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NATURAL VIBRATIONS AND STABILITY OF SHELLS WITH ARBITRARY CROSS-SECTION, CONTAINING FLOWING FLUID

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Abstract. *The focus of the paper is on the numerical analysis of natural vibrations and stability of cylindrical shells with arbitrary cross-sections, completely or partially filled with a quiescent or flowing fluid. A three-dimensional problem is considered and solved by the finite element method. The motion of the shell is described by applying the variational principle of virtual displacements, which incorporates the linearized Bernoulli equation for evaluating the hydrodynamic pressure. A compressible inviscid fluid is considered in the framework of the potential theory. The governing equations are converted using the Bubnov–Galerkin method. To demonstrate the possibilities of three-dimensional numerical simulation two problems on the natural vibrations and stability of elliptical cylindrical shells are investigated.*

1 INTRODUCTION

Over the last few decades the interaction of cylindrical (mainly circular) shells with a fluid has been the focus of extensive investigation made with the use of different approaches including simulations by the numerical methods [1]. Some studies have been carried out based on the finite-element simulation or using a combined approach, in which the structure displacements are determined by the finite-element method while the fluid behavior is investigated by the method of boundary integral equations. A short review of the papers realizing different numerical approaches is given in [2]. With the exception of paper [3] the proposed solutions are two-dimensional. Most of them are written in the framework of axisymmetric problem formulations, which restricts the class of examined problems to the bodies of revolution. Although in practice the axisymmetric shells are the most frequently occurring structures, they are subjected to 3D loads and therefore their interaction with fluids may give rise to different types of spatial vibration modes and instability. The analytical solutions for such systems are rather complicated and their number is limited. In this paper, the dynamic behavior of cylindrical shells containing a quiescent or flowing fluid is investigated based on the results of three-dimensional finite-element simulation. As an example two problems have been investigated numerically to estimate the influence of the fill level on the natural frequencies, vibration modes and stability of thinwalled elliptical cylindrical shells.

2 MATHEMATICAL AND NUMERICAL FORMULATIONS OF THE PROBLEM

Let us consider horizontally oriented, thin wall noncircular (ellipsoidal) cylindrical shells with semi-axes R_z and R_y, length L and thickness h, interacting with the internal steady-state flow of compressible fluid (Figure 1). The deformations arising in the shell under hydrodynamic action are considered to be small. The influence of the boundary layer and dynamic effects on the free surface in the case of structures partially filled with liquid are neglected.

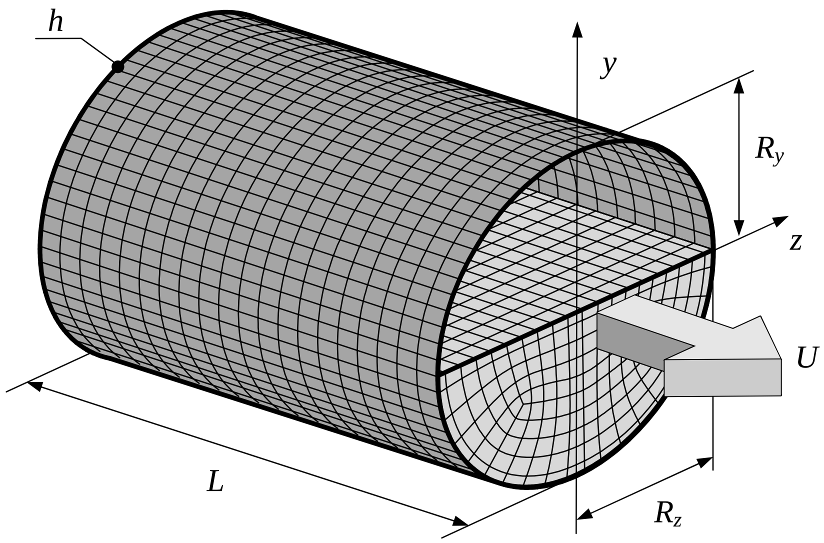

Figure 1: Scheme and an example of a finite-element discretization.

A potential flow of liquid medium occupying volume V_f, in case of small perturbations is described by the wave equation

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{2U}{c^2}\frac{\partial^2\phi}{\partial t\partial x} + M^2\frac{\partial^2\phi}{\partial x^2}, \tag{1}$$

where U is the flow velocity, c is the speed of sound in the medium, $M = U/c$ is the Mach number, (x, y, z) are the Cartesian coordinates.

The perturbation velocity potential at the inlet and outlet of the structure obey the following boundary conditions:

$$x = 0: \ \phi = 0; \qquad x = L: \ \partial\phi/\partial x = 0. \tag{2}$$

The impermeable boundary condition applied to the fluid-shell interface $S_\sigma = S_f \cap S_s$ is written as

$$\frac{\partial\phi}{\partial n} = \left(\frac{\partial\boldsymbol{\delta}}{\partial t} + U\frac{\partial\boldsymbol{\delta}}{\partial x}\right)\cdot\mathbf{n}, \tag{3}$$

where S_f is the bounding surface of the fluid volume V_f; S_s is the shell surface; $\mathbf{n}$ is normal vector to the surface S_σ; $\boldsymbol{\delta} = \left\{\begin{array}{ccc} u & v & w \end{array}\right\}^{\mathrm{T}}$ is the vector of shell displacements; u, v and w are the meridional, circumferential and normal components of the displacement vector.

Using the Bubnov–Galerkin method, we reduce a partial differential equation for the perturbation velocity potential Eq. (1) together with boundary conditions Eq. (2) and impermeability condition Eq. (3) to the system of equations, which can be written in matrix form as

$$\left(\mathbf{K}_f + \mathbf{A}_f^c\right)\boldsymbol{\phi} + \mathbf{M}_f\ddot{\boldsymbol{\phi}} - \mathbf{C}_f^c\dot{\boldsymbol{\phi}} - \mathbf{C}_f\dot{\boldsymbol{w}} - \mathbf{A}_f\boldsymbol{w} = 0, \tag{4}$$

where

$$\mathbf{K}_f = \sum_{m_f}\int\limits_{V_f}\left(\frac{\partial\mathbf{F}^{\mathrm{T}}}{\partial x}\frac{\partial\mathbf{F}}{\partial x} + \frac{\partial\mathbf{F}^{\mathrm{T}}}{\partial y}\frac{\partial\mathbf{F}}{\partial y} + \frac{\partial\mathbf{F}^{\mathrm{T}}}{\partial z}\frac{\partial\mathbf{F}}{\partial z}\right)dV, \quad \mathbf{M}_f = \sum_{m_f}\int\limits_{V_f}\frac{1}{c^2}\mathbf{F}^{\mathrm{T}}\mathbf{F}dV,$$

$$\mathbf{A}_f = \sum_{m_\sigma}\int\limits_{S_\sigma}U\mathbf{F}^{\mathrm{T}}\frac{\partial\bar{\mathbf{N}}}{\partial x}dS, \quad \mathbf{A}_f^c = -\sum_{m_f}\int\limits_{V_f}M^2\frac{\partial\mathbf{F}^{\mathrm{T}}}{\partial x}\frac{\partial\mathbf{F}}{\partial x}dV,$$

$$\mathbf{C}_f = \sum_{m_\sigma}\int\limits_{S_\sigma}\mathbf{F}^{\mathrm{T}}\bar{\mathbf{N}}dS, \quad \mathbf{C}_f^c = -\sum_{m_f}\int\limits_{V_f}\frac{2U}{c^2}\frac{\partial\mathbf{F}^{\mathrm{T}}}{\partial x}\mathbf{F}dV.$$

Here m_f, m_σ is the number of finite elements used to decompose the fluid domain V_f and surface domain S_σ, respectively; $\mathbf{F}$, $\bar{\mathbf{N}}$ are the shape functions for the perturbation velocity potential and the normal component of the vector of shell displacements.

Simulation of the shell of arbitrary geometry is carried out on the assumption that its curvilinear surface is approximated to sufficient accuracy by a set of plane tetragonal elements. The strains are calculated using the relations of the Kirchhoff–Love's thin shells theory. A mathematical formulation of the dynamic shell problem has been developed by applying the variational principle of virtual displacements, which incorporates the equation of hydrodynamic pressure exerted on the structure by the fluid and the work done by the inertial forces. It can be expressed in the matrix form as follows

$$\int\limits_{S_s} \delta\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{T}dS + \int\limits_{V_s} \rho_s \delta\boldsymbol{d}^{\mathrm{T}}\ddot{\boldsymbol{d}}dV - \int\limits_{S_\sigma} \delta\boldsymbol{d}^{\mathrm{T}}\boldsymbol{P}dS = 0, \quad \boldsymbol{P} = \left\{ \begin{array}{cccccc} 0 & 0 & p & 0 & 0 & 0 \end{array} \right\}^{\mathrm{T}}, \tag{5}$$

$$p = -\rho_f \left(\frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial x} \right). \tag{6}$$

Here: $\boldsymbol{\varepsilon}$, $\boldsymbol{T}$, $\boldsymbol{d}$, $\boldsymbol{P}$ are the vectors of the generalized strains, forces and moments, displacements and surface loads, respectively; ρ_s is the density of the shell material.

With the use of the procedure of the finite element method and expression for pressure Eq. (6), equation (5) is reduced to the following relation

$$\mathbf{K}_s\boldsymbol{d} + \mathbf{M}_s\ddot{\boldsymbol{d}} + \rho_f\mathbf{C}_s\dot{\boldsymbol{\phi}} + \rho_f\mathbf{A}_s\boldsymbol{\phi} = 0. \tag{7}$$

where

$$\mathbf{K}_s = \sum_{m_s}\int\limits_{S_s} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}dS, \qquad \mathbf{M}_s = \sum_{m_s}\int\limits_{V_s} \rho_s\mathbf{N}^{\mathrm{T}}\mathbf{N}dV,$$
$$\mathbf{C}_s = \sum_{m_\sigma}\int\limits_{S_\sigma} \mathbf{F}^{\mathrm{T}}\bar{\mathbf{N}}dS, \qquad \mathbf{A}_s = \sum_{m_\sigma}\int\limits_{S_\sigma} U\bar{\mathbf{N}}^{\mathrm{T}}\frac{\partial\mathbf{F}}{\partial x}dS.$$

Here m_s is the number of finite elements used to decompose the shell surface; $\mathbf{N}$ is the shape function of the finite element; $\mathbf{B}$ is the matrix relating the strains ε to the nodal displacements; $\mathbf{D}$ is the matrix of elastic constants for an isotropic material.

The vector of nodal displacements of the shell element has six components

$$\boldsymbol{\delta}_l = \left\{ \begin{array}{cccccc} u_l & v_l & w_l & \theta_{x_l} & \theta_{y_l} & \theta_{z_l} \end{array} \right\}^{\mathrm{T}}, \qquad l = \overline{1,4},$$

where $\theta_x, \theta_y, \theta_z$ are the angular displacements with respect to the corresponding axis of the coordinate system on the lateral surface of the shell. For approximation of membrane displacements we use the linear relations and for approximation of bending displacements we use non-conforming shape functions [4].

To solve the problem of dynamic behavior of a shell conveying fluid, it is necessary to solve simultaneously two systems of equations (4) and (7)

$$\mathbf{K}\left\{ \begin{array}{cc} \boldsymbol{d} & \boldsymbol{\phi} \end{array} \right\}^{\mathrm{T}} + \mathbf{M}\left\{ \begin{array}{cc} \ddot{\boldsymbol{d}} & \ddot{\boldsymbol{\phi}} \end{array} \right\}^{\mathrm{T}} + \mathbf{C}\left\{ \begin{array}{cc} \dot{\boldsymbol{d}} & \dot{\boldsymbol{\phi}} \end{array} \right\}^{\mathrm{T}} + \mathbf{A}\left\{ \begin{array}{cc} \boldsymbol{d} & \boldsymbol{\phi} \end{array} \right\}^{\mathrm{T}} = 0,$$

where

$$\mathbf{K} = \left[\begin{array}{cc} \mathbf{K}_s & 0 \\ 0 & \mathbf{K}_f \end{array} \right], \quad \mathbf{M} = \left[\begin{array}{cc} \mathbf{M}_s & 0 \\ 0 & \mathbf{M}_f \end{array} \right],$$
$$\mathbf{C} = \left[\begin{array}{cc} 0 & \rho_f\mathbf{C}_s \\ -\mathbf{C}_f & -\mathbf{C}_f^c \end{array} \right], \quad \mathbf{A} = \left[\begin{array}{cc} 0 & \rho_f\mathbf{A}_s \\ -\mathbf{A}_f & \mathbf{A}_f^c \end{array} \right].$$

Here $\mathbf{K}$ is the stiffness matrix, $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the hydrodynamic damping matrix and $\mathbf{A}$ is the hydrodynamic stiffness matrix.

Representing the perturbed motion of the shell and the fluid as $\boldsymbol{d} = \boldsymbol{q}\exp(i^*\omega t)$, $\boldsymbol{\phi} = \boldsymbol{f}\exp(i^*\omega t)$, where $\boldsymbol{q}$, $\boldsymbol{f}$ are some functions of coordinates, $i^* = \sqrt{-1}$, and $\omega = \lambda_1 + i^*\lambda_2$ is the characteristic quantity, we finally arrive at

$$\left(\mathbf{K} - \omega^2\mathbf{M} + i^*\omega\mathbf{C} + \mathbf{A}\right)\left\{\begin{array}{cc} \boldsymbol{q} & \boldsymbol{f} \end{array}\right\}^{\mathrm{T}} = 0. \tag{8}$$

The resulting system of equations (8) can be reduced to

$$\left[\begin{array}{cc} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\left(\mathbf{K}+\mathbf{A}\right) & -\mathbf{M}^{-1}\mathbf{C} \end{array}\right]\boldsymbol{x} = \omega\boldsymbol{x}, \tag{9}$$

where $\mathbf{I}$ is the unit matrix, $\boldsymbol{x} = \left\{\boldsymbol{q}, \boldsymbol{f}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{f}}\right\}^{\mathrm{T}}$. To increase accuracy of calculations, the solution to the eigenvalue problem Eq. (9) is found by making use of the Lanczos algorithm for sparse matrices.

3 NUMERICAL RESULTS

The developed finite-element algorithm was used to estimate the influence of the filling level, the ratio of the ellipse semi-axes $\alpha = R_z/R_y$ and the ratio of the shell length L to its radius R_y on the natural frequencies, vibration modes and boundary of hydroelastic stability of the elliptic cylindrical shells containing a quiescent or a flowing fluid. In all numerical computations the shell surface was approximated by 1440 elements (72 along the circumference and 20 along its length). The coupled system in the case when the shell is completely filled with the fluid has about 15000 degrees of freedom.

3.1 Natural vibrations of shells with a quiescent fluid

The first example is the problem of natural vibrations of the horizontal elliptical cylindrical shell clamped at both ends and containing a motionless fluid (elasticity modulus $E = 2.05\times10^{11}$ N/m^2; Poisson's ratio $\nu = 0.3$; $\rho_s = 7800$ kg/m^3; $R_y = 0.07725$ m; $h = 1.5\times10^{-3}$ m; $L/R_y = 2.99$, $\rho_f = 1000$ kg/m^3, $c = 1500$ m/s). The numerical experiments have shown that even a small amount of fluid in such configurations leads to an essential decrease in the lowest vibration frequency of the system and reaches 60% for the minimum fill level $k = 0.25$ (Figure 2,*a*). Here the value of $k = V_f/V_i$, where V_i is the volume of the shell interior, specifies the level of shell filling with a fluid.

Compared to circular ($\alpha = 1$) shells, the low natural frequencies of shells with elliptical cross sections at $\alpha < 1$ monotonically decrease over the entire range of the parameter k and at $\alpha > 1$ acquire the asymptotic character. This means that starting with a certain fill level they become insensitive to the level growth (Figure 2,*a*) In the analysis of the effect of the ellipticity parameter, the size of the vertical semi-axis R_y is assumed invariable. In this case, the variation of the ellipticity parameter is related to a monotonic change in the cross sectional area of the elliptical shell and fluid level at a fixed value of k. However, the dependence of the minimal natural frequency of vibrations on the ellipticity parameter α is of non-monotonic character (Figure 2,*b*) and reaches the maximum value at $\alpha < 1$.

The vibration modes corresponding to low frequencies of partially filled horizontal elliptical shells have one distinguishing feature. Irrespective of the fluid level the maximum half-wave height along the circumference is always observed on the part of the lateral shell surface that interacts with the fluid (Figure 3) whereas the displacements of "dry" surface are much less. As

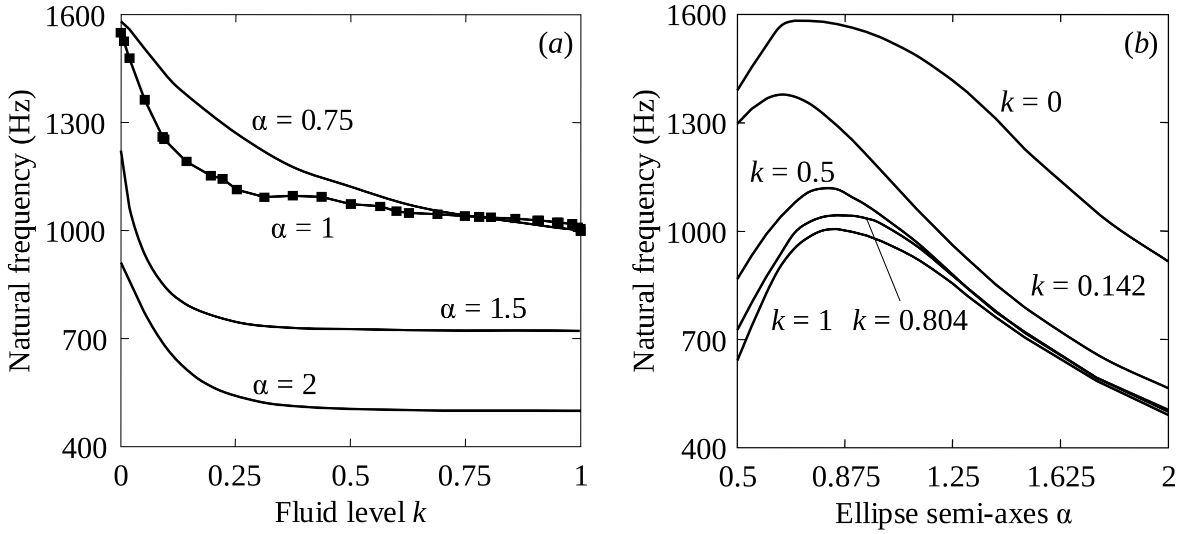

Figure 2: The low natural frequencies of clamped-clamped elliptical shells for (*a*) different fluid levels and (*b*) different ratio of ellipse semi-axes.

a rule, such specific behavior is not common to high frequency modes but is frequently observed in circular cylindrical shells as was shown in papers [5, 6].

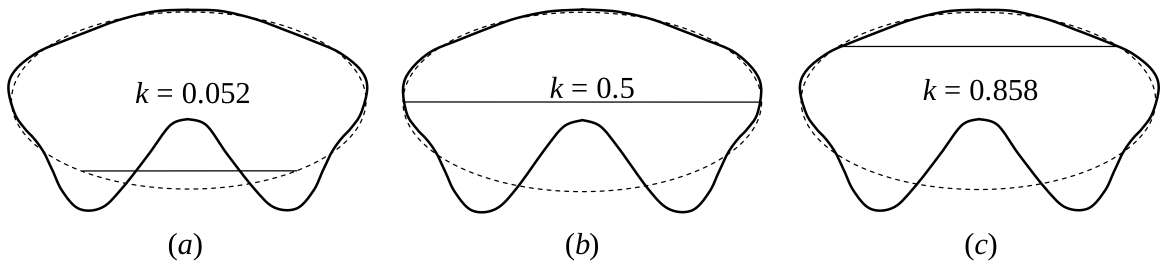

Figure 3: The vibration modes of clamped-clamped partially filled elliptical shell, $\alpha = 2$: (*a*) — $\omega_1 = 673.4$ Hz; (*b*) — $\omega_1 = 505.2$ Hz; (*c*) — $\omega_1 = 500.1$ Hz.

3.2 Stability of elliptical shells, containing flowing fluid

As the second example we consider an elliptical cylindrical shell ($E = 2.06 \times 10^{11}$ N/m^2, $\nu = 0.3$, $\rho_s = 7850$ kg/m^3, $h = 2 \times 10^{-3}$ m, $L/R_y = 2$, $R_y = 0.2$ m) simply supported at both ends ($x = 0, L$: $v = w = \theta_y = \theta_z = 0$) and interacting with the internal fluid flow. The analysis of the obtained results is performed using the dimensionless velocity $\Lambda = U/\xi$, where $\xi = [E/\rho_s\,(1-\nu^2)]^{0.5}$. It is known [1] that under the prescribed boundary conditions, an increase in the flow velocity may lead to a loss of shell stability by divergence. In this case the minimum natural frequency of the system decreases until it becomes zero at $\Lambda = \Lambda_D$. Moreover, this scenario is associated with the appearance of two imaginary parts of this mode, one of them being negative.

From the results displayed in Figure 4 it follows that the hydroelestic stability boundary for elliptical cylindrical shells essentially depends on the linear dimensions of the structure. For example, an increase in the length of the shell completely filled with a fluid results in a decrease of the critical velocities corresponding to the divergence instability Λ_D (Figure 4,*a*). It should be noted that starting with $L/R \geq 6$ there is a qualitative and quantitative difference in the critical (instability -induced) velocities between the shells with elliptical and circular ($\alpha = 1$) cross-sections. This discrepancy is partially caused by the difference in the fluid volumes but for the most part is due to geometrical peculiarities of the structures having non-circular cross-sections.

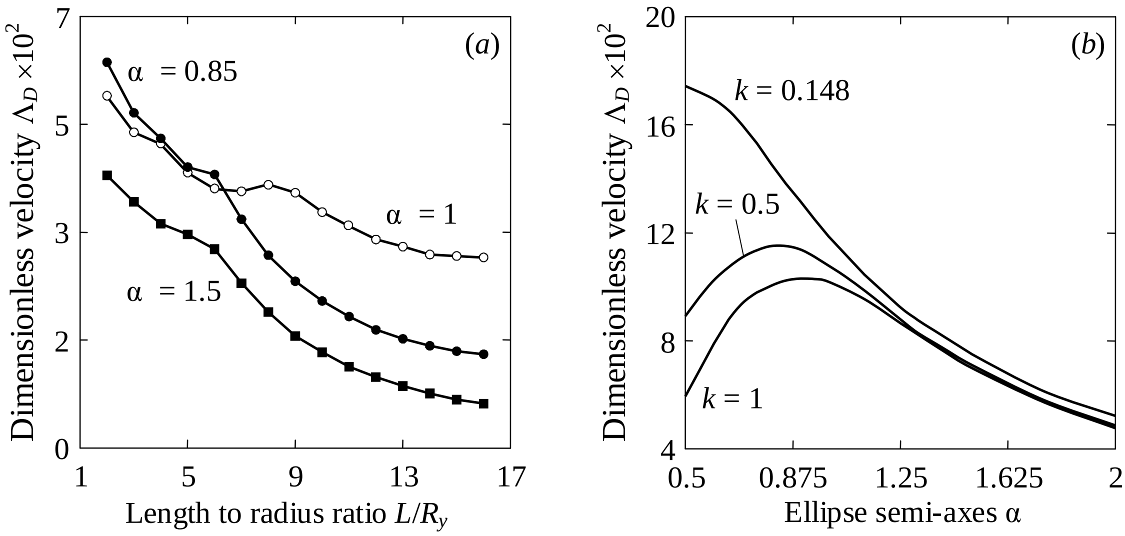

Figure 4: The dependence of dimensionless divergent velocities Λ_D of elliptical cylindrical shells on: (*a*) — ratio L/R_y, $k = 1$; (*b*) — ratio of ellipse semi-axes α and fluid level k.

Figure 4,*b* presents the critical velocities corresponding to the divergence Λ_D of the elliptical cylindrical shell ($E = 2.05 \times 10^{11}$ N/m^2; $\nu = 0.3$; $\rho_s = 7800$ kg/m^3; $R_y = 0.07725$ m; $h = 1.5 \times 10^{-3}$ m; $L/R_y = 2.99$), clamped at both ends for different fluid levels and ratios of semi-axes of the ellipse. From the data presented it follows that for the examined configurations a decrease in the fluid level k leads to a growth of the critical fluid velocities producing no effect on the type of stability loss. The boundary of hydroelastic stability changes essentially if the cross-section of the elliptical cylindrical shell partially filled with a fluid is ”compressed” relative a circular contour ($\alpha < 1$). In the case of well-stretched cross-sections ($\alpha > 1.25$), situations may arise when the critical velocities for the onset of divergence instability are actually independent of the fluid level.

4 CONCLUSIONS

A mathematical problem has been proposed to investigate the hydroelastic interaction of cylindrical shells of arbitrary cross-sections with inviscid fluids. The numerical algorithm developed in the framework of the mathematical mode has been used to evaluate natural frequencies, spatial vibration modes and critical velocities of the internal fluid flow, at which the thin-walled structure loses stability. The performance of the algorithm has been tested by applying it to the problems of shells with elliptical cross-sections. The obtained results show that the algorithm can be effectively used to analyze a wide spectrum of parameters, which may cause qualitative changes in the dynamic characteristics and stability boundary of the shell-fluid systems.

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