal force excitation is related to the topology of the vertical evolution of the curve, that is, the distribution of the elementary segments along the vertical axis.*
iii) *The energy cover due to the vertical force excitation is related to the topology of the horizontal evolution of the curve, that is, the distribution of the elementary segments along the horizontal axis.*

Now we are dealing with two distinct problems:

1. *Direct problem*. We say that the problem is direct if it is given an ordered sequence of curves following a certain formation law. All curves are defined on a constant interval $(0,L_0)$. The dynamic technique examines the existence of fractal topologies of the given sequence.

2. *Inverse problem*. We say that the problem is inverse if it is given a certain element, the master curve, of a hypothetically sequence that may have a fractal structure. The dynamical technique is applied to samples cut off sequentially forming a set of curves $C_1,C_2,...C_n$ whose projections on the horizontal axis follow a decreasing sequence $L_1>L_2>\ldots>L_n$ starting with the master curve. The dynamical technique is applied to find out the topology embedded in the master curve, if any.

Definition:

A set of plane curves is an entirely fractal set if and only if the three dynamical covers reveals the fractal topology of the set.

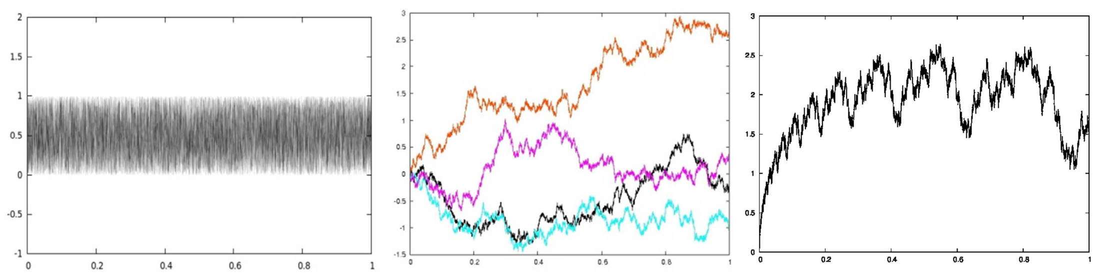
A given master curve is an entirely fractal curve if and only if the sequence of samples obtained from the master curve reveals a fractal topology embedded in the given master curve for the three distinct dynamical covers.

A set is said to be perfectly fractal if it is entirely fractal and if any of the curves belonging the set is an entirely fractal curve.

SELECTED EXAMPLES.

As introduced in the previous sections the dynamical characterization shows that the Koch quadric is a perfectly fractal curve. For this section we have selected three examples to show the application of the energy cover criterion (Figure 6). Two examples refer to random generations – random walk and white noise – and the third one is the Weierstrass-Mandelbrot curve which is a singular curve generated by an infinity sum [4]:

$$W(t)=\lim_{m\to\infty}\left[\sum_{n=-m}^{m}\frac{1-\cos b^{n}t}{b^{(2-D)n}}\right]$$



White noise Random walk Weiertrass-Mandelbrot

Fig.6. Three particular curves, random walk D=1.5 ; White noise D=2.0 and the Weierstrass-Mandelbrot curve with D=1.5

Let us consider the Weierstrass-Mandelbrot curve truncated at m=100 with b=1.5 and D=1.5. Figure 7 represents the curve after fifteen interactions [5]. The energy covers for the three excitations M, H and V estimate coherent dynamical fractal dimensions:

D_M=1.463734; D_V=1.462550; D_V=1,467751

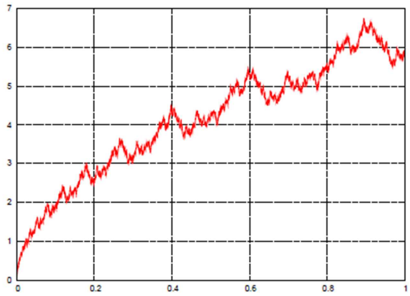


Fig.7. Generation of the sequence of curves approaching the Weierstrass-Mandelbrot curve for m=100 after 15 interactions.

The errors are less than 2.5%. All the covers indicate a coherent fractal characterization. The sequence representing the Weierstrass-Mandelbrot curve is entirely fractal. Table 1 shows the dynamic dimensions for other values of D. The errors remain below ±6%. Taking a large number of terms for the approximating sequences probably the precision could be improved. In any case it is possible to say that all the approximations represent entirely fractal sequences.

	D=1.0	D=1.1	D=1.3	D=1.5	D=1.7	D=1.9	D=2.0
D_M	1.032688	1.096462	1.288016	1.463734	1.611834	1.7988095	2.020544
D_H	1.044311	1.110245	1.290945	1.462550	1.609598	1.801783	1.998374
D_V	1.047202	1.112634	1.293410	1.467751	1.616178	1.808980	2.051467

Tablel 1. Dynamical dimension for the Weierstrass-Mandelbrot curve evaluated with a converging sequence of curves. Direct problem.

Now if we take the largest term in the Weierstrass-Mandelbrot approximating sequence and try to determine the embedded topology using the sample technique, the inverse problem, the following values are obtained:

	D=1.0	D=1.1	D=1.3	D=1.5	D=1.7	D=1.9	D=2.0
D_M	1.005330	1.025217	1.031615	1.013655	1.000221	1.000394	0.999985
D_H	1.031976	1.151303	1.351264	1.510396	1.675107	1.894361	2.011184
D_V	1.010868	1.050767	1.051465	1.011211	0.970399	0.999697	1.008233

Table 2. Dynamical dimension for the Weierstrass-Mandelbrot curve using a sequence of samples cutoff from a master curve. Inverse problem.

For the inverse problem therefore only the energy cover corresponding to the horizontal force reveals the fractal characteristic of the curve. Therefore the Weierstrass-Mandelbrot curves do not constitute a perfect fractal set. For the other two cases, random walk and white noise we have similar results as shown in table 3.

	Random walk		White noise	
	Direct problem	Inverse problem	Direct problem	Inverse problem
D_M	1.465275	1.000000	1.965198	1.013887
D_H	1.498594	1.508150	1.983766	2.033592
D_V	1.462351	0.999995	1.981602	1.0011220

Table 3. Dynamical dimension for random walk and white noise, direct and inverse prolems.

The random walk is defined in the domain [0,1]. The curve was determined considering 2^{14} points and at each step the function deviates of an amount 0.01 upwards or downwards with equal probability. The white noise represents the random distribution in the interval [0,1] along the x-axis of 2^{14} points within the interval [0,1] in the y-axis. As for the Weierstrass-Mandelbrot curve the white noise and the random walk are not perfect fractal sets. For the inverse problem only the energy cover corresponding to the horizontal force characterizes a fractal topology embedded in the master curve.

CONCLUSION

The dynamical determination of the fractal topology of plane curves provides much more complete information about the character of the fractal structure. It is possible using the dynamic technique to distinguish the topological characteristics of the curve length independently from the distribution of the elementary segments along the horizontal and vertical axis. A curve of the type represented in Figure 8 is not fractal with respect to the total length although it could present fractal characteristics with respect to the distribution along the vertical axis.

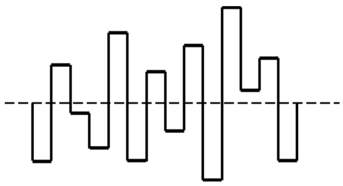


Fig.8. Particular curve with fractal topological distribution along the vertical axis

Note that the mechanical and geometric characteristics of the wires representing the springs of the harmonic oscillators and the mass as well are kept invariants for all terms of the sequences. For the more general case where these characteristics vary for each term of the sequence the fractal characteristic may be considerably modified.

The dynamical characterization of fractal curves provides more refined information as compared with other approximation methods to determine the fractal characteristic of plane curves. Since the three distinct energy covering are topologically independent it is possible to detected different fractal formation for particular curves. Affine similarity for instance is a good example where the dynamic dimension may be of great help. Two different similarity scales corresponding to two orthogonal axes may be detected using the dynamic dimension technique. It is also important to mention that the method exposed here may be useful to find the fractal characteristics of fibers and membranes through experiments on material samples. That is the inverse problem may be solved experimentally. Another important application is the determination of the fine fractal structure of proteins. In this case the problem is more complex since we have to deal with harmonic oscillators with at least six degrees of freedom.

The central focus of this paper is to explore possible fractal characteristics of fractal curves using the dynamic response of a set of harmonic oscillators built up according to the geometry of the fractal set. Since the results are promising it is now necessary to explore the deeper analytical meaning underneath the numerical experiments. We believe that new roads concerning the analysis of energy measures and the respective topological consequence should be explored. Also experimental research may be developed to check the efficiency of this method to determine the fractal characteristics of certain materials.

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