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RESONANCE PASSAGE OF CYCLIC SYMMETRIC STRUCTURES

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Abstract. *The resonance passage of cyclic symmetric structures that are excited by accelerating travelling waves is studied. Many mechanical applications that have cyclic symmetry are subject to vibration, e.g. bladed turbine disks or gears. Dynamic loads can cause damages of the structure. Hence it is important to predict the vibration response. Most work done on this field is dealing with periodic excitation. Thus, a steady state solution can be found. To calculate the vibration response of discretized cyclic symmetric structures that are subject to travelling waves the matrices describing the structure can be reduced to only one cyclic segment, while applying complex constraints.*

This paper shows that the assumption of cyclic symmetry can be extended for the case of a linearly changing excitation frequency. Therefor, the analytical solution for a single degree of freedom system that is excited by a changing frequency is developed. The character of the analytical solution shows that the assumption of cyclic symmetry can be made while the excitation is an accelerating travelling wave. This is exemplarily shown using a six degree of freedom system comprising three cyclic sectors. The model is reduced to only one cyclic symmetric segment.

To give a further confirmation that the assumption of cyclic symmetry is reasonable the vibration response of the full model and the reduced model are compared. The full model is calculated using a time step integration. The results show an exact agreement.

1 INTRODUCTION

Many rotationally periodic mechanical applications as turbines are subject to high dynamic loads. These loads can cause high cycle fatigue damages. Thus it is very important to predict vibration response. Turbines are cyclic symmetric structures that are excited by travelling waves. THOMAS showed in [1], that these structures can be reduced to only one cyclic segment to calculate the vibration response while excitation is periodic. Taking advantage of the cyclic symmetry the number of degrees of freedom is significantly reduced.

Turbines are also exposed to resonance passages. Figure 1 is a classical CAMPBELL-diagram. The rotational frequency of the rotor f_r is proportional to t. The resonance passages of different eingine orders are illustrated. Before the operation point is reached the resonance frequency is passed. The acceleration of the excitation frequency depends on the engine order.

Analytical methods to solve the ordinary differential equation (ODE) of a linear single degree of freedom system (SDOF-system) that is subject to transient excitation are illustrated in e.g. [2, 3, 4].

Few works discuss the vibration response of run-ups and run-downs of turbine applications as e.g. [5, 6, 7]. The works [8, 9, 10] take into account that the structure is cyclic symmetric. They use numerical approaches. But there is no work that uses analytical solutions for the calculation of the resonance passage.

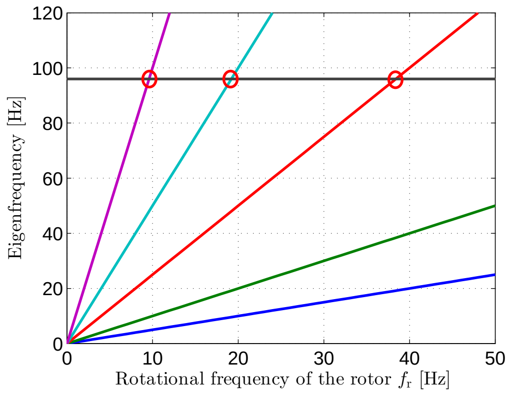

Figure 1: Different engine orders in the CAMPBELL diagram

2 MATHEMATICAL DISCRIPTION

2.1 Accelerating travelling wave

The ODE of an SDOF-system is given by

$$m\ddot{x} + d\dot{x} + kx = f(t), \tag{1}$$

where $f(t)$ represents a particular transient excitation namely the sine sweep, with

$$f(t) = \hat{f}\mathrm{e}^{\mathrm{i}(\frac{\alpha t^2}{2}+\phi)}. \tag{2}$$

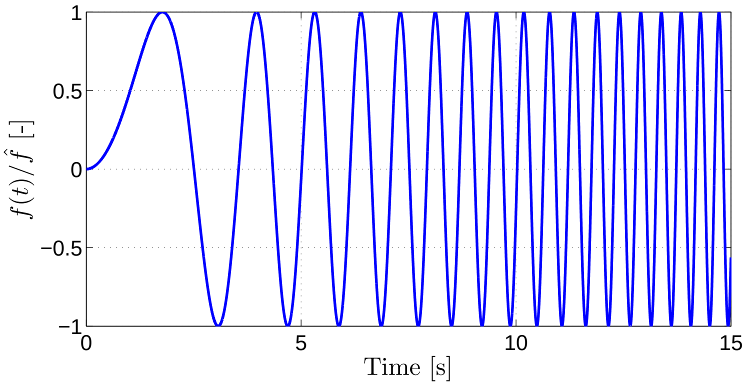

Figure 2: Sine sweep

where α is the angular acceleration, ϕ is the phase angle, and $\hat{f}$ is the excitation force amplitude. Figure 2 shows a sine sweep. The excitation frequency is proportional to the time. The time-dependent instantaneous frequency of the excitation $\Omega(t)$ results from the derivation of the phase angle,

$$\Omega(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\alpha t^2}{2} + \phi\right) = \alpha t, \tag{3}$$

where $\alpha > 0$ represents a run-up and $\alpha < 0$ represents a run-down. While extending the SDOF-system to a multi degree of freedom system (MDOF-system) Eq. 1 is written by

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t), \tag{4}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{x}$ is the displacement, and $\mathbf{f}(t)$ represents the excitation. While describing a cyclic symmetric structure the matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ in Eq. 4 have a block symmetric character. Due to the change of the rotational frequency of the rotor $\mathbf{f}(t)$ represents the accelerating travelling wave with

$$f_i(t) = \hat{f}\mathrm{e}^{\mathrm{i}(\frac{\alpha t^2}{2} + \phi_i)}, \tag{5}$$

where i decribes a discrete point of the structure and ϕ_i is the corresponding angular in the local coordinate system that rotates with the structure, see Figure 3.

2.2 Analytical solution

Using the variation of parameters the solution of Eq. 1 can be obtained by

$$x(t) = \Re\left\{\frac{\hat{f}}{\eta m}\int_0^t \mathrm{e}^{-\mathrm{i}(\frac{\alpha}{2}\tau^2) - \frac{\gamma}{2}(t-\tau)} \sin(\eta(t-\tau))\mathrm{d}\tau\right\}, \tag{6}$$

with

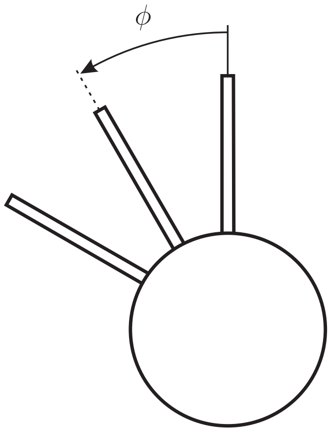

Figure 3: Angular ϕ in the local coordinate system

$$\gamma = \frac{d}{m} \tag{7}$$

and

$$\eta = \sqrt{\left(\frac{k}{m}\right) - \frac{\gamma^2}{4}}. \tag{8}$$

The integral in Eq. 6 cannot be evaluated in a closed form. To determine the solution, Eq. 6 is rewritten as

$$\underline{x}(t) = \frac{\hat{f}(\mathrm{i}-1)}{2\eta\sqrt{\alpha}m}\left[\mathrm{e}^{-\underline{v}^2}\int_{\underline{v}_0}^{\underline{v}}\mathrm{e}^{\underline{v}^2}\mathrm{d}\underline{v} + \mathrm{e}^{-\underline{u}^2}\int_{\underline{u}_0}^{\underline{u}}\mathrm{e}^{\underline{u}^2}\mathrm{d}\underline{u})\right]\mathrm{e}^{-\mathrm{i}\theta(t)}. \tag{9}$$

The complex integration limits $\underline{u}$ and $\underline{v}$ are defined as

$$\underline{u}(\tau) = \frac{1-\mathrm{i}}{2\sqrt{\alpha}}\left(\alpha\tau - \eta + \mathrm{i}\frac{\gamma}{2}\right), \tag{10}$$

and

$$\underline{v}(\tau) = \frac{\mathrm{i}-1}{2\sqrt{\alpha}}\left(\alpha\tau + \eta + \mathrm{i}\frac{\gamma}{2}\right), \tag{11}$$

where $\underline{u}_0 = \underline{u}(0)$ and $\underline{v}_0 = \underline{v}(0)$. By introduction of the FADEEVA function or complex error function

$$\underline{w}(\underline{z}) = e^{-\underline{z}^2}\left(1 + \frac{2\mathrm{i}}{\sqrt{\pi}}\int_0^{\underline{z}}\mathrm{e}^{\underline{z}^2}\mathrm{d}\underline{z}\right), \tag{12}$$

where $\underline{z}$ is an admissible complex number. The solution can be written as

$$\underline{x}(t) = \underbrace{\frac{\hat{f}(\mathrm{i}-1)}{4\eta m}\sqrt{\frac{\pi}{\alpha}}\left[\underline{w}(\underline{u}) + \underline{w}(\underline{v}) - \underline{w}(\underline{u}_0)\mathrm{e}^{\underline{u}_0^2-\underline{u}^2} - \underline{w}(\underline{v}_0)\mathrm{e}^{\underline{v}_0^2-\underline{v}^2}\right]}_{\underline{A}(t)}\mathrm{e}^{-\mathrm{i}\theta(t)}. \tag{13}$$

The expression $\underline{A}(t)$ does not depend on the angular difference ϕ and is the complex amplitude of the system.

2.3 Application to an MDOF-system with cyclic symmetry

Using the solution in Eq. 13 a cyclic symmetric system can be reduced to one cyclic sector. To show examplarily that the assumption is resonable the six degree of freedom system in Figure 4 is examined.

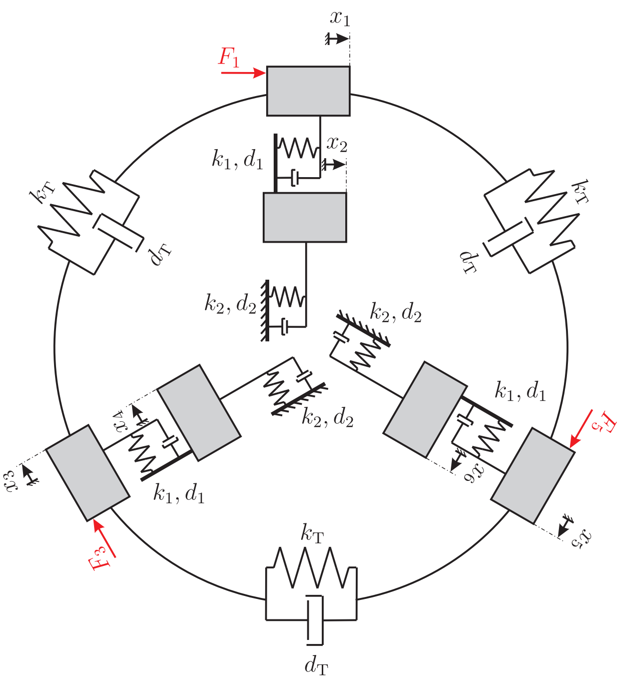

Figure 4: Six degree of freedom system

The system has three cyclic symmetric sectors. Thus, the phase difference is $\Delta\psi = \frac{2\pi}{3}$. The system can be described by Eq. 4 where the excitation is an accelerating travelling wave with

$$f(t) = \left[\hat{f}\mathrm{e}^{\mathrm{i}\left(\frac{1}{2}\alpha t^2\right)}, 0, \hat{f}\mathrm{e}^{\mathrm{i}\left(\frac{1}{2}\alpha t^2-\Delta\psi\right)}, 0, \hat{f}\mathrm{e}^{\mathrm{i}\left(\frac{1}{2}\alpha t^2+\Delta\psi\right)}, 0\right]^{\mathrm{T}}. \tag{14}$$

The matrices of the system are block symmetric and can be written as

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_2 \end{pmatrix}, \tag{15}$$

$$\mathbf{C} = \begin{pmatrix} d_1 + 2d_\mathrm{T} & -d_1 & -d_\mathrm{T} & 0 & -d_\mathrm{T} & 0 \\ -d_1 & d_1 + d_2 & 0 & 0 & 0 & 0 \\ -d_\mathrm{T} & 0 & d_1 + 2d_\mathrm{T} & -d_1 & -d_\mathrm{T} & 0 \\ 0 & 0 & -d_1 & d_1 + d_2 & 0 & 0 \\ -d_\mathrm{T} & 0 & -d_\mathrm{T} & 0 & d_1 + 2d_\mathrm{T} & -d_1 \\ 0 & 0 & 0 & 0 & -d_1 & d_1 + d_2 \end{pmatrix}. \tag{16}$$

and

$$\mathbf{K} = \begin{pmatrix} k_1 + 2k_\mathrm{T} & -k_1 & -k_\mathrm{T} & 0 & -k_\mathrm{T} & 0 \\ -k_1 & k_1 + k_2 & 0 & 0 & 0 & 0 \\ -k_\mathrm{T} & 0 & k_1 + 2k_\mathrm{T} & -k_1 & -k_\mathrm{T} & 0 \\ 0 & 0 & -k_1 & k_1 + k_2 & 0 & 0 \\ -k_\mathrm{T} & 0 & -k_\mathrm{T} & 0 & k_1 + 2k_\mathrm{T} & -k_1 \\ 0 & 0 & 0 & 0 & -k_1 & k_1 + k_2 \end{pmatrix} \tag{17}$$

Since the phase difference in Eq. 13 only appears in $\theta(t)$ and does not affect the amplitude it can be assumed that the vibration response can be expressed by only one cyclic sector as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \mathrm{e}^{+\mathrm{i}\Delta\psi} = \begin{pmatrix} x_5 \\ x_6 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\Delta\psi}. \tag{18}$$

By applying complex constraints to the damping and the stiffness matrix the system can be reduced with

$$\mathbf{C}_\mathrm{red} = \begin{pmatrix} d_2 + 2d_\mathrm{T} - d_\mathrm{T}\mathrm{e}^{\mathrm{i}+\Delta\psi} - d_\mathrm{T}\mathrm{e}^{\mathrm{i}-\Delta\psi} & -d_2 \\ -d_2 & d_1 + d_2 \end{pmatrix} \tag{19}$$

and

$$\mathbf{K}_\mathrm{red} = \begin{pmatrix} k_2 + 2k_\mathrm{T} - k_\mathrm{T}\mathrm{e}^{\mathrm{i}+\Delta\psi} - k_\mathrm{T}\mathrm{e}^{\mathrm{i}-\Delta\psi} & -k_2 \\ -k_2 & k_1 + k_2 \end{pmatrix}. \tag{20}$$

The remaining equation of motion has two degrees of freedom and can be decoupled according to a method of KRYLOV as shown in [3]. Therefor, the order of the differential equation is reduced to one and the resulting eigenvectors are used for the decoupling. Hereby two SDOF-systems are obtained. The resulting solution is calculated using Eq. 13.

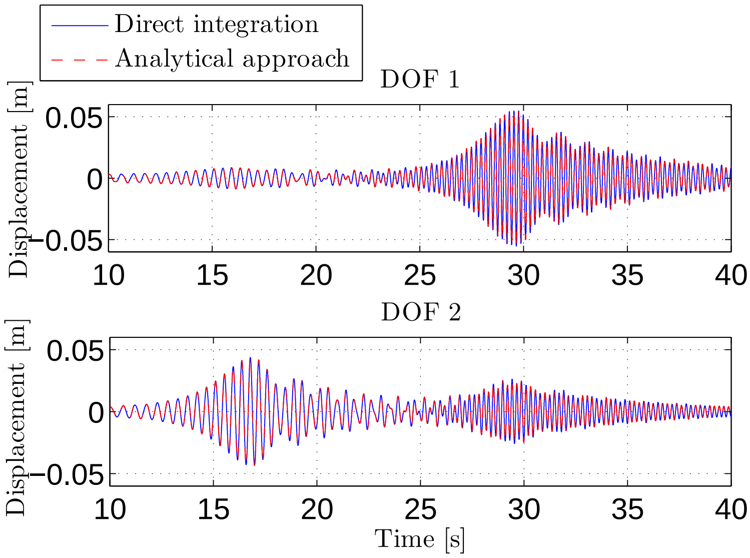

Figure 5: Results of the six degree of freedom system

3 RESULTS

To show that the calculated results are reasonable the whole model is computed using a time step integration. The data are compared to the results of the analytical solution of the cyclic sector with complex constraints, see Figure 5. The comparison shows an exact agreement.

The computation cost of the analytical approach is much lower than of the time-step-integration. The time step integration is using the RUNGE-KUTTA-FEHLBERG algorithm. The computation was done using Simulink. The algorithm to calculate the complex error function $w(z)$ is implemented from [11]. The analytical solution is computed in Matlab. The calculation time of the example above is given by Table 1.

Analytical approach	1.47 s
Direct integration	9.89 s

Table 1: Computation cost

A further advantage of the anlytical method is that the ampliutde can be calculated directly by the absolute value of the complex amplitude with

$$A(t) = |\underline{A}(t)|\,, \tag{21}$$

as shown in Figure 6. The size of the amplitude of the vibration response is the relevant information for most technical applications. While using the analytical approach the time discretization can be significantly reduced to calculate the amplitude. For a good approximation it is sufficient to compute one point per quasi period. In contrary to the direct integration each point can be calculated separately and does not depend on the step before. Hereby the calculated number of points is minimized.

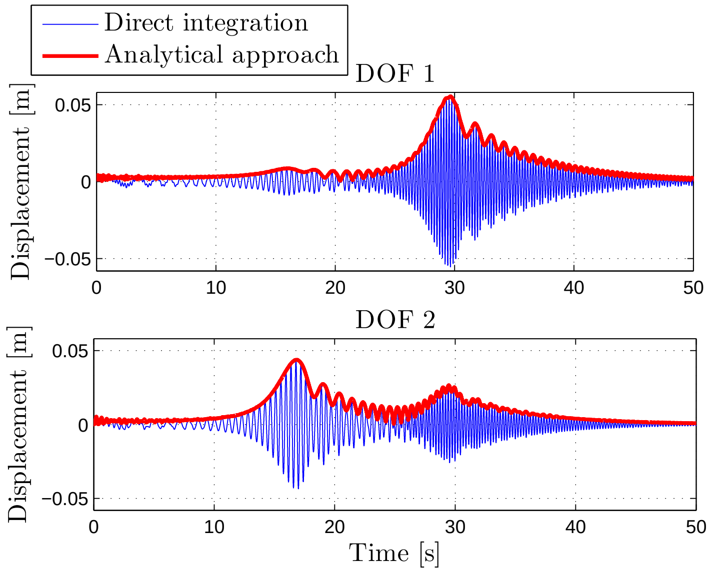

Figure 6: Calculation of the amplitude

4 CONCLUSIONS

While the number of run-ups and run-downs of turbines is rising the vibration response of the resonance passage becomes more important. Since turbines are cyclic symmetric structures it is shown that the assumption of cyclic symmetry can be extended to the excitation by accelerating travelling waves. The cyclic symmetric reduction and the analytical method to calculate resonance passage were successfully combined and the degrees of freedom of the system were reduced. The results are validated by time step integration. It could be shown that the computation cost were significantly reduced.

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