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IN-PLANE VIBRATIONS OF CURVED BEAMS WITH VARIABLE CROSS-SECTIONS CARRYING ADDITIONAL MASS

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Abstract. *Linear in-plane vibrations of curved beams are investigated using finite elements method. Curvature of beams is taken circular and the cross-sections are taken circular and rectangular. Uniform, unsymmetrical, and symmetrically tapered beams are considered and the natural frequencies are obtained for different boundary conditions. Elongation and bending effects are taken for in-plane vibrations. An additional mass on beam is also considered and its effect to natural frequencies is investigated for varying cross-sections.*

Especially, symmetrically tapered beams can be used as a carrier leg in industry. The load on beam can be a machine, a mechanical device or a motor. If the load produces a harmonic vibration, the frequency of it can be considered as a forcing frequency and if that frequency coincides the natural frequency of the beam resonance occurs. So, the natural frequencies should be known in order to avoid the high amplitudes and damages of the resonance.

In this work, the amount and the location of the additional load is changed and the effects on natural frequencies are investigated for symmetrically tapered beams which can be used as carrier legs. The changes of natural frequencies are determined and interpreted.

1 INTRODUCTION

Vibration analysis of curved beams is an important subject in mechanics due to its various applications. Especially curved beams with variable cross-sections have been widely used to satisfy modern architectural and structural requirements. They can be used as gears, pumps and turbines, ships, in horizontally curved continuous bridges or in the design of ribs, etc. There have been many studies about the curved beam vibrations. However, some of them are about the curved beams with variable cross-sections. Kawakami et al. [1] presented an approximate method to study the analysis for the in-plane and out-of-plane free vibrations of horizontally curved beams with arbitrary shapes and variable cross-sections. They indicate that the characteristic equation for free vibration can be derived by applying the Green function, which is obtained as a discrete type solution of differential equations governing the flexural behaviour of the curved beam under the action of a concentrated load. Krishnan and Suresh [2] investigated the effect of shear deformation and rotary inertia on natural end cross over frequencies of curved beams by using a simple cubic linear beam element. They analyzed both uniform and non-uniform (varied cross-section) beams. Free and forced in plane vibrations of circular arches with variable cross-sections are investigated by Tong et al. [3] using the Kirchoff assumptions for thin beams. In-plane free vibrations of circular arches are investigated by Liu and Wu [4] using the generalized differential quadrature rule (GDQR). Arches with uniform, continuously varying and stepped cross-sections are presented to illustrate the validity and accuracy of the GDQR. Karami and Malekzadeh [5] analyzed in plane free vibration of circular arches with varying cross-sections. By developing a differential quadrature method they aimed to obtain the higher order natural frequencies more accurately. Shin et al. [6] analyzed the vibration of a circular arch with variable cross-section using differential transformation and generalized differential quadrature. They applied the generalized differential quadrature method (GDQM) and differential transformation method (DTM) for vibration analysis. Yang et al. [7] investigated free in-plane vibration of uniform and non-uniform curved beams with variable curvatures, including the effects of the axis extensibility, shear deformation and rotary inertia by using extended-Hamilton principle. Firouz-Abadi et al. [8] presented an asymptotic solution to transverse free vibrations of variable-section beams by using Wentzel, Kramers, Brillouin (WKB) approximation. Ece et al. [9] studied vibration of an isotropic beam which has a variable cross-section. The results show that the non-uniformity in the cross-section influences the natural frequencies and mode shapes. Chen [10] developed the differential quadrature element method (DQEM) in-plane vibration analysis model of arbitrarily curved beam structures. Rafezy and Howson [11] investigated the vibration of doubly asymmetric, three-dimensional structures comprising wall and frame assemblies with variable cross-section. A boundary element method is developed by Sapountzakis [12] for the non-uniform torsional vibration problem of doubly symmetric composite bars of arbitrary variable cross-section. Tüfekçi and Doğruer [13] analyzed free out-of-plane vibrations of a circular arch with uniform cross-section by taking into effects of transverse shear and rotary inertia due to the both flexural and torsional vibrations by using the initial value method. Huang et al. [14] investigated out-of-plane dynamics of beams with arbitrarily varying curvature and cross-section by dynamic stiffness matrix method.

In this study, the linear free in-plane vibrations of uniform and variable cross-section beams are analyzed by Finite Element Method. The curvature of beams is circular and the cross-sections are taken circular and rectangular. The natural frequencies and mode shapes are obtained for different boundary conditions. An additional mass on beam is also considered and its effect to natural frequencies is investigated.

2 MODELLING AND GOVERNING EQUATIONS

Curved beam is modeled as a finite element as shown in Figure 1. *X, Y*, and *Z* are global coordinates, and u_c ,v_c, w_c are the tangential, radial and out-of-plane displacements for the curved beam respectively. Curved beam lies in *X-Y* plane, *s* is tangential coordinate and γ is the arch angle of one finite element.

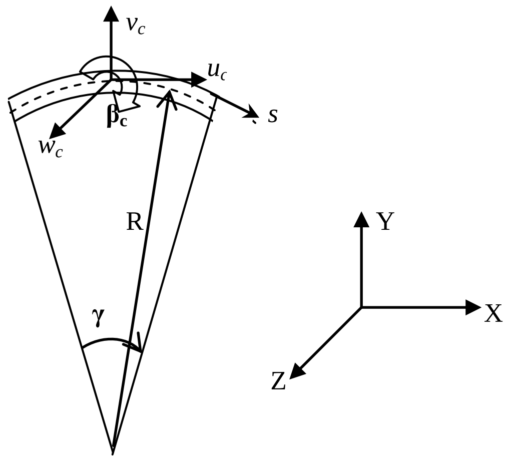


Figure 1: Curved beam element.

The in-plane elastic and kinetic energy equations of the curved beam can be expressed as follows

$$U_c = \frac{1}{2} E \int_s \left[A\varepsilon_c^2 + I\kappa_c^2 \right] ds \qquad T_c = \frac{1}{2} \rho \int_s \left[A(\dot{u}_c^2 + \dot{v}_c^2) + I\dot{\beta}_c^2 \right] ds \tag{1}$$

where *A* is the cross-sectional area, *E* is modulus of elasticity, *I* is mass moment of inertia. The beam is assumed as Bernoulli-Euler type, so the shear and rotary inertia effects are not taken into account. $(\dot{\ })$ denotes differentiation with respect to time *t*. In-plane strain, net cross-sectional rotation and curvature change terms are;

$$\varepsilon_c = \frac{\partial u_c}{\partial s} + \frac{v_c}{R} \quad , \quad \beta_c = \frac{\partial \mathrm{v_c}}{\partial \mathrm{s}} - \frac{u_c}{R} \quad , \quad \kappa_c = \frac{\partial \beta_c}{\partial s} = \frac{\partial^2 \mathrm{v}_c}{\partial s^2} - \frac{1}{R}\frac{\partial u_c}{\partial s} \tag{2}$$

Displacement vector for one finite element is taken as follows

$$[V]_{ce}^T = [u_{ci} \quad \alpha_{ci} \quad v_{ci} \quad \theta_{ci} \quad u_{ci+1} \quad \alpha_{ci+1} \quad v_{ci+1} \quad \theta_{ci+1}] \tag{3}$$

$$\text{where} \quad \alpha_{ci} = \partial u_{ci} / \partial s \quad \theta_{ci} = \partial v_{ci} / \partial s \quad \Psi_{ci} = \partial w_{ci} / \partial s \tag{4}$$

Four degrees of freedom are taken for each node of elements. By following the finite element procedure, the stiffness and inertia matrices are obtained for in-plane vibrations.

3 FREE VIBRATIONS

The total energy in the system is constant as follows

$$\{\dot{V}\}^T [M]\{\dot{V}\} + \{V\}^T [K]\{V\} = 0 \qquad (5)$$

where $\{V\}$ denotes global displacement vector, $[K]$ and $[M]$ are global stiffness and inertia matrices. Then, one obtains the eigenvalue equation giving the natural frequencies

$$\left| [K] - \omega_n^2 [M] \right| = 0 \qquad (6)$$

At this part, the in-plane free vibrations of uniform, unsymmetrical (Figure 2) and symmetrical (Figure 3) tapered curved beams are analyzed. The beams are rectengular cross-sectioned and circularly curved. The height of the cross-section is taken h_0 at the beginning, h_c at the crown, and h_1 at the end of the unsymmetric beam, while it is h_0 at both the beginning and the end, h_c at the crown of the symmetrically tapered beam. The width is taken constant. R is the radius of the curvature, and θ_0 is the arc angle. By using a MATLAB computer program which was performed by the authors, the dimensionless in-plane natural frequencies of uniform and tapered beams are obtained.

3.1 The frequencies of uniform and tapered beams

First five frequencies of uniform curved beams at different end conditions are shown in Table 1. The arc angle is 90^0.

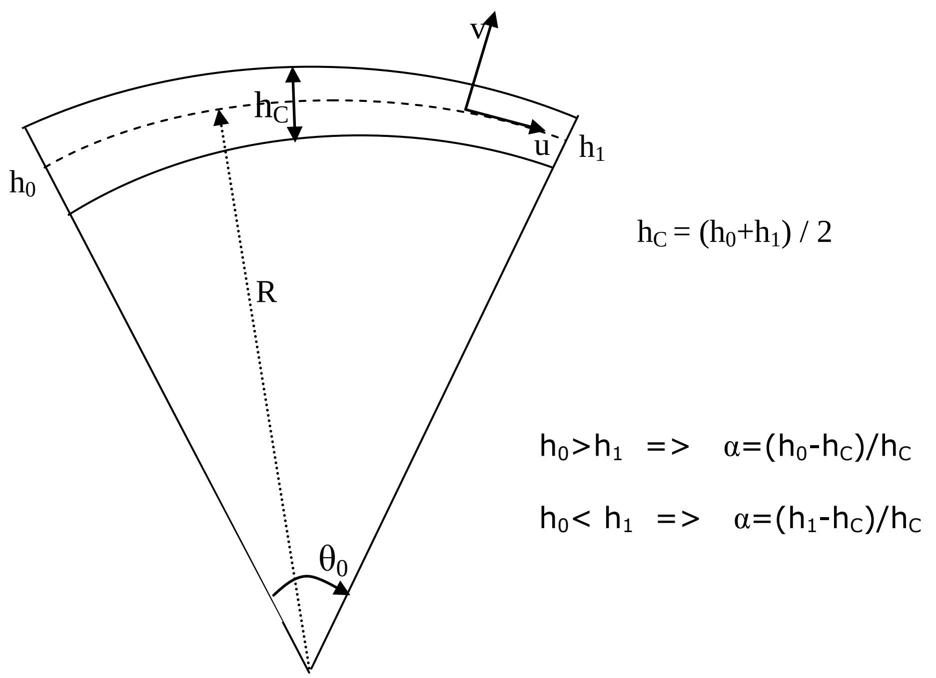


Figure 2: Unsymmetrical tapered beam.

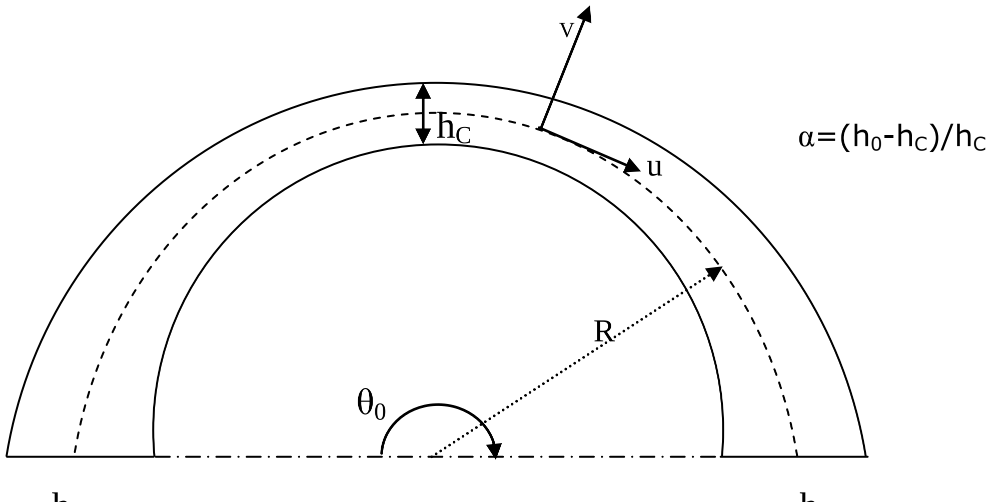


Figure 3: Symmetrical tapered beam.

	H-H			C-C		C-H	
Mode	Present	Ref (2)	Ref (5)	Present	Ref (2)	Present	Ref (2)
1	33.9600	33.93	33.958	55.8241	55.82	44.0921	44.05
2	79.9433	79.42	79.851	106.6959	104.28	92.9838	91.82
3	152.1632		152.14	193.0209		171.8175	
4	237.8999		237.06	284.6115		260.8579	
5	349.4972		349.39	409.7734		378.8652	

Table 1: Lowest frequency coefficients ($\Theta_o^2\ \lambda_1$) of a uniform circular arch. $\theta_0=90^0$

In Table 2, non-dimensional fundamental frequencies of a uniform circular curved beam are seen at different arch angles. The results are represented with the results of literature. The vibration of unsymmetric tapered beam is also analyzed. As seen in Figure 2, α is the taper parameter. In Table 3, the fundamental frequencies of unsymmetric tapered beam are presented for clamped-clamped and hinged-hinged boundary conditions. α is taken 0.1 and 0.2. The dimensionless frequencies are obtained with the general formula of

$$\chi_i=\omega_i\sqrt{(\rho A_c\ /\ EI_c)} \tag{7}$$

A_c is the rectangular cross-section's area of the curved beam at the crown and I_c is the moment of inertia of the beam at the crown as well. In Table 4 and Table 5, first two frequencies of unsymmetrical tapered beam are shown for 0.2 and 0.4 taper parameters. The end conditions are clamped-free, hinged-free, and free-free, respectively. Symmetrical tapered beam (Figure 3) is also analyzed for clamped-clamped and hinged-hinged end conditions. The results are presented at Table 6, at different arc angles. α is taken 0.1, 0.2 and 0.3. The results are good enough in comparison with the literature.

3.2 The additional mass

Symmetrically tapered beam can be used as a carrier leg in industry. The load on beam can be a machine, a mechanical device or a motor. If the load produces a harmonic vibration, the frequency of it can be considered as a forcing frequency and if that frequency coincides the natural frequency of the beam resonance occurs. So, the natural frequencies should be known in order to avoid the high amplitudes and damages of the resonance. Of course the amount and the location of the load effects the natural frequencies. These factors are analyzed at this part of this work.

θ_0	C-C Present	C-C Ref (4)	H-H Present	H-H Ref (4)	C-F Present	C-F Ref (4,5)	H-F Present	H-F Ref (4,5)	F-F Present	F-F Ref (4,5)
10^o	2021.9736	2021.9893	1293.4951	1293.5037	115.4952	115.4953	503.9852	503.9888	733.6396	733.6490
20^o	503.5457	503.5498	321.5126	321.5148	28.9274	28.9274	124.4229	124.4238	182.7995	182.8019
30^o	222.3699	222.3718	141.5320	141.5331	12.8964	12.8964	54.1992	54.1996	80.8016	80.8026
40^o	123.9753	123.9764	78.5574	78.5580	7.2857	7.2857	29.6902	29.6905	45.1131	45.1137
60^o	53.7396	53.7402	33.6258	33.6261	3.2784	3.2784	12.3433	12.3434	19.6498	19.6501
80^o	29.2172	29.2175	17.9639	17.9641	1.8763	1.8763	6.4318	6.4319	10.7728	10.7730
90^o	22.6248		13.7635	13.7637	1.4982	1.4982	4.8867	4.8868	8.3911	8.3912
100^o	17.9259	17.9262	10.7760	10.7761	1.2278	1.2279	3.8080	3.8080	6.6960	6.6961
120^o	11.8474	11.8476	6.9267	6.9268	0.8762	0.8762	2.4564	2.4564	4.5088	4.5088
140^o	8.2314	8.2315	4.6533	4.6534	0.6647	0.6647	1.6885	1.6885	3.2121	3.2122
160^o	5.9273	5.9274	3.2179	3.2179	0.5282	0.5282	1.2202	1.2202	2.3882	2.3882
180^o	4.3844	4.3844	2.2667	2.2667	0.4352	0.4352	0.9188	0.9188	1.8372	1.8372

Table 2: Non-dimensional fundamental frequencies of a uniform circular curved beam under different boundary conditions.

	$\alpha = 0.1$						$\alpha = 0.3$			
θ_0	C-C Present	C-C Ref (4,5)	C-C Ref(6)	H-H Present	H-H Ref (4,5)	H-H Ref (6)	C-C Present	C-C Ref (4,5)	H-H Present	H-H Ref (4,5)
10^o	2015.9664	2016.983	2017.0	1289.0582	1290.485	1290.5	1972.9043	1975.996	1263.8007	1265.767
20^o	502.0516	502.3033	502.30	320.6068	320.7631	320.76	491.3307	492.0978	314.1095	314.6084
30^o	221.7102	221.8216	221.82	141.1266	141.2012	141.20	216.9793	217.3162	138.2680	138.4839
40^o	123.6069	123.6698	123.67	78.3355	78.3731	78.373	120.9698	121.1592	76.7392	76.8588
50^o	78.2186	78.2575	78.258	49.2871	49.3123	49.312	76.5497	76.6699	48.2787	48.3545
60^o	53.5804	53.6075	53.607	33.5288	33.5461	33.546	52.4386	52.5209	32.8387	32.8904
80^o	29.1308	29.1456		17.9114	17.9206		28.5118	28.5564	17.5356	17.5649
90^o	22.5581			13.7232			22.0797		13.4332	
120^o	11.8129			6.9059			11.5643		6.7560	
180^o	4.3720			2.2595			4.2842		2.2058	

Table 3: Non-dimensional fundamental frequencies of unsymmetric and circular curved beam under different boundary conditions.

θ_0	C-F Mode 1 Present	C-F Mode 1 Ref (4)	C-F Mode 2 Present	C-F Mode 2 Ref (4)	H-F Mode 1 Present	H-F Mode 1 Ref (4)	H-F Mode 2 Present	H-F Mode 2 Ref (4)	F-F Mode 1 Present	F-F Mode 1 Ref (4)	F-F Mode 2 Present	F-F Mode 2 Ref (4)
10^o	88.4052	88.4809	660.2410	660.9595	472.4727	472.9768	1598.8379	1600.4856	731.9557	732.6887	2009.7222	2011.7617
20^o	22.1600	22.1631	163.0379	163.2150	116.3778	116.5046	397.8714	398.2820	182.3894	182.5740	501.6036	502.1133
40^o	5.5759	5.5839	38.9308	38.9737	27.5538	27.5837	97.7553	97.8563	45.0227	45.0682	124.5819	123.7981
60^o	2.5092	2.5140	16.1928	16.2108	11.3417	11.3540	42.3218	42.3656	19.6177	19.6259	54.7742	54.4349
80^o	1.4356	1.4399	8.4366	8.4461	5.8519	5.8585	23.0277	23.0516	10.7601	10.7708	30.3543	30.1703
120^o	0.6741		3.2139		2.2001		9.4094		4.5081		12.9454	
180^o	0.3360		1.1906		0.8111		3.5455		1.8395		5.2788	

Table 4: Frequencies of first two modes of an unsymmetrical and circular curved beam under different boundary conditions. (α = 0.2)

θ_0	C-F Mode 1 Present	C-F Mode 1 Ref (4)	C-F Mode 2 Present	C-F Mode 2 Ref (4)	H-F Mode 1 Present	H-F Mode 1 Ref (4)	H-F Mode 2 Present	H-F Mode 2 Ref (4)	F-F Mode 1 Present	F-F Mode 1 Ref (4)	F-F Mode 2 Present	F-F Mode 2 Ref (4)
10^o	63.4309	63.6881	587.8368	589.2368	433.2210	434.2458	1535.2312	1538.5761	731.9557	732.6887	2009.7222	2011.7617
20^o	15.8938	15.9544	144.8012	145.1512	106.4238	106.6792	381.7646	382.5978	182.3894	182.5740	501.6036	502.1133
40^o	4.0057	4.0211	34.2849	34.3693	24.9676	25.0294	93.5785	93.7838	45.0227	45.0682	124.5819	123.7981
60^o	1.8077	1.8116	14.1085	14.1441	10.1639	10.1892	40.4061	40.4953	19.6177	19.6259	54.7742	54.4349
80^o	1.0344	1.0385	7.2743	7.2932	5.1893	5.2028	21.9311	21.9798	10.7601	10.7708	30.3543	30.1703
120^o	0.4846		2.7270		1.9204		8.9215		4.5081		12.9454	
180^o	0.2447		0.9979		0.6984		3.3379		1.8395		5.2788	

Table 5: Frequencies of first two modes of an unsymmetrical and circular curved beam under different boundary conditions. (α = 0.4)

θ_0	α = 0.1 C-C Present	α = 0.1 C-C Ref (4)	α = 0.1 H-H Present	α = 0.1 H-H Ref (4)	α = 0.2 C-C Present	α = 0.2 C-C Ref (4)	α = 0.2 H-H Present	α = 0.2 H-H Ref (4)	α = 0.3 C-C Present	α = 0.3 C-C Ref (4)	α = 0.3 H-H Present	α = 0.3 H-H Ref (4)
10^o	2149.7072	2149.7593	1357.2028	1357.2106	2275.3034	2275.3957	1419.0421	1419.0535	2399.0299	2399.1619	1479.2441	1479.2608
20^o	535.4369	535.4500	337.3869	337.3889	566.7961	566.8193	352.7956	352.7985	597.6898	597.7229	367.7964	367.8006
30^o	236.5123	236.5183	148.5481	148.5490	250.4200	250.4304	155.3584	155.3598	264.1223	264.1371	161.9885	161.9904
40^o	131.9054	131.9088	82.4730	82.4735	139.7048	139.7107	86.2737	86.2745	147.3900	147.3984	89.9738	89.9750
50^o	83.5051	83.5073	51.9091	51.9095	88.4771	88.4809	54.3167	54.3172	93.3769	93.3824	56.6606	56.6613
90^o	24.1437		14.4791		25.6396		15.1739		27.1153		15.8505	
180^o	4.7210		2.3948		5.0545		2.5195		5.3851		2.6411	

Table 6: Non-dimensional fundamental frequencies of a symmetrical tapered curved beam for different α values.

As the first application, the load is considered as a point mass at the crown of the half circle symmetrically tapered beam (Figure 4). The first five in-plane natural frequencies of different amount of additional mass are shown in Table 7 for different boundary conditions. The beam is divided into 150 finite elements. The mid-point is the right side of the 75th element. So the numbers lie from 0 to 150 from left to right as well.

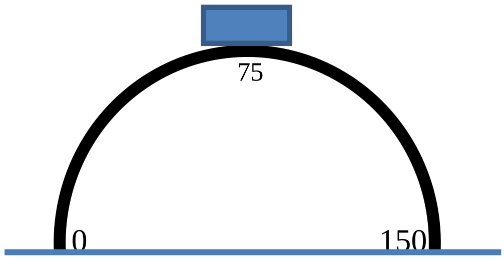


Figure 4: Additional mass is at the crown (mid-point)

Boundary Condition	Mode	Amount of the Additional Mass				
		No mass	**50 kg**	**100 kg**	**150 kg**	**200 kg**
C-C	1	5.0546	4.8388	4.6485	4.4789	4.3266
	2	10.8698	9.6371	8.7233	8.0186	7.4565
	3	19.9084	19.9019	19.8965	19.8918	19.8878
	4	30.3660	28.8794	27.9844	27.4014	26.9957
	5	43.8876	43.6700	43.4888	43.3355	43.2042
H-H	1	2.5195	2.4126	2.3182	2.2341	2.1585
	2	7.6897	6.9472	6.3731	5.9156	5.5414
	3	15.4044	15.3790	15.3577	15.3395	15.3237
	4	25.0495	23.6614	22.7863	22.1972	21.7777
	5	37.2827	37.1249	36.9936	36.8827	36.7878

Table 7: The dimensionless in-plane natural frequencies of half circled tapered beam ($\alpha = 0.2$)

As it's seen in the tables, the amount of additional mass slightly reduces the natural frequencies. But if we look at Figure 5 which shows the changes of Mode 1 and Mode 2 of the clamped-clamped beam, the distance between the frequencies decreases due to the amount of additional mass.

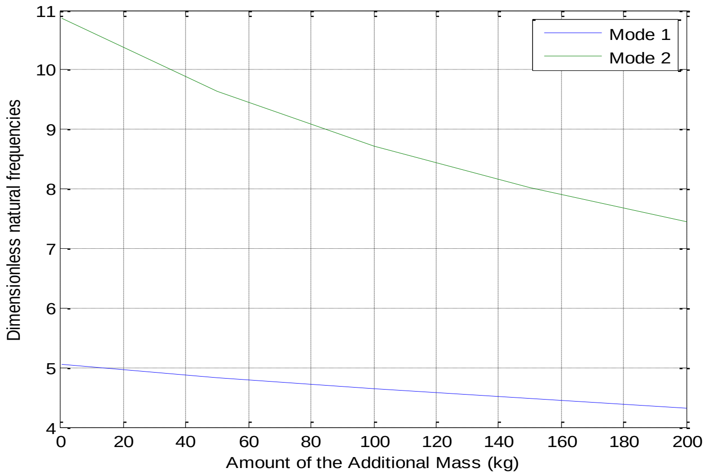


Figure 5: The changes of natural frequencies of first two modes (C - C)

In Table 8, the natural frequencies of in-plane vibrations are investigated for the mass is at crown (Position 1) and as another application, the mass is divided into two symmetric points.

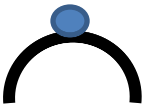
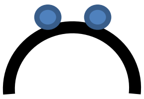
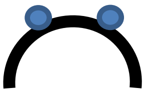
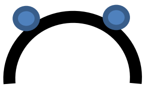
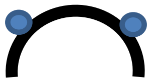
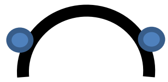
Mode	Position 1	Position 2	Position 3	Position 4	Position 5	Position 6
	Mid-point (crown) 	70th-80th node 	60th-90th node 	50th-100th node 	40th-110th node 	30th-120th node 
1	4.7865	4.7558	4.5848	4.4904	4.6025	4.8948
2	8.4526	8.6856	9.9943	11.0010	10.4653	10.1418
3	20.8586	19.6894	17.0212	18.3488	20.1304	18.6387
4	28.6019	29.6699	30.7484	24.8215	27.8153	30.7533
5	45.3085	41.2372	42.0384	44.7369	38.4451	44.6201
Additional mass: 150 kg - α=0.3						

Table 8: The in-plane natural frequencies of the beam with different mass locations (C-C)

If we assume that the beam is a carrier leg and the mass is a motor or any device which causes vibration, the natural frequencies of system should be known to avoid resonance. First and second modes are more important for the system because it is easier for the forcing frequency to coincide the first and second natural frequencies more than the further modes. So, as seen at Table 8, locating the half masses at different nodes changes the natural frequencies. At this point, we determine that the gap between 1^{st} and 2^{nd} modes is wider than the others at Position 4 (Figure 7). So this location is more convenient than the others in order to avoid resonance.

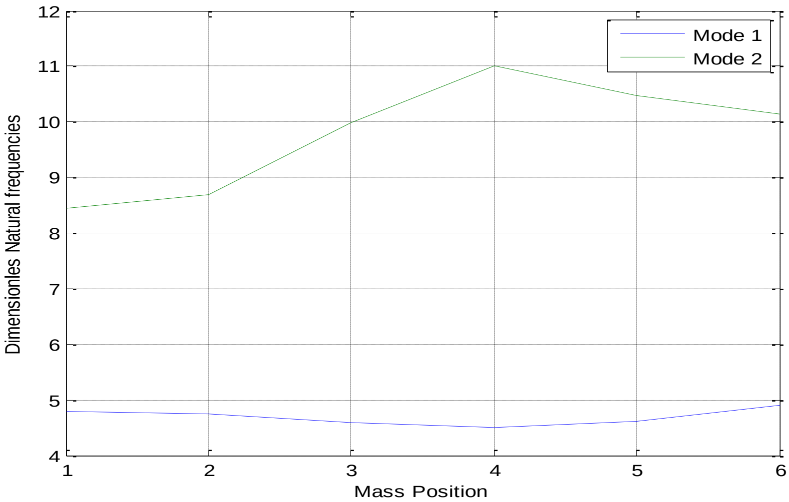


Figure 7. Natural Frequencies due to the mass location (C - C) , α=0.3

As making calculations, similar situation is observed for hinged-hinged boundary condition. Position 4 is still more convenient to avoid the resonance. But the results are quite different for Clamped-Sliding beam system. The gap between first two frequencies makes minimum at Position 1 (mass at the crown) which seems the best location at this case. (Table 9)

Mode	Position 1	Position 2	Position 3	Position 4	Position 5	Position 6
1	1.3398	1.3423	1.3591	1.3822	1.3976	1.3932
2	4.8963	4.8626	4.6643	4.5480	4.6677	4.9783
3	9.7629	9.9836	11.1339	11.6540	10.7891	10.6117
4	21.0054	19.8949	17.5057	18.8982	20.4925	18.8599
5	28.6733	29.8406	31.6007	26.4699	29.6316	31.2107
Additional mass: 150 kg - α=0.3						

Table 9: The in-plane natural frequencies of the beam with different mass locations (C - S)

4 CONCLUSIONS

- The uniform and tapered curved beams are analyzed. Free in-plane natural frequencies are obtained.
- Symmetrical and unsymmetrical tapered beam vibrations are examined separately and the results of different boundary conditions are compared with the literature.
- As a new approach, symmetrical tapered beam is assumed as a carrier leg and an additional load is located on the beam. The load can be supposed as a device or a motor which can produce vibration. The forcing frequency can coincide with the natural frequency of the system. So, this situation is resonance which can cause damages on the system because of high amplitudes. So, for that reason, it is important to make vibration analysis on the system.
- The analysis is made by changing the amount and the location of the mass. The changes of natural frequencies are investigated and interpreted. The most convenient mass positions are determined to avoid resonance.

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