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IDENTIFICATION OF THE MATERIAL PROPERTIES OF MICROISOTROPIC MATERIALS

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Abstract. *In the present work, a vibration problem of a rectangular plate is considered to identify the upper bounds of the unknown material moduli of the plate which is assumed microisotropic. The frequencies are obtained by extending the Ritz Method to microisotropic case. Three dimensional (3-D) vibration analysis shows that some additional frequencies occur among the classical frequencies as characterizing the microisotropic effects. It is also observed that these additional frequencies disappear and only the classical frequencies remain by the increasing values of microisotropic constants beyond some certain limits. The inverse problem is established for the identification of the upper bounds of the microisotropic constants as an optimization problem where an error function is minimized.*

1 INTRODUCTION

It is well known that the material response to external effects closely depends on to the inner structure of the material, so the linear theory of elasticity is unable to explain the behavior of the material having complex inner structures. Among some others, Eringen's micromorphic [1,2] theory is the one which is widely used to explain the behavior of such materials.

In the passage from macroscopic to microscopic approach, a very serious difficulty occurs in engineering problems as the lack of the knowledge in the statistical thermodynamics and mechanics. This seriously limits the development of the microscaled models because of the unknown material properties. With other words, micromorphic models have great advantages in modeling physical realities with the minimum ambiguity and arbitrariness, but unfortunately they are computationally and mathematically inefficient in practice. To solve this very important problem of the theory, several further simplifications are considered like micropolar solids [3] which reduces the number of the unknown material coefficients from eighteen to six in linear case which makes the theory more applicable to several problems. But this approach is not very convenient to describe some microstructures like the medium having microcracks and microvoids which is the case in micro damaged materials. Microstretch theory assumes that each material point can make also micro elongation in addition to micro rotation. There are nine material moduli in this case and this approach may be more reliable than micropolar theory to model micro damaged materials [4]. Kiris and Inan used more simplified version of this theory to describe micro-damaged bodies called micro-elongation theory [5]. This theory has only five material constants. But its application to real engineering materials is limited.

Another approach is microisotropy introduced by Koh [6]. It may be considered as the one between two limits of the microstructural theories, namely micromorphic theory on one end and micro elongation theory on the other end. In this approach, a kind of an "isotropy" is considered as the coincidence of the principal directions of the stress and proper strain quantities. This assumption reduces the number of unknown material properties from eighteen to twelve. A subsequent formulation of the problem with respect to the principal directions of the microstrain is proposed by Koh [7] which requires the analysis of nine equations in nine unknowns instead of the original twelve equations containing eighteen unknown material moduli of micromorphic theory.

The present study aims to determine the upper bounds of the material properties of linear homogeneous microisotropic materials (defined by Koh [6] or "micro-co-axial" case preferred by the authors) by considering an inverse problem where an error function is minimized by the use of similar procedure given in [8-11].

The wave propagation problem in micropolar and microstretch media are discussed in [10-15] and it is shown that two and three new waves appear, respectively, which do not exist in the classical theory. Thus, it is obvious to expect that some additional waves will also appear in the present problem. Then, following Zhou et al. [16], the frequency analysis for microisotropic case is carried on. For this purpose, small-strain, 3-D microisotropic theory and Ritz method are extended to the present problem. The detailed procedure is given in authors' earlier works [10-11]. Here, the key point is that the values of classical frequencies begin to deviate while the additional frequencies arising from microisotropic character of the medium move out among the classical frequencies after some threshold values of these constants. This phenomenon tells us that the microstructure become more dominant and starts to affect the macro properties. Thus, we construct an optimization problem where an error function related to the difference between calculated and referenced frequencies is minimized and as a result of this optimization problem, the threshold values are obtained as the upper bounds of the microisotropic properties.

2 FUNDAMENTAL EQUATIONS OF MICROISOTROPIC MEDIUM

In general, co-axiality is defined as the coincidence of the principle directions of stress and strain tensors locally. The extension of this concept to microstructure is given by Koh [6] and defined as microisotropy. Thus, it is expected that the principle directions of some particular strain measures would coincide with suitable stress measures in micro structures. [We prefer to call this case as micro-co-axiality, but we will keep the definition given by Koh [6] through the article.]

To find the convenient sets of stress-strain measures, we may observe the pairing in energy equation which may be written as

$$\rho\dot{\varepsilon} = t_{km}v_{m,k} - \sigma_{km}\nu_{mk} + t_{kmn}\nu_{mn,k} + q_{k,k} + \rho h. \tag{1}$$

Here, the main stress measures are the asymmetric stress tensor t_{km}, the relative stress tensor σ_{km}, and the stress moment tensor, t_{kmn}. ρ is mass density, v_k is velocity vector, ν_{km} is gyration tensor, q_k is heat flux vector and h is heat source per unit mass. We may observe from the above energy expression that the proper strain measures are the macro strain e_{km}, related to $v_{k,m}$; the micro-displacement gradient ϕ_{km}, related to ν_{km} and the micro-deformation gradient $\phi_{mn,k}$ related to $\nu_{mn,k}$. Then the postulate of principle directions' coincidence may be summarized as follows as given by Koh [6-7]:

The principle directions of symmetric part of the stress tensor $t_{(km)}$ coincide with the principle directions of the macro strain e_{km}. One of the results of this condition is the coincidence of the principle directions of symmetric part of the relative stress tensor $\sigma_{(km)}$ with the micro-strain $\phi_{(km)}$, and vice versa.

The principle directions of symmetric stress moment tensor $t_{k(mn)}$ coincide with the symmetric microdeformation gradient tensor $\phi_{(mn)k}$.

The principle values of each stress measures above and corresponding strain measures are independent of the other strain and stress measures.

By these special conditions of microisotropy, the constitutive equations were obtained by Koh as follows [6-7],

$$\begin{aligned}
t_{(kl)} &= (\lambda+\sigma_1)I_e\,\delta_{kl} + 2(\mu+\sigma_2)e_{kl}\\
t_{[kl]} &= \sigma_{[kl]} = 2\sigma_5\left(w_{lk} + \phi_{[kl]}\right)\\
\sigma_{(kl)} &= \sigma_1 I_\phi\delta_{kl} + 2\sigma_2\,\phi_{(kl)} - 2\sigma_5 e_{kl}\\
t_{k(lm)} &= \tau_3 I_{\phi,k}\,\delta_{lm} + (\tau_7+\tau_{10})\phi_{(lm),k}\\
t_{k[lm]} &= (\tau_7-\tau_{10})\phi_{[lm],k} + 2\tau_9(\phi_{[lk],m} - \phi_{[mk],l}) + 2\tau_4(\phi_{[ln],n}\delta_{km} - \phi_{[mn],n}\delta_{kl})
\end{aligned} \tag{2}$$

Here, $\lambda, \mu, \sigma_1, \sigma_2, \sigma_5, \tau_3, \tau_4, \tau_7, \tau_9$ and τ_{10} are the material coefficients and σ_i and τ_i are defined as given in [1,2] and λ and μ are Lamé constants. t_{kl} and σ_{kl} are macro and micro stress tensors, t_{klm} denotes stress-moment tensor, $m_{kl} = e_{lmn}t_{knm}$ is moment tensor, $\varepsilon_{kl} = u_{(k,l)}$ is strain tensor, $\phi_{(kl)}$ is micro strain tensor, u_k is displacement vector and $w_{lk} = \frac{1}{2}\left(u_{l,k} - u_{k,l}\right)$ and $\phi_{[kl]}$ are macro and micro rotations tensors, respectively.

Now, defining following abbreviations,

$$A_1 = \lambda + \sigma_1, \quad A_2 = \mu + \sigma_2, \quad A_3 = \sigma_5, \quad A_4 = -\sigma_1, \quad A_5 = -\sigma_2,$$
$$B_1 = \tau_3, \quad 2B_2 = \tau_7 + \tau_{10}, \quad B_3 = 2\tau_4 + 2\tau_9 + \tau_7 - \tau_{10}, \quad B_4 = -2\tau_4, \quad B_5 = -2\tau_9 \tag{3}$$

field equations for a microisotropic medium may be written as,

$$(A_1 + A_2 - A_3)u_{l,lk} + (A_2 + A_3)u_{k,ll} + 2A_3\varepsilon_{mlk}\phi_{m,l} + \rho f_k = \rho \frac{\partial^2 u_k}{\partial t^2}$$
$$B_1\phi_{mm,nn}\delta_{kl} + 2B_2\phi_{(kl),mm} - A_4\phi_{mm}\delta_{kl} - 2A_5\phi_{(kl)} + \rho f_{(kl)} = \frac{1}{2}\rho j \frac{\partial^2 \phi_{(kl)}}{\partial t^2} \tag{4}$$
$$2B_3\phi_{k,ll} + 2(B_4 + B_5)\phi_{l,lk} - 4A_3(r_k + \phi_k) - \rho l_k = \rho j \frac{\partial^2 \phi_k}{\partial t^2}$$

where $r_k = \frac{1}{2} e_{klm}u_{m,l}$ and $\phi_k = \frac{1}{2} e_{klm}\phi_{lm}$ are macro and micro rotation vectors and f_k, $f_{(kl)}$ and l_k are body force, symmetric body moment and body couple, respectively and j is micro inertia per unit mass.

Now, by the use of general energy density expression given in (1), the internal energy density for such a medium may be written as the sum of five different energies

$$W = W_1 + W_2 + W_3 + W_4 + W_5 \tag{5}$$

where,

$$W_1 = \frac{1}{2}\left[A_1\, e_{kk}e_{ll} + 2A_2\, e_{kl}e_{lk}\right], \quad W_2 = 2A_3(r_k + \phi_k)(r_k + \phi_k)$$
$$W_3 = \frac{1}{2}\left[A_4\, \phi_{kk}\phi_{ll} + 2A_5\, \phi_{(kl)}\phi_{(lk)}\right], \quad W_4 = \frac{1}{2}\left[B_1\, \phi_{kk,m}\phi_{ll,m} + 2B_2\, \phi_{(kl),m}\phi_{(lk),m}\right] \tag{6}$$
$$W_5 = B_3\, \phi_{k,l}\phi_{k,l} + B_4\, \phi_{l,k}\phi_{k,l} + B_5\, \phi_{k,k}\phi_{l,l}$$

Here, W_1 shows the macro straining part, W_2 corresponds micro and macro rotational parts. W_3, W_4 and W_5 show micro straining part, micro stain gradient part and micro rotation gradient part of the internal energy density, respectively.

The kinetic energy density for this problem is given as

$$K = \frac{1}{2}\,\rho \dot{u}_k\dot{u}_k + \frac{1}{2}\rho\, j\dot{\phi}_{kl}\dot{\phi}_{kl} + \frac{1}{2}\rho\, j\dot{\phi}_k\dot{\phi}_k \quad . \tag{7}$$

3 3D VIBRATION ANALYSIS OF A MICROISOTROPIC PLATE

In this part of the work, we investigate the vibration problem of a rectangular plate which is modeled by microisotropic theory. The total potential energy, U, and total kinetic energy, K of the plate may be written as,

$$U = \frac{1}{2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\int_{-h/2}^{h/2}\Big\{A_1\Lambda_1^2 + 2A_2\left[\Lambda_2 + 2\Lambda_3\right] + 4A_3\Lambda_4 + A_4\Lambda_5^2 + 2A_5\left[\Lambda_6 + 2\Lambda_7\right]$$
$$+ B_1\left[\left(\Lambda_{5,1}\right)^2 + \left(\Lambda_{5,2}\right)^2 + \left(\Lambda_{5,3}\right)^2\right] + 2B_2\left[\Lambda_8 + 2\Lambda_9\right] + 2B_3\left[\Lambda_{10} + \Lambda_{11}\right] \tag{8}$$
$$+ 2B_4\left[\Lambda_{10} + 2\Lambda_{12}\right] + 2B_5\Lambda_{13}^2\Big\}\, dzdydx$$

and

$$T=\frac{\rho}{2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\int_{-h/2}^{h/2}\left\{\left[\left(\frac{\partial u_1}{\partial t}\right)^2+\left(\frac{\partial u_2}{\partial t}\right)^2+\left(\frac{\partial u_3}{\partial t}\right)^2\right]+j\left[\left(\frac{\partial \phi_{(11)}}{\partial t}\right)^2+\left(\frac{\partial \phi_{(22)}}{\partial t}\right)^2+\left(\frac{\partial \phi_{(33)}}{\partial t}\right)^2\right.\right.$$
$$\left.\left.+2\left(\left(\frac{\partial \phi_{(12)}}{\partial t}\right)^2+\left(\frac{\partial \phi_{(13)}}{\partial t}\right)^2+\left(\frac{\partial \phi_{(23)}}{\partial t}\right)^2\right)+\left(\frac{\partial \phi_1}{\partial t}\right)^2+\left(\frac{\partial \phi_2}{\partial t}\right)^2+\left(\frac{\partial \phi_3}{\partial t}\right)^2\right]dzdydx \tag{9}$$

Here, a, b, and h show the plate dimensions and

$$\Lambda_1=e_{11}+e_{22}+e_{33},\quad \Lambda_2=(e_{11}^2+e_{22}^2+e_{33}^2),\quad \Lambda_3=(e_{12}^2+e_{13}^2+e_{23}^2),$$
$$\Lambda_4=(r_1+\phi_1)^2+(r_2+\phi_2)^2+(r_3+\phi_3)^2,\quad \Lambda_5=\phi_{11}+\phi_{22}+\phi_{33},\quad \Lambda_6=\phi_{(11)}^2+\phi_{(22)}^2+\phi_{(33)}^2,$$
$$\Lambda_7=\phi_{(12)}^2+\phi_{(13)}^2+\phi_{(23)}^2,\quad \Lambda_{10}=\phi_{1,1}^2+\phi_{2,2}^2+\phi_{3,3}^2,\quad \Lambda_{13}=\phi_{1,1}+\phi_{2,2}+\phi_{3,3},$$
$$\Lambda_8=\phi_{(11),1}^2+\phi_{(11),2}^2+\phi_{(11),3}^2+\phi_{(22),1}^2+\phi_{(22),2}^2+\phi_{(22),3}^2+\phi_{(33),1}^2+\phi_{(33),2}^2+\phi_{(33),3}^2,$$
$$\Lambda_9=\phi_{(12),1}^2+\phi_{(13),1}^2+\phi_{(23),1}^2+\phi_{(12),2}^2+\phi_{(13),2}^2+\phi_{(23),2}^2+\phi_{(12),3}^2+\phi_{(13),3}^2+\phi_{(23),3}^2,$$
$$\Lambda_{11}=\phi_{1,2}^2+\phi_{1,3}^2+\phi_{2,1}^2+\phi_{2,3}^2+\phi_{3,1}^2+\phi_{3,2}^2,\quad \Lambda_{12}=\phi_{1,2}\phi_{2,1}+\phi_{1,3}\phi_{3,1}+\phi_{2,3}\phi_{3,2}. \tag{10}$$

Now, assuming harmonic-time dependence, we may write

$$\left\{u_k(x,y,z,t),\phi_{(kl)}(x,y,z,t),\phi_k(x,y,z,t)\right\}$$
$$=\left\{U_k(x,y,z),\Phi_{(kl)}(x,y,z),\Phi_k(x,y,z)\right\}\;e^{i\omega t};\quad k,l=1,2,3 \tag{11}$$

Here, ω denotes the natural frequency. Introducing non-dimensional parameters,

$$\xi=\frac{2x}{a},\quad \eta=\frac{2y}{b},\quad \zeta=\frac{2z}{h}, \tag{12}$$

the minimum energy functional of the microisotropic plate may be written as

$$\Pi=V-T. \tag{13}$$

Here,

$$V=\frac{h}{4\alpha_1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left\{A_1\bar{\Lambda}_1^2+2A_2\bar{\Lambda}_2+(A_2+A_3)\left(\bar{\Lambda}_3+\bar{\bar{\Lambda}}_3\right)+2(A_2-A_3)\bar{\bar{\bar{\Lambda}}}_3\right.$$
$$+B_1\left(\bar{\gamma}_1^2+\bar{\bar{\gamma}}_2^2+\bar{\bar{\bar{\gamma}}}_3^2\right)+2B_2\left(\bar{\Lambda}_8+\bar{\bar{\Lambda}}_8+\bar{\bar{\bar{\Lambda}}}_8\right)+4B_2\left(\bar{\Lambda}_9+\bar{\bar{\Lambda}}_9+\bar{\bar{\bar{\Lambda}}}_9\right)+2(B_3+B_4)\bar{\Lambda}_{10}$$
$$\left.+2B_3\bar{\Lambda}_{11}+4B_4\bar{\Lambda}_{12}+2B_5\bar{\Lambda}_{13}^2\right\}d\zeta\,d\eta\,d\xi+\frac{b\,h}{2}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}A_3\,\bar{\Lambda}_4\,d\zeta\,d\eta\,d\xi$$
$$+\frac{a\,b\,h}{16}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left\{4A_3\bar{\bar{\Lambda}}_4+A_4\bar{\bar{\Lambda}}_5^2+2A_5\bar{\Lambda}_6+4A_5\bar{\Lambda}_7\right\}d\zeta\,d\eta\,d\xi, \tag{14}$$

$$T=\frac{\rho}{16}a\,b\,h\,\omega^2\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left\{\left[U_1^2+U_2^2+U_3^2\right]+j\left[\left(\Phi_{(11)}^2+\Phi_{(22)}^2+\Phi_{(33)}^2\right)\right.\right.$$
$$\left.\left.+2\left(\Phi_{(12)}^2+\Phi_{(13)}^2+\Phi_{(23)}^2\right)+\left(\Phi_1^2+\Phi_2^2+\Phi_3^2\right)\right]\right\}d\zeta\,d\eta\,d\xi\;. \tag{15}$$

and

$$\bar{\Lambda}_1 = \bar{e}_{\xi\xi} + \bar{e}_{\eta\eta} + \bar{e}_{\zeta\zeta}, \quad \bar{\Lambda}_2 = \bar{e}^2_{\xi\xi} + \bar{e}^2_{\eta\eta} + \bar{e}^2_{\zeta\zeta}, \quad \bar{\Lambda}_3 = {}_1\bar{e}^2_{\xi\eta} + {}_1\bar{e}^2_{\xi\zeta} + {}_1\bar{e}^2_{\eta\zeta},$$
$$\bar{\bar{\Lambda}}_3 = {}_2\bar{e}^2_{\xi\eta} + {}_2\bar{e}^2_{\xi\zeta} + {}_2\bar{e}^2_{\eta\zeta}, \quad \bar{\bar{\bar{\Lambda}}}_3 = {}_1\bar{e}_{\xi\eta}\,{}_2\bar{e}_{\xi\eta} + {}_1\bar{e}_{\xi\zeta}\,{}_2\bar{e}_{\xi\zeta} + {}_1\bar{e}_{\eta\zeta}\,{}_2\bar{e}_{\eta\zeta},$$
$$\bar{\Lambda}_4 = \left({}_2\bar{e}_{\eta\zeta} - {}_1\bar{e}_{\eta\zeta}\right)\bar{\Phi}_\xi + \left({}_1\bar{e}_{\xi\zeta} - {}_2\bar{e}_{\xi\zeta}\right)\bar{\Phi}_\eta + \left({}_2\bar{e}_{\xi\eta} - {}_1\bar{e}_{\xi\eta}\right)\bar{\Phi}_\zeta, \quad \bar{\bar{\Lambda}}_4 = \bar{\Phi}^2_\xi + \bar{\Phi}^2_\eta + \bar{\Phi}^2_\zeta,$$
$$\bar{\Lambda}_5 = \left(\bar{\Phi}_{\xi\xi} + \bar{\Phi}_{\eta\eta} + \bar{\Phi}_{\zeta\zeta}\right), \quad \gamma_1 = \bar{\Lambda}_{5,\xi}, \quad \gamma_2 = \alpha_1\bar{\Lambda}_{5,\eta}, \quad \gamma_3 = \frac{\alpha_1}{\alpha_2}\bar{\Lambda}_{5,\zeta},$$
$$\bar{\Lambda}_6 = \bar{\Phi}^2_{\xi\xi} + \bar{\Phi}^2_{\eta\eta} + \bar{\Phi}^2_{\zeta\zeta}, \quad \bar{\Lambda}_7 = \bar{\Phi}^2_{(\xi\eta)} + \bar{\Phi}^2_{(\xi\zeta)} + \bar{\Phi}^2_{(\eta\zeta)}, \quad \bar{\Lambda}_8 = \bar{\Phi}^2_{\xi\xi,\xi} + \bar{\Phi}^2_{\eta\eta,\xi} + \bar{\Phi}^2_{\zeta\zeta,\xi},$$
$$\bar{\bar{\Lambda}}_8 = \alpha_1^2\left(\bar{\Phi}^2_{\xi\xi,\eta} + \bar{\Phi}^2_{\eta\eta,\eta} + \bar{\Phi}^2_{\zeta\zeta,\eta}\right), \quad \bar{\bar{\bar{\Lambda}}}_8 = \frac{\alpha_1^2}{\alpha_2^2}\left(\bar{\Phi}^2_{\xi\xi,\zeta} + \bar{\Phi}^2_{\eta\eta,\zeta} + \bar{\Phi}^2_{\zeta\zeta,\zeta}\right),$$
$$\bar{\Lambda}_9 = \bar{\Phi}^2_{(\xi\eta),\xi} + \bar{\Phi}^2_{(\xi\zeta),\xi} + \bar{\Phi}^2_{(\eta\zeta),\xi}, \quad \bar{\bar{\Lambda}}_9 = \alpha_1^2\left(\bar{\Phi}^2_{(\xi\eta),\eta} + \bar{\Phi}^2_{(\xi\zeta),\eta} + \bar{\Phi}^2_{(\eta\zeta),\eta}\right),$$
$$\bar{\bar{\bar{\Lambda}}}_9 = \frac{\alpha_1^2}{\alpha_2^2}\left(\bar{\Phi}^2_{(\xi\eta),\zeta} + \bar{\Phi}^2_{(\xi\zeta),\zeta} + \bar{\Phi}^2_{(\eta\zeta),\zeta}\right), \quad \bar{\Lambda}_{10} = \bar{\Phi}^2_{\xi,\xi} + \alpha_1^2\bar{\Phi}^2_{\eta,\eta} + \frac{\alpha_1^2}{\alpha_2^2}\bar{\Phi}^2_{\zeta,\zeta},$$
$$\bar{\Lambda}_{11} = \alpha_1^2\bar{\Phi}^2_{\xi,\eta} + \frac{\alpha_1^2}{\alpha_2^2}\bar{\Phi}^2_{\xi,\zeta} + \bar{\Phi}^2_{\eta,\xi} + \frac{\alpha_1^2}{\alpha_2^2}\bar{\Phi}^2_{\eta,\zeta} + \bar{\Phi}^2_{\zeta,\xi} + \alpha_1^2\bar{\Phi}^2_{\zeta,\eta},$$
$$\bar{\Lambda}_{12} = \alpha_1\bar{\Phi}_{\xi,\eta}\bar{\Phi}_{\eta,\xi} + \frac{\alpha_1}{\alpha_2}\bar{\Phi}_{\xi,\zeta}\bar{\Phi}_{\zeta,\xi} + \frac{\alpha_1^2}{\alpha_2^2}\bar{\Phi}_{\eta,\zeta}\bar{\Phi}_{\zeta,\eta}, \quad \bar{\Lambda}_{13} = \bar{\Phi}_{\xi,\xi} + \alpha_1\bar{\Phi}_{\eta,\eta} + \frac{\alpha_1}{\alpha_2}\bar{\Phi}_{\zeta,\zeta} \tag{16}$$

and

$$\begin{gathered}\bar{e}_{\xi\xi} = \bar{U}_{\xi,\xi}, \quad \bar{e}_{\eta\eta} = \alpha_1\bar{U}_{\eta,\eta}, \quad \bar{e}_{\zeta\zeta} = \frac{\alpha_1}{\alpha_2}\bar{U}_{\zeta,\zeta}, \quad {}_1\bar{e}_{\xi\eta} = \alpha_1\bar{U}_{\xi,\eta}, \quad {}_2\bar{e}_{\xi\eta} = \bar{U}_{\eta,\xi}, \\ {}_1\bar{e}_{\xi\zeta} = \frac{\alpha_1}{\alpha_2}\bar{U}_{\xi,\zeta}, \quad {}_2\bar{e}_{\xi\zeta} = \bar{U}_{\zeta,\xi}, \quad {}_1\bar{e}_{\eta\zeta} = \frac{\alpha_1}{\alpha_2}\bar{U}_{\eta,\zeta}, \quad {}_2\bar{e}_{\eta\zeta} = \bar{U}_{\zeta,\eta}, \quad \alpha_1 = \frac{a}{b}, \quad \alpha_2 = \frac{h}{b}.\end{gathered} \tag{17}$$

Following [10, 16], we use triplicate series of Chebyshev polynomials multiplied by admissible functions which satisfy the boundary conditions of the plate for each amplitude functions of the Eq. (11), i.e.,

$$\begin{aligned}U_m(\xi,\eta,\zeta) &= F_{U_m}(\xi,\eta)\sum_{i,j,k=1}^{\infty} A^m_{ijk}P_i(\xi)P_j(\eta)P_k(\zeta), \qquad m = 1,2,3 \\ \phi_{(mn)}(\xi,\eta,\zeta) &= F_{\phi_{(mn)}}(\xi,\eta)\sum_{i,j,k=1}^{\infty} B^{(mn)}_{ijk}P_i(\xi)P_j(\eta)P_k(\zeta), \quad m,n = 1,2,3 \\ \Phi_m(\xi,\eta,\zeta) &= F_{\Phi_m}(\xi,\eta)\sum_{i,j,k=1}^{\infty} C^m_{ijk}P_i(\xi)P_j(\eta)P_k(\zeta). \qquad m = 1,2,3\end{aligned} \tag{18}$$

Here, one dimensional i^{th} Chebyshev polynomial and the boundary functions are taken as given in Refs. [10,16].

Now, minimizing the energy functional Eq. (13) with respect to the unknown coefficients of Chebyshev polynomials gives the following eigenvalue problem

$$\left(\mathbf{K} - \Omega^2\,\mathbf{M}\right)\mathbf{Z} = \mathbf{0}. \tag{19}$$

Here, $\Omega = \omega a\sqrt{\rho}$ and the column vector $\mathbf{Z}$ is written with its sub-column vectors as

$$\mathbf{Z} = \{\mathbf{A}^1, \mathbf{A}^2, \mathbf{A}^3, \mathbf{B}^{11}, \mathbf{B}^{22}, \mathbf{B}^{33}, \mathbf{B}^{(12)}, \mathbf{B}^{(13)}, \mathbf{B}^{(23)}, \mathbf{C}^1, \mathbf{C}^2, \mathbf{C}^3\}^T, \tag{20}$$

and, each sub-column vector, for example $\mathbf{A}^1$ is in the following form

$$\mathbf{A}^1 = \{A^1_{111}, \ldots, A^1_{11N}, \ldots, A^1_{1k1}, \ldots, A^1_{1kN}, \ldots, A^1_{I11}, \ldots, A^1_{IJK}\}. \tag{21}$$

The rigidity and the mass matrices are defined in terms of sub-matrices as the followings,

$$\mathbf{K} = \begin{pmatrix}
\mathbf{K}_{u_1u_1} & \mathbf{K}_{u_1u_2} & \mathbf{K}_{u_1u_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{u_1\phi_2} & \mathbf{K}_{u_1\phi_3} \\
\mathbf{K}_{u_1u_2} & \mathbf{K}_{u_2u_2} & \mathbf{K}_{u_2u_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{u_2\phi_1} & \mathbf{0} & \mathbf{K}_{u_2\phi_3} \\
\mathbf{K}_{u_1u_3} & \mathbf{K}_{u_2u_3} & \mathbf{K}_{u_3u_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{u_3\phi_1} & \mathbf{K}_{u_3\phi_2} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\phi_{11}\phi_{11}} & \mathbf{K}_{\phi_{11}\phi_{22}} & \mathbf{K}_{\phi_{11}\phi_{33}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\phi_{11}\phi_{12}} & \mathbf{K}_{\phi_{22}\phi_{22}} & \mathbf{K}_{\phi_{22}\phi_{33}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\phi_{11}\phi_{33}} & \mathbf{K}_{\phi_{22}\phi_{33}} & \mathbf{K}_{\phi_{33}\phi_{33}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\phi_{(12)}\phi_{(12)}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\phi_{(13)}\phi_{(13)}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\phi_{(23)}\phi_{(23)}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{K}_{u_2\phi_1} & \mathbf{K}_{u_3\phi_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\phi_1\phi_1} & \mathbf{K}_{\phi_1\phi_2} & \mathbf{K}_{\phi_1\phi_3} \\
\mathbf{K}_{u_1\phi_2} & \mathbf{0} & \mathbf{K}_{u_3\phi_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\phi_1\phi_2} & \mathbf{K}_{\phi_2\phi_2} & \mathbf{K}_{\phi_2\phi_3} \\
\mathbf{K}_{u_1\phi_3} & \mathbf{K}_{u_2\phi_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\phi_1\phi_3} & \mathbf{K}_{\phi_2\phi_3} & \mathbf{K}_{\phi_3\phi_3}
\end{pmatrix} \tag{22}$$

and

$$\mathbf{M} = \begin{pmatrix}
\mathbf{M}_{u_1u_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{M}_{u_2u_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{M}_{u_3u_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\phi_{11}\phi_{11}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\phi_{22}\phi_{22}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\phi_{33}\phi_{33}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\phi_{(12)}\phi_{(12)}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\phi_{(13)}\phi_{(13)}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\phi_{(23)}\phi_{(23)}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\phi_1\phi_1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\phi_2\phi_2} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\phi_3\phi_3}
\end{pmatrix} \tag{23}$$

To provide the shortness, the expressions of the above sub-matrices are not given here. Just for sample, we gave only few of them,

$$\begin{aligned}
&\mathbf{K}_{u_1u_1} = (A_1 + 2A_2) D^{1,1}_{u_1ku_1\underline{k}} G^{0,0}_{u_1lu_1\underline{l}} H^{0,0}_{m\underline{m}} + (A_2 + A_3)\alpha_1^2 \left(D^{0,0}_{u_1ku_1\underline{k}} G^{1,1}_{u_1lu_1\underline{l}} H^{0,0}_{m\underline{m}} + \frac{1}{\alpha_2^2} D^{0,0}_{u_1ku_1\underline{k}} G^{0,0}_{u_1lu_1\underline{l}} H^{1,1}_{m\underline{m}} \right), \\
&\mathbf{K}_{u_1u_2} = \alpha_1 \left[A_1 D^{1,0}_{u_1ku_2\underline{k}} G^{0,1}_{u_1lu_2\underline{l}} H^{0,0}_{m\underline{m}} + (A_2 - A_3) D^{0,1}_{u_1ku_2\underline{k}} G^{1,0}_{u_1lu_2\underline{l}} H^{0,0}_{m\underline{m}} \right], \mathbf{K}_{u_1\phi_2} = a \frac{\alpha_1}{\alpha_2} A_3 D^{0,0}_{u_1k\phi_2\underline{k}} G^{0,0}_{u_1l\phi_2\underline{l}} H^{1,0}_{m\underline{m}}, \\
&\mathbf{K}_{pq} = B_1 (D^{1,1}_{pkq\underline{k}} G^{0,0}_{plq\underline{l}} H^{0,0}_{m\underline{m}} + \alpha_1^2 D^{0,0}_{pkq\underline{k}} G^{1,1}_{plq\underline{l}} H^{0,0}_{m\underline{m}} + \frac{\alpha_1^2}{\alpha_2^2} D^{0,0}_{pkq\underline{k}} G^{0,0}_{plq\underline{l}} H^{1,1}_{m\underline{m}}) + \frac{a^2}{4} A_4 D^{0,0}_{pkq\underline{k}} G^{0,0}_{plq\underline{l}} H^{0,0}_{m\underline{m}} \\
&((p = \phi_{11}, q = \phi_{22}), (p = \phi_{11}, q = \phi_{33}), (p = \phi_{22}, q = \phi_{33})), \quad \mathbf{M}_{qq} = \frac{1}{4} D^{0,0}_{qkq\underline{k}} G^{0,0}_{qlq\underline{l}} H^{0,0}_{m\underline{m}} \ \ (q = u_1, u_2, u_3),
\end{aligned} \tag{24}$$

Similar forms may be found in [10] which are given for microstretch case.

CONSTRUCTION OF THE OPTIMIZATION PROBLEM AND THE NUMERICAL RESULTS

The construction of the optimization problem is based on to the comparison of calculated and reference frequencies' spectrum. First, we must underline the fact that there is no data for the experimental frequencies of the plate modeled by microisotropic or any other micromorphic theories. Thus, we consider the frequencies of the plate made of Gauthier's [17] material which are obtained by the classical theory as the experimental frequencies or the reference frequencies of the present problem. The frequencies' spectrum of the plate modeled by microisotropic theory contains both the classical frequencies and the additional micro frequencies due to the waves appear in microisotropic theory as the case in microstretch theory [10]. The dependence of the micro frequencies on to the microisotropic constants is more sensitive than the classical frequencies. As a result, when the microisotropic constants are getting bigger, the values of micro frequencies rapidly increase and move out among the classical frequencies' spectrum and the classical frequencies almost all remain same in value unless the microisotropic constants exceed some certain threshold values. After they exceed these threshold values, all additional frequencies move out from the spectrum and also the classical frequencies begin to deviate. This phenomenon tells us that the microstructure become more dominant and starts to affect the macro properties. Thus, an optimization problem as an inverse problem may be constructed where an error function related to the difference between calculated and the reference frequencies is minimized and then the threshold values of microisotropic constants are obtained as the upper bounds of the microisotropic elastic constants as in authors' previous work for microstretch constants [10].

The optimization problem is constructed by superposing two objective functions; first one is minimizing the difference between calculated and reference classical frequencies and second one is minimizing the number of the additional frequencies due to microisotropic character of the medium and we may write,

$$\underset{\text{subject to } \mathbf{X}}{Minimize} \quad \tau_1 \sum_{i=1}^{I} \left(\frac{f_i^{ref} - f_i^{cal}(\mathbf{X})}{f_i^{ref}} \right)^2 + \tau_2 \; length(B^{micro}(\mathbf{X})) . \tag{25}$$

Here, f_i^{ref} denotes i^{th} reference frequencies obtained from classical elasticity, $f_i^{cal}(\mathbf{X})$ denotes i^{th} calculated frequencies obtained from microisotropic theory, $B^{micro}(\mathbf{X})$ is the set of additional micro frequencies occurring in the spectrum (first I number of frequencies), τ_1 and τ_2 satisfying $\tau_1 + \tau_2 = 1$, are the weights which identify first and second objective functions are dominant [10] and the parameter vector $\mathbf{X}$ contains the unknown microisotropic material constants and is given as

$$\mathbf{X} = \{\sigma_1, \sigma_2, \sigma_3, B_1, B_2, B_3, B_4, B_5\} . \tag{26}$$

The optimization problem given by Eq. (25) is solved by combining the direct search algorithm (DSA) and the genetic algorithm (GA) [10]. The material of the plate is considered as a Gauthier material [17] composed of aluminum matrix within randomly distributed epoxy spheres. The geometrical and material properties of the plate and the reference frequencies obtained from classical theory are given in Table.1.

Material	Aluminum matrix with randomly distributed epoxy spheres		
Young Modulus (E)	5.31 GPa	Poisson ratio (ν)	0.4

Density(ρ)	2192 kg/m^3	Micro inertia (j)	$1.96\times10^{-7}\,m^2$
Dimensions($a\times b\times t$)	$1\times1\times0.1\,m^3$	Boundary Conditions	FFFF (F=Free)
Ref. Frequencies	1.19833, 1.80347, 2.39605, 3.06437, 3.06437, 5.38536, 5.52594, 5.52594, 5.80731, 6.75343, 7.56143, 8.26271, 8.26271, 8.53479, 8.53479, 8.54115, 9.91288, 10.0431, 10.646, 10.9261 (Hz)		

Table 1: The properties of the plate made of Gauthier's material [17].

The solution of the optimization problem gives the upper bounds of unknown microisotropic constants as given in Table.2.

$\sigma_1 = 0.384\,10^{-4}\,GPa,\ \sigma_2 = 0.955\,10^{-4}\,GPa,\ \sigma_3 = 0.32\,10^{-3}\,GPa,\ B_1 = 0.422\,10^{-6}\,GN,$ $B_2 = 0.161\,10^{-5}\,GN,\ B_3 = 0.478\,10^{-7}\,GN,\ B_4 = 0.933\,10^{-6}\,GN,\ B_5 = 0.785\,10^{-7}\,GN$	
Obtained Freq. (Hz) with above microisotropic parameters: 1.19872, 1.80387, 2.39592, 3.06502, 3.06502, 5.38636, 5.52609, 5.52609, 5.80834, 6.75367, 7.56152, 8.26272, 8.26272, 8.53594, 8.53594, 8.54129, 9.91331, 10.043, 10.646, 10.9266	Error $3.57516\ 10^{-7}$

Table 2: The final results of microisotropic elastic constants for Gauthier composite plate [17] obtained from Direct Search Method, $\mathbf{X} = \{\sigma_1, \sigma_2, \sigma_3, B_1, B_2, B_3, B_4, B_5\}$.

4 CONCLUSIONS

In the present paper, a procedure is given to determine the upper bounds of the unknown material properties of micro isotropic bodies.

As a result, present analysis shows that:

- Some additional frequencies appear among the classical frequencies due to the micro structure.
- By the increasing values of the micro properties, the micro additional frequencies start to move out from the frequencies' spectrum. In the mean time the classical frequencies stay almost unchanged.
- After some certain threshold values of micro elastic constants, most of the additional frequencies leave the spectrum and some small chances occur in the values of the classical frequencies.
- If further more increase is considered beyond these threshold values of micro elastic constants, all additional frequencies due to micro structure move out the frequency spectrum and the deviation in the values of the classical frequencies become more noticeable. This is an expected result. Because, after exceeding the obtained values of microisotropic elastic constants, the material loses its microstructural character and reflects the macro character of the material.

We have to point out here that to find more specific values for the micro constants, we need to know some numerical data for the micro frequencies of the microisotropic material obtained by experimental analysis which needs more sophisticated equipments and experimental techniques. Since no such experimental data is available in the literature, we used the frequencies obtained from classical 3-D analysis which give us only the upper bounds for micro constants.

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