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A COMPUTATIONAL MODEL FOR STRUCTURAL VIBRATION PROBLEMS.

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Abstract. *This paper deals with the development of a computational model in freeFEM++ for the analyzes of structural vibration problems in civil engineering structures during earthquakes. freeFEM++ is an open source program, based on the finite element method, which could be helpful to study and improving the analyzes of the dynamic response of civil engineering structures due to seismic loads.*

The most general approach for dynamic analyzes is the direct numerical integration of the dynamic equilibrium equations on the weak formulation. The resolution of this dynamic problem must satisfy two steps: space integration and time integration.

For the first step are approached the fundamentals of numerical methods of space integration and is used finite element method.

For the second step is used a time-domain numerical method - Newmark's method - to determinate dynamic response of strutural systems. After the solution is defined at time "$t - \Delta t$", the method determinate the solution at time "t". This method requires the solution of the linearized dynamic equilibrium equations at each time step.

Additionally some essential concepts are introduced for the study of experimental results obtained by performing vibration tests.

Finally, is studied a physical model of a two story building, based on the analyzes of results of an ambient vibration test and on the development of two numerical models (in freeFEM++ and SAP2000) for comparison and analysis of results and validation of the computational tool developed. After the calibration of numerical models using experimental results, it's carried out a dynamic analysis of numerical models subjected to seismic action, with subsequent comparison and analysis of results.

1 Introduction

In the past, civil engineering structures were dimensioned using only static analysis, considering static loads along time (eg gravity load, such as dead weight and permanent loads). With the time, the computational advance made it possible the consideration of dynamic analysis, giving importance to a number of phenomenon which vary in value or direction over time. Thus, in addition to the permanent loads, such as the movement of people and machinery engines on slabs, the impact of wind on tall buildings, road traffic on bridges and the occurrence of earthquakes become fundamental in the design of structures.

A better knowledge of dynamic loads enabled a more detailed study and analysis of the structural dynamic behaviour.

A better knowledge of dynamic loads and structural dynamic behaviour has allowed the design of more slender and flexible solutions (improving the aesthetic appearance of structures) and the development of numerical models more reliable (Figure 1 (b)).

Thus, it is intended to approach and develop some methodologies for studying problems of structural dynamics through computational tool in *freeFEM++*, version 3.20 [3].

In this work it was studied a scale model of the structure of a two story building, based on the development of numerical models in *freeFEM++* (Figure 2(b)) and in *SAP2000*, version 15 [Computers and Structures, Inc. 2011]. Vibration tests were carried out on the physical model in order to calibrate the numerical models. Finally, a dynamic analysis was performed in both numerical models, considering a seismic load.

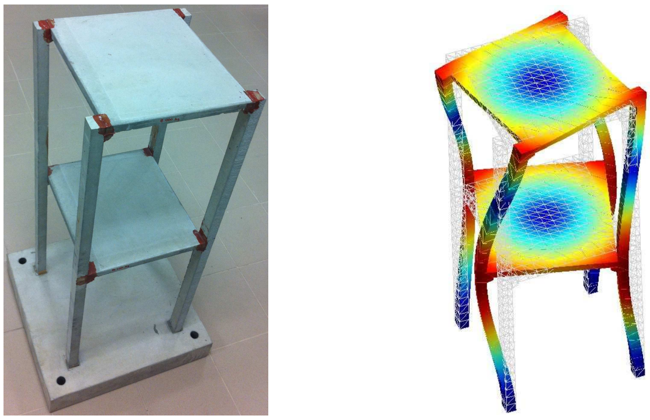
(a) (b)

Figure 1: *Burj Khalifa Bin Zayid*: (a) building completed in 2009 in Dubai, United Arab Emirates and (b) corresponding numerical model in finite element.

2 Methodology

In the analysis of structural dynamic behaviour with multiple degrees of freedom (MDOF), instead of one vibrating mass, is considered N oscillating masses.

The system with N equations of dynamics, considering a model with accelerations imposed on the base level equal in all supports, is obtained with linear dynamic equilibrium equa-



(a) (b)

Figure 2: Case study: (a) Two storey building strutural model and (b) corresponding finite element model developed with *freeFEM++*.

tions written as:

$$\left\{ \begin{array}{l} \underline{m}\,\ddot{\underset{\sim}{u}}\,(t) + \underline{c}\,\dot{\underset{\sim}{u}}\,(t) + \underline{k}\,\underset{\sim}{u}\,(t) = -a_b \underline{m}\,\underset{\sim}{l}\,(t) \\ \text{Initial and boundary conditions} \end{array} \right. \tag{1}$$

Where $\underline{m}$, $\underline{c}$ and $\underline{k}$ are respectively the mass, damping and stiffness matrix of the model, $\ddot{\underset{\sim}{u}}(t)$, $\dot{\underset{\sim}{u}}(t)$ and $\underset{\sim}{u}(t)$ are respectively the acceleration, velocity and displacement vectors, and $\underset{\sim}{l}$ is the influence vector that relates the degrees of freedom of the structure with the imposed accelerations on the base level (a_b).

The damping matrix $\underline{c}$ can be obtained using the MEF (same way as the mass matrix $\underline{m}$), although it is usual to use the concept of Rayleigh damping. This concept is defined by a proportionality to mass and stiffness matrices, $\underline{m}$ and $\underline{k}$, so that dumping matrix $\underline{c}$ is also a "full" matrix, given by:

$$\underline{c} = c_1\,\underline{m} + c_2\,\underline{k} \tag{2}$$

Where c_1 and c_2 are the constants of the Rayleigh damping matrix which quantifies the ratio between the mass matrices $\underline{m}$ and stiffness $\underline{k}$, respectively.

For the two story building example were assumed $c_1 = 0,01$ and $c_2 = 0,0001$.

2.1 Undamped free vibration of multiple degree of freedom system

To obtain the natural frequencies and mode shapes of a system with multiple degrees of freedom, it proceeds an analysis of the movements in free vibration, assuming no external forces applied to the structure, and no damping effects of the material. Thus, the system of equations to solve is the following:

$$\left\{ \begin{array}{l} \underline{m}\,\ddot{\underset{\sim}{u}}\,(t) + \underline{k}\,\underset{\sim}{u}\,(t) = \underset{\sim}{0} \\ \text{Initial and boundary conditions} \end{array} \right. \tag{3}$$

The solution of previous system of equations can be obtained with the transition of structural coordinates (corresponding to the displacements of the multiple degrees of freedom of the structure) in modal coordinates [5], given by:

$$\underset{\sim}{u}\,(t) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} a_1\, \cos{(\omega_1\, t)} + b_1\, \sin{(\omega_1\, t)} \\ a_2\, \cos{(\omega_2\, t)} + b_2\, \sin{(\omega_2\, t)} \end{bmatrix} = \underset{\sim}{\phi_n}\, u_n^*(t) \tag{4}$$

Where $\underset{\sim}{\phi_n}$ corresponds to a modal configuration of the structure and does not vary with time, and $u_n^*(t)$ corresponds to displacement along time in each degree of freedom given by the sinusoidal function, defined by:

$$u_n^*\,(t) = a_n\, \cos{(\omega_n\, t)} + b_n\, \sin{(\omega_n\, t)} \tag{5}$$

The constants a_n and b_n can be determined from the initial conditions.

After replacing the expression (4) in the equilibrium equation to solve (3), it is necessary to solve a problem of eigenvalues and eigenvectors, defined by:

$$|\underline{k} - \lambda\, \underline{m}|\, \underset{\sim}{\phi_n} = \underset{\sim}{0} \tag{6}$$

Where eigenvalues given by λ correspond to the square of the angular frequencies (ω_n^2) and eigenvectors $\underset{\sim}{\phi_n}$ correspond to vectors with the mode shapes (modal configuration of the structure for each frequency).

2.2 Boundary conditions

In this work it is used the boundary conditions in displacement Dirichlet $\underset{\sim}{u} = \underset{\sim}{p}$, which allow the restriction of movements in certain directions and points (punctual elastic restraints).

2.3 Integration in Space. Finite Element Method.

The aim of this method is the approximation to a continuous domain analized by a finite number of sub-domains called finite elements. This approximation is given by an equation − a fundamental approach of FEM −, where is adopted a set of interpolation functions − shape functions − that defines (approximately) the displacement field in each finite element, according to the displacement of the nodal points [4].

With this approximation, obtains the elementary equation of dynamic equilibrium, given by:

$$\underline{m}^e\, \underset{\sim}{\ddot{u}}^e(t) + \underline{c}^e\, \underset{\sim}{\dot{u}}^e(t) + \underline{k}^e\, \underset{\sim}{u}^e(t) = \underset{\sim}{f}^e(t) \tag{7}$$

The global matrices of mass, stiffness and damping ($\underline{m}$, $\underline{k}$ and $\underline{c}$) and the global vector of external forces ($\underset{\sim}{f}$) are obtained by overlapping the elementary matrices and vector. This assembly allows to obtain the dynamic equilibrium equation for the entire global domain analyzed, which corresponds to a system of dependent differential equations of 2nd order, to solve for the initial conditions established. Thus, in the numerical resolution of the problem of initial values and boundary (1), after the first stage corresponding to the integration in space, the dynamic analysis is reduced to the resolution of following initial value problem:

$$\begin{cases} \underline{m}\, \underset{\sim}{\ddot{u}}(t) + \underline{c}\, \underset{\sim}{\dot{u}}(t) + \underline{k}\, \underset{\sim}{u}(t) = \underset{\sim}{f}(t) \\ \text{Initial conditions} \end{cases} \tag{8}$$

The vector $\underset{\sim}{u}$ contains the displacements in the three degrees of freedom in all the nodes of the structure, and the vector $\underset{\sim}{f}$ corresponds to the vector of nodal forces, equivalent to densities, superficial or concentrated forces aplied in elements.

2.4 *FreeFEM++* formulation

For the implementation of the FEM in *freeFEM++*, it is necessary to deduce the weak formulation of the problem of initial values and boundary (1), which can be obtained with variational formulation or PTV.

That can be written abbreviated:

$$\int_V \underset{\sim}{v}^T \mathrm{m} \ddot{\underset{\sim}{u}}\, dV + \int_V \underset{\sim}{v}^T \mathrm{c} \dot{\underset{\sim}{u}}\, dV + \int_V \big(\lambda\, \mathrm{Div}(\underset{\sim}{v}) \mathrm{Div}(\underset{\sim}{u}) + 2\,\mu\, \varepsilon(\underset{\sim}{v})^T \varepsilon(\underset{\sim}{u})\big)\, dV = \int_V \underset{\sim}{v}^T \underset{\sim}{f}\, dV \quad (9)$$

Where the parameters μ and λ corresponding to the Lamé coefficients defined by:

$$\mu = G = \frac{E}{2\,(1+\nu)} \qquad \lambda = \frac{E\,\nu}{(1+\nu)\,(1-2\nu)} \quad (10)$$

2.5 Integration in Time. Newmark's Method

The Newmark's expressions, that permit the calculation of the displacements $\underset{\sim}{u}$ and the velocities $\dot{\underset{\sim}{u}}$ for the certain instant time $(t + \Delta t)$, are given by:

$$\underset{\sim}{u}_{t+\Delta t} = \underset{\sim}{u}_t + \dot{\underset{\sim}{u}}_t \Delta t + \left(\frac{1}{2} - \beta\right) \Delta t^2 \ddot{\underset{\sim}{u}}_t + \beta \Delta t^2 \ddot{\underset{\sim}{u}}_{t+\Delta t} \quad (11)$$

$$\dot{\underset{\sim}{u}}_{t+\Delta t} = \dot{\underset{\sim}{u}}_t + (1-\alpha)\, \Delta t \ddot{\underset{\sim}{u}}_t + \alpha \Delta t \ddot{\underset{\sim}{u}}_{t+\Delta t} \quad (12)$$

Constants α and β define the acceleration variation within each period of time Δt and determine the stability and accuracy of this method.

On those equations (11) and (12) adds up the dynamic equilibrium equation (8) for period of time $t + \Delta t$, given by:

$$\underline{m}\, \ddot{\underset{\sim}{u}}_{t+\Delta t} \;+\; \underline{c}\, \dot{\underset{\sim}{u}}_{t+\Delta t} \;+\; \underline{k}\, \underset{\sim}{u}_{t+\Delta t} \;=\; \underset{\sim}{f}_{t+\Delta t} \quad (13)$$

The group of equations (11), (12) and (13) creates a method for obtaining the numerical solution of the problem dynamic (8). Therefore, after the known of displacement, velocity, acceleration and external force applied at time t, substituting the equations (11) and (12) into equation (ref-taylor9) obtains the acceleration $\ddot{\underset{\sim}{u}}_{t+\Delta t}$. Once known $\ddot{\underset{\sim}{u}}_{t+\Delta t}$, using the equations (11) and (12) is possible to calculate the displacement $\underset{\sim}{u}_{t+\Delta t}$ and the velocity $\dot{\underset{\sim}{u}}_{t+\Delta t}$.

As mentioned in [2], the Newmark's method is stable if:

$$\beta \leq \frac{1}{2} \leq \alpha \text{ , when } \Delta t \leq \frac{\sqrt{2}}{\omega_{max}\sqrt{\alpha - 2\beta}} \quad (14)$$

Where ω_{max} is the maximum frequency of natural vibration, undamped, of the structure. In this case is guaranteed that there is a constant $C > 0$ such that, for any instant time t, we have:

$$||u_t|| \leq C \qquad\qquad t \in [0, T] \quad (15)$$

Additionally, it guarantees the convergence of the method is the stiffness matrix $\underline{k}$ is symmetric and positive definite, ie, all the eigenvalues are positive.

2.5.1 Algorithm

Although there are several algorithms for calculating the numerical method in the literature, it was decided to develop an integration process according to [1], then:

1. For initial conditions (initial displacement and velocity vectors, $\underset{\sim}{u}_0$ and $\dot{\underset{\sim}{u}}_0$) and initial external force vector $\underset{\sim}{f}_0$;

2. Solve
$$\ddot{\underset{\sim}{u}}_0 = \underline{m}^{-1} \left(\underset{\sim}{f}_0 - \underline{c}\, \dot{\underset{\sim}{u}}_0 - \underline{k}\, \underset{\sim}{u}_0 \right) \tag{16}$$

3. Define an integration step Δt constant in the iterative process;

4. Determine an effective stiffness matrix $\underline{k}_e$:
$$\underline{k}_e = \underline{k} + \frac{\alpha}{\beta\, \Delta t}\, \underline{c} + \frac{1}{\beta\, \Delta t^2}\, \underline{m} \tag{17}$$

5. Determine two auxiliary matrices $\underline{A}_1$ and $\underline{A}_2$:
$$\underline{A}_1 = \frac{1}{\beta\, \Delta t}\, \underline{m} + \frac{\alpha}{\beta}\, \underline{c} \qquad ; \qquad \underline{A}_2 = \frac{1}{2\,\beta}\, \underline{m} + \Delta t \left(\frac{\alpha}{2\,\beta} - 1 \right) \underline{c} \tag{18}$$

6. Iterative process:

   Calculate the variations in displacement, velocity and acceleration in each period of time Δt, caused by the variation of external forces $\underset{\sim}{f}$.

   (a) Solving the equation of dynamic equilibrium for the calculation of $\Delta \underset{\sim}{f}_t$:
   $$\Delta \underset{\sim}{f}_t = \left(\underset{\sim}{f}_t - \underset{\sim}{f}_{t-\Delta t} \right) + \underline{A}_1\, \dot{\underset{\sim}{u}}_{t-\Delta t} + \underline{A}_2\, \ddot{\underset{\sim}{u}}_{t-\Delta t} \tag{19}$$

   (b) Displacement, velocity and acceleration variations:
   $$\begin{aligned} \Delta \underset{\sim}{u}_t &= \underline{k}_e^{\,-1}\, \Delta \underset{\sim}{f}_t \\ \Delta \dot{\underset{\sim}{u}}_t &= \frac{\alpha}{\beta\, \Delta t}\, \Delta \underset{\sim}{u}_t - \frac{\alpha}{\beta}\, \dot{\underset{\sim}{u}}_{t-\Delta t} + \Delta t \left(1 - \frac{\alpha}{2\,\beta} \right) \ddot{\underset{\sim}{u}}_{t-\Delta t} \\ \Delta \ddot{\underset{\sim}{u}}_t &= \frac{1}{\beta\, \Delta t^2}\, \Delta \underset{\sim}{u}_t - \frac{1}{\beta\, \Delta t}\, \dot{\underset{\sim}{u}}_{t-\Delta t} - \frac{1}{2\,\beta}\, \ddot{\underset{\sim}{u}}_{t-\Delta t} \end{aligned}$$

   (c) Update the variables $\underset{\sim}{u}_t$, $\dot{\underset{\sim}{u}}_t$ and $\ddot{\underset{\sim}{u}}_t$:
   $$\underset{\sim}{u}_t = \underset{\sim}{u}_{t-\Delta t} + \Delta \underset{\sim}{u}_t \quad ; \quad \dot{\underset{\sim}{u}}_t = \dot{\underset{\sim}{u}}_{t-\Delta t} + \Delta \dot{\underset{\sim}{u}}_t \quad ; \quad \ddot{\underset{\sim}{u}}_t = \ddot{\underset{\sim}{u}}_{t-\Delta t} + \Delta \ddot{\underset{\sim}{u}}_t \tag{20}$$

For the two story building example were assumed $\alpha = 1/2$ and $\beta = 1/4$.

2.6 Seismic actions considered

As external force was considered a seismic action on Ox direction. It was considered a record with accelerations at the base $a_b(t)$ (Figure 3) from the classic earthquake that occurred in *El Centro*, California, United States of America in 1940. This seismic record has the value of acceleration at the base measured every 0.01 seconds, with a total duration of 40 seconds.

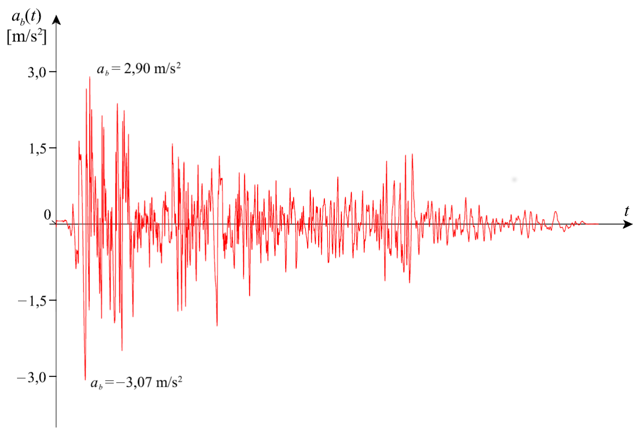


Figure 3: Graphical representation of the seismic acceleration record obtained in the earthquake in *El Centro*, 1940, California, USA.

Moreover, it was also studied the dynamic behaviour of the structure when subjected to another seismic record, with different dynamic characteristics and intensity. Thus, it was considered a record with accelerations at the base $a_b(t)$ (Figure 4) from the earthquake that occurred in *Mexico city*, Mexico in 1985. This seismic record has the value of acceleration at the base measured every 0.01 seconds, with a total duration of 60 seconds.

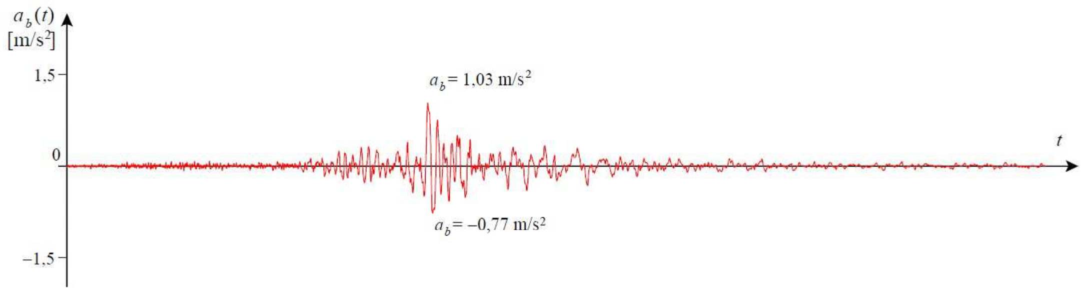


Figure 4: Graphical representation of the seismic acceleration record obtained in the earthquake in *Mexico city*, 1985, Mexico.

The comparison of numerical models was obtained through the determination of the displacement in direction Ox of a point on the upper slab of the model (Figure 5), compatible with the meshes developed in programs *freeFEM++* and *SAP2000*.

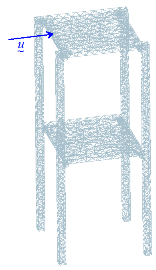


Figure 5: Indication of the point to consider for determining the displacement along time. Coordinates (0.03, 0.215, 1.05).

3 Analysis of results

After the development of numerical models in *freeFEM++* and *SAP2000* were obtained the first natural frequencies and the respective mode shapes, which allow to define the points to instrument the physical model on an ambient vibration test. After the test, knowing the values of natural frequencies, the numerical models were calibrated in order to approximate the value of the first natural frequency to the value obtained in the physical model[1] ($f_1 = 23,3$ Hz). This calibration has been achieved by varying the modulus of elasticity in each numerical model. Table 1 shows the values of natural frequencies of vibration adjusted to the physical model and the corresponding modulus of elasticity considered.

N	**Physical model [Hz]**	**FreeFEM ++ [Hz]**	Error dif. [%]	**SAP 2000 [Hz]**	Error dif. [%]
1	23,3	23,30	0,0	23,30	0,0
2	25,3	25,45	0,6	25,50	0,8
3	40,6	39,34	3,1	39,19	3,5
4	78,4	73,39	6,4	73,29	6,5
5	96,1	88,90	7,5	88,81	7,6
E [GPa]		36,5		35,8	

Table 1: Comparison of the results of the natural frequencies of vibration of numerical models with the physical model.

3.1 Dynamic analysis

After the calibration of the numerical models, in order to compare the results obtained in *freeFEM++* with the results obtained in *SAP2000*, shall be calculated absolute error given by the expression:

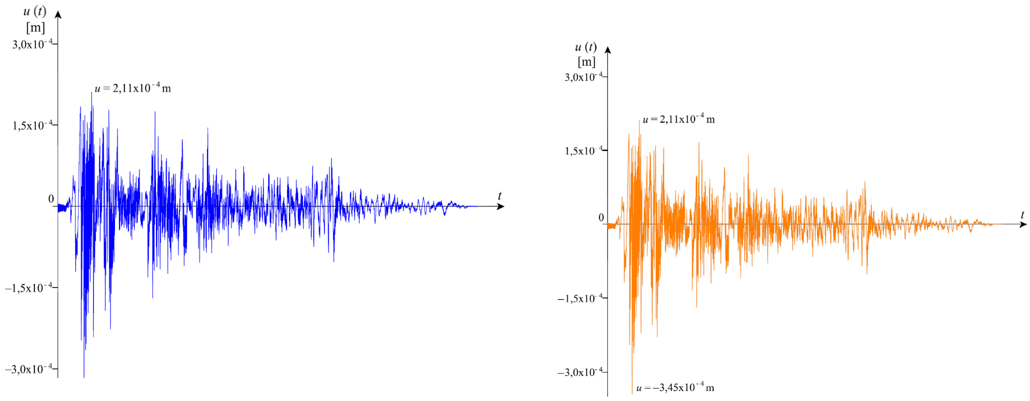
$$e_{absolute} = |u_{SAP2000} - u_{freeFEM++}| \tag{21}$$

As were considered two different seismic records as external force, were obtained two different dynamic analysis: *El Centro* analysis and *Mexico city* analysis.

[1]It was decided because first mode of vibration is what best characterizes the response of a structure.

3.1.1 *El Centro* analysis

Considering the record with accelerations at the base $a_b(t)$ shown in Figure 3, and the Newmark's method, were obtained in *freeFEM++* the displacements shown in Figure 6(a) . For *El Centro* analysis using the numerical model in *SAP2000* were obtained the displacements shown in Figure 6(b) .

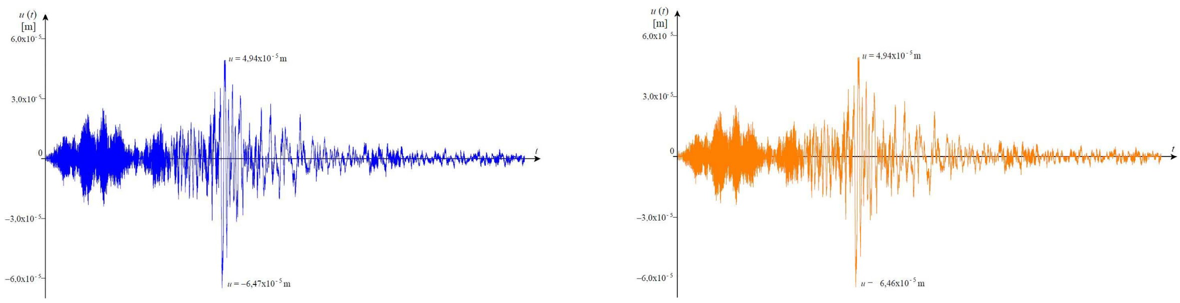


(a) *freeFEM++* (b) *SAP2000*

Figure 6: Graphic representation of displacement of the point on the upper slab along time for *El Centro* analysis.

3.1.2 *Mexico city* analysis

Considering the record with accelerations at the base $a_b(t)$ shown in Figure 4, were obtained in *freeFEM++* the displacements shown in Figure 7 (a). On the numerical model in *SAP2000*, for dynamic analysis were obtained the displacements shown in Figure 7 (b).



(a) *freeFEM++* (b) *SAP2000*

Figure 7: Graphic representation of displacement along time of the point on the upper slab for *Mexico city* analysis.

3.2 Analysis and comparison of results

By representing the error in *El Centro* analysis (Figure 9) is concluded that there is more discrepancies in the period up to 5 seconds. This period is where the seismic action and the responses calculated in both models have maximum values.

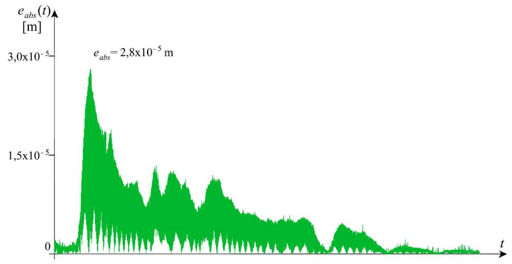


Figure 8: Graphic representation of the absolute error between the results in *freeFEM++* and *SAP2000*, in *El Centro* analysis.

However, analysing the Figure 6(a) and Figure 6(b), it can be concluded that both models have similar progress, being able to compare the relative error between peak values obtained in both models (Table 2).

Values	***freeFEM++* [m]**	***SAP2000* [m]**	**Relative error [%]**
Maximum	2,11 $\times 10^{-4}$	2,11 $\times 10^{-4}$	0,11
Minimum	-3,48 $\times 10^{-4}$	-3,45 $\times 10^{-4}$	0,85

Table 2: Maximum relative error of the numerical models for *El Centro* analysis.

In *Mexico city* analysis, error graphic provides constant peaks over time, with approximate value of $1,0 \times 10^{-6}$ m, (Figure 9).

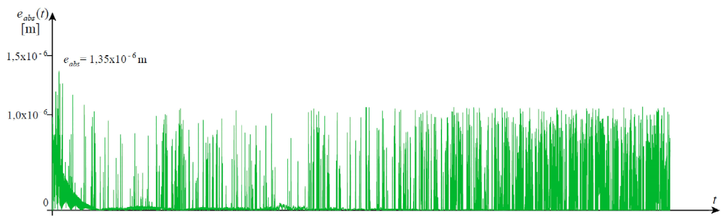


Figure 9: Graphic representation of the absolute error between the results in *freeFEM++* and *SAP2000*, in *Mexico city* analysis.

However, analysing the Figure 7 (a) and Figure 7 (b), once again both models have similar progress, being able to compare the relative error between peak values obtained in both models (Table 3).

Values	***freeFEM++* [m]**	***SAP2000* [m]**	**Relative error [%]**
Maximum	4,94 $\times 10^{-5}$	4,94 $\times 10^{-5}$	0,07
Minimum	-6,47 $\times 10^{-5}$	-6,46 $\times 10^{-5}$	0,08

Table 3: Maximum relative error of the numerical models for *Mexico city* analysis.

4 Conclusions

The study of the dynamic behaviour of civil engineering structures submitted to seismic loads continues to be a topic of high importance and represents a challenge to find and develop more efficient tools for their analysis.

In this paper an alternative tool were presented (program *freeFEM++*) to solve problems of structural dynamics, using the finite element method (FEM). To show the applicability of this tool were presented briefly, the fundamentals of structural dynamics, the principles and methodology of integration in space using FEM, and even the method of time integration - the Newmark's method.

To apply the concepts discussed along this work, were studied a physical model of a two story building. In this study were developed numerical models in *freeFEM++* and *SAP2000* and made a vibration test to identify the modal parameters of the physical model (namely, the natural frequencies and the corresponding mode shapes). By comparing the results of the two models it was concluded that the methodology developed in *freeFEM++* has a good approximation to the numerical results obtained with the program *SAP2000*.

Comparing the numerical results with experimental results obtained in the vibration test, there is a need to calibrate a parameter (the elastic modulus) in numerical models to adjustment of the values of natural frequencies. After adjusting the numerical values identified experimentally in the physical model, it was concluded that the methodology developed in *freeFEM++* obtains, in general, minor differences when compared with the differences obtained in *SAP2000*.

Once calibrated the numerical models, proceeded to two dynamic analyses of the two story building (using seismic records in the form of histories of acceleration obtained during the occurrence of two earthquakes - in *El Centro*, 1940, USA, and another in *Mexico City*, 1985, Mexico) using the Newmark's method implemented in *freeFEM++* and available in *SAP2000* for time integration. Analysing and comparing the histories of displacements, obtained at a common point on meshes used in both numerical models, there were small variations in the results, which can be associated with several aspects, among which are:

- differences in consideration of the boundary conditions, which differs between the two numerical models;
- the adoption of different types of finite elements in both numerical models, which forced to generate different meshes.

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