

DETECTING PHASE SYNCHRONIZATION IN COUPLED CHAOTIC NONCOHERENT OSCILLATORS BY USING COMPLEX WAVELET TRANSFORM

Maria Teodora Ferreira¹, Rosangela Follmann², Elbert E. N. Macau*³, Margarete O. Domingues⁴

^{1, 2, 3, 4} National Institute for Space Research - INPE, Sao Jose dos Campos
^{1, 2, 4} mteodoraf25, rosangelafollmann, margarete.oliveira.domingues@gmail.com
³ elbert@lac.inpe.br

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Abstract. *Phase synchronization in nonidentical coupled chaotic systems appears for some conditions in which weak coupling causes the systems time evolution to lock in phase to one another, while their amplitudes remain chaotic and uncorrelated. To identify this phenomenon, given a signal it is necessary to measure properly its phase. If a system has a dominant peak in the power spectrum, there are several methods that allow a proper definition for phase. However, if the signal has a broad-band spectrum, which is typical for non-coherent signal, then the measurement of the phase may be a challenge. Phase is defined as an increasing function of the time. The standard method for measuring phase does not complain with this requirement. In this work we present an innovative method for measuring phase that complies with the requirement of increasing in time. This method is based on the Dual Tree Complex Wavelet Transform, which is a form of a discrete wavelet transform that generates complex coefficient by using a dual tree wavelet filters to obtain their real and imaginary parts. The proposed approach is robust and computationally efficient. Furthermore, it shows flexibility and in principle is applicable to any experimental time series.*

1 INTRODUCTION

The phenomenon of synchronization has been observed in various natural systems, such as in heart cells, applause, flashing of the Southeast Asian fireflies, chirping of crickets, and more [1, 2, 3]. Basically, synchronization is understood as the mutual adjustment of the oscillations of periodic oscillators due to some sort of interaction between them [4].

Over the last decades, considerable progress has been made toward generalizing the concept of synchronization, allowing it to encompass chaotic oscillators [4, 5, 6, 7, 8, 9]. In this work our interest is *phase synchronization*, which occurs mainly for weak coupling with the amplitudes of the oscillators remaining uncorrelated while their phases evolve in step with each other over time [8, 9].

Detecting phase synchronization of chaotic systems requires a clear and unambiguous definition of phase for the oscillators in order to test the condition $\Delta\phi(t) = |\phi_1(t) - \phi_2(t)| < const < 2\pi$, where $\phi_1(t)$ and $\phi_2(t)$ are the phases of systems 1 and 2, respectively. As far as obtaining the phase of the oscillators is concerned, several methods have been proposed and the majority of them have as the input the projection of the attractor on a proper surface of section: Hilbert transform, Poincaré surface of section [7], curvature and recurrence plots [10, 11], localized sets [12], a method based on least squares analysis to detect the phase [13], short-time Fourier transforms and the continuous complex wavelet transform methodologies [14, 15, 16, 17]. However, none of these methods works properly for all the situations. In particular, the methodologies that use continuous complex wavelet transform are based on the complex Morlet wavelet to obtain the phase of a chaotic series yielding good results for coherent systems [15, 17]. However, these approach implies a high computational cost and the results may be not easy to understand, due to its redundancy framework when applied to large time series.

To overcome these difficulties, we propose here an approach to extract the phase of chaotic systems based on a efficient wavelet transform - the Dual-Tree Complex Wavelet Transform (DT-CWT).

2 METHODOLOGY

The Wavelet Transform (WT) is a linear transformation that can be used for the analysis of nonstationary signals in order to extract signal variations in frequency and to detect structures temporally and/or spatially localized [18]. Therefore, the WT provides a time-frequency representation of the signal using the multiresolution technique by which different frequencies are analyzed with different time resolutions [19].

In theory there are two forms of WT - continuous and discrete. When the parameters of scale and translation are continuous we have the so called Continuous Wavelet Transform (CWT), which transforms a one-dimensional signal (time) into a two-dimensional representation (time, scale) that is highly redundant. Otherwise, one have the Discrete Wavelet Transform (DWT) in which the parameters are discretely sampled.

The CWT has been used in previous works in order to calculate the phase of chaotic systems and to evaluate phase synchronization in coupled Rössler systems [14, 15, 16, 17].

In this paper we will use the *dual-tree* approach, described in [20], which is a relatively recent enhancement to the Discrete Wavelet Transform (DWT), with important additional properties: it is nearly shift invariant and directionally selective in two and higher dimensions.

2.1 Dual-Tree Complex Wavelet Transform

Mathematically, any finite energy signal $x(t)$ can be decomposed in terms of basis functions, let us say here $\phi_{j,n}(t)$, and their respective wavelet functions associated $\psi_{j,n}(t)$ via the following expansion:

$$x(t) = \sum_{n \in \mathbb{Z}} c(j, n) \phi_{j,n}(t) + \sum_{j,n \in \mathbb{Z}} d(j, n) \psi_{j,n}(t), \quad (1)$$

wherein $c(j, n)$ are scale coefficients and $d(j, n)$ are wavelet coefficients

$$c(j, n) = \langle x, \phi_{j,n} \rangle = \int x(t) \phi_{j,n}(t) dt \quad (2)$$

$$d(j, n) = \langle x, \psi_{j,n} \rangle = \int x(t) \psi_{j,n}(t) dt. \quad (3)$$

The coefficients $c(j, n)$ and $d(j, n)$ are calculated by using a very efficient, algorithm based on convolutions of the analyzed signal $x(n)$ with a discrete-in-time low-pass filter $h_0(n)$, a high-pass filter $h_1(n)$ and downsampling operations. Two channels are used in the implementations by filter bank, one being associated with the filter $h_0(n)$ and their scaling functions $c(j, n)$, while the other is associated with filter $h_1(n)$ and their wavelet coefficient $d(j, n)$. This is called the Mallat algorithm or Mallat-tree decomposition [19].

Despite its applicability and computational efficiency, DWT has four fundamental drawbacks, which are: oscillations, shift variance, aliasing and lack of directionality [20]. Since wavelets are bandpass functions, the wavelet coefficients tend to oscillate positive and negative around singularities and this considerably complicates the wavelet-based processing, making singularity extraction and signal modeling, in particular, very challenging [20]. In [20] a possible solution to resolve these deficiencies is presented by making use of complex wavelets $\psi_c(t) = \psi_h(t) + \imath \psi_g(t)$, wherein $\psi_h(t)$ denotes the real part and $\psi_g(t)$ the imaginary part.

The DT-CWT employs two real DWT's. The first DWT (upper filter bank or upper tree) consists of low-pass filters $h_0(n)$ and high-pass filters $h_1(n)$, while the second DWT (lower filter bank or lower tree) is composed by low-pass filters $g_0(n)$ and high-pass filters $g_1(n)$. Each DWT is composed by two different filter sets - so that each one satisfies the conditions for perfect reconstruction. Furthermore, they are jointly constructed so that the overall transformed is approximately analytical.

Considering a real signal $x(n)$, the real parts of the complex wavelets and of the complex wavelets coefficients, i.e., $\psi_h(t)$ and $d_h(j, n)$, respectively are the outputs of the first DWT. Now, considering the second DWT we have as output the imaginary parts of the complex wavelets and of the complex wavelets coefficients $\psi_g(t)$ and $d_g(j, n)$, respectively (see Figure 1(a)). So, considering the two DWT's, we obtain the complex wavelets as $\psi_c(t) = \psi_h(t) + \imath \psi_g(t)$, while the complex wavelets coefficients are $d_c(j, n) = d_h(j, n) + \imath d_g(j, n)$.

The Figure (1) shows a schematic representation of the decomposition of the real signal $x(n)$ using DT-CWT. In Figure (1)(a) the signal is decomposed in one scale ($j = 1$) and (b) in three scales ($j = 1, 2, 3$).

In order to satisfy the conditions for perfect reconstruction, the filters are designed such that the complex wavelets $\psi_c(t) := \psi_h(t) + \imath \psi_g(t)$ are approximately analytic [20]. To do this, they are designed so that $\psi_g(t)$ is approximated by the Hilbert pair of $\psi_h(t)$, in other words $\psi_g(t) \approx \mathcal{H}\{\psi_h(t)\}$, where \mathcal{H} denotes the Hilbert transform [21, 22, 23].

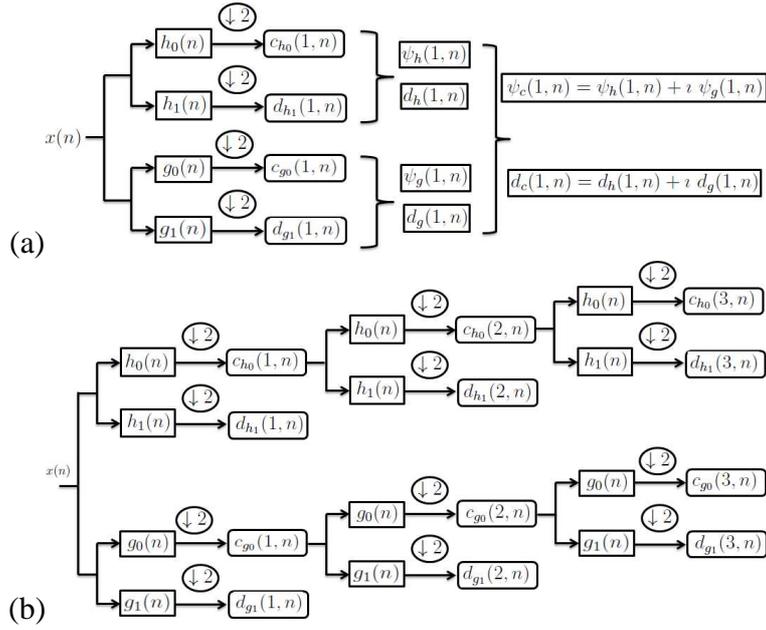


Figure 1: Schematic representation of the decomposition of the signal $x(n)$ in: (a) one scale ($j = 1$) and (b) three scales ($j = 1, 2, 3$), using the DT-CWT.

2.2 Phase Computation

In order to calculate the phase of a chaotic system using the approach of DT-CWT, the series for the variable x of the System 1, i.e, $x_1(n)$, is analyzed by DT-CWT multiscale. From this analysis we obtain the complex wavelet coefficients $d_c(j, n)$ at each scale j . After obtaining this coefficients, the energy $E(j, n)$ at each scale j is calculated as the square root of the modulus of complex wavelet coefficients, i.e, $E(j, n) = |d_c(j, n)|^2$. After the energy is calculated for each scale, the maximum energy $\max_j E(j, n) = E(J, n)$ is found in order to localize the scale J in which decomposed signal has its maximum energy. The scale J associated to the maximum energy is used to compute the phase $\phi(J, n) = \arctan\left(\frac{d_g(J, n)}{d_h(J, n)}\right)$ in a specific time (see Figure 2(a)). Subsequently, the same procedure is applied to the series of the variable x of the system 2, i.e, $x_2(n)$.

Then we calculated the phase of each system, $\phi_1(J, n)$ and $\phi_2(J, n)$. Next, the phase difference between the systems is calculated as $\Delta\phi(J, n) = \phi_1(J, n) - \phi_2(J, n)$ which allows one to evaluate if the phase synchronization condition is holding or not (see Figure 2(b)).

The Figure 2(a) is a schematic representation of the method for calculating the phase of a chaotic system by using the approach of DT-CWT. Figure 2(b) illustrates a schematic representation for the application of the method to two series $x_1(n)$ and $x_2(n)$, which are the series in the variable x of the system 1 and 2, respectively.

3 APPLICATION AND RESULTS

Here, this methodology is applied to data series generated by two non-identical Rössler systems [24] in non-phase-coherent regime and coupled bidirectionally by variable x . We use the Runge-Kutta 4th order method with integration time step equal to 0.01. The system is given by equations (4) and the parameters considered are based in [4].

$$\dot{x}_{1,2} = -\omega_{1,2} y_{1,2} - z_{1,2} + \eta (x_{2,1} - x_{1,2}) \quad (4)$$

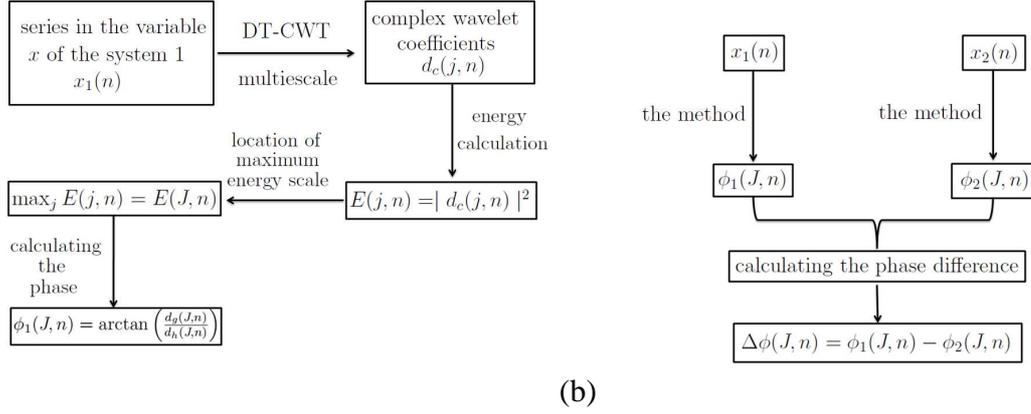


Figure 2: Schematic representation in (a) of the proposed method for calculating the phase using the x variable of a chaotic system and (b) of the application of the method in two series $x_1(n)$ and $x_2(n)$.

$$\begin{aligned}\dot{y}_{1,2} &= \omega_{1,2} x_{1,2} + 0.2925 y_{1,2} \\ \dot{z}_{1,2} &= 0.4 + z_{1,2} (x_{1,2} - 8.5)\end{aligned}$$

where $\omega_1 = 0.98$ and $\omega_2 = 1.02$.

The number of points in the time series analyzed is 2^{23} . For each system, using the x variable, the DT-CWT method was applied considering 23 scales of decomposition. Then, the maximum energy scale is localized and the phase is calculated on this scale. After that, we have the phase of each system, so that the phase difference between the systems is calculated in order to check the condition for phase synchronization.

The Fig. 3 illustrates in (a) a projection of the attractor in the xy plane; Scale versus the maximum energy in each scale considering a coupling strength of (b) 0.05, (c) 0.15, (d) 0.2; phase at $t = [1000, 2000]$ in the scale of maximum energy considering a coupling strength of (e) 0.05 and (f) 0.15 and (g) phase at $t = [1000, 1050]$ in the scale of maximum energy considering a coupling strength of 0.2.

The Fig 4 illustrates the phase difference using in (a) the wavelet method; (b) the traditional method for calculate the phase [8] and (c) is shown a zoom, $t = [0, 600]$ of the phase difference between systems considering $\eta = 0.15$ and using the traditional method and the wavelet method proposed. For small intensity of coupling $\eta = 0.05$, the phase difference increases with time characterizing no phase synchronization. By increasing more the coupling ($\eta = 0.15$), some plateaus of phase synchronization appear and when $\eta = 0.2$ the phase synchronization occurs. Note in Fig 4(c) that the traditional method can not find the plateaus of phase synchronization rightly. It is worth mentioning that the plateaus of the phase synchronization were properly verified on their associated series.

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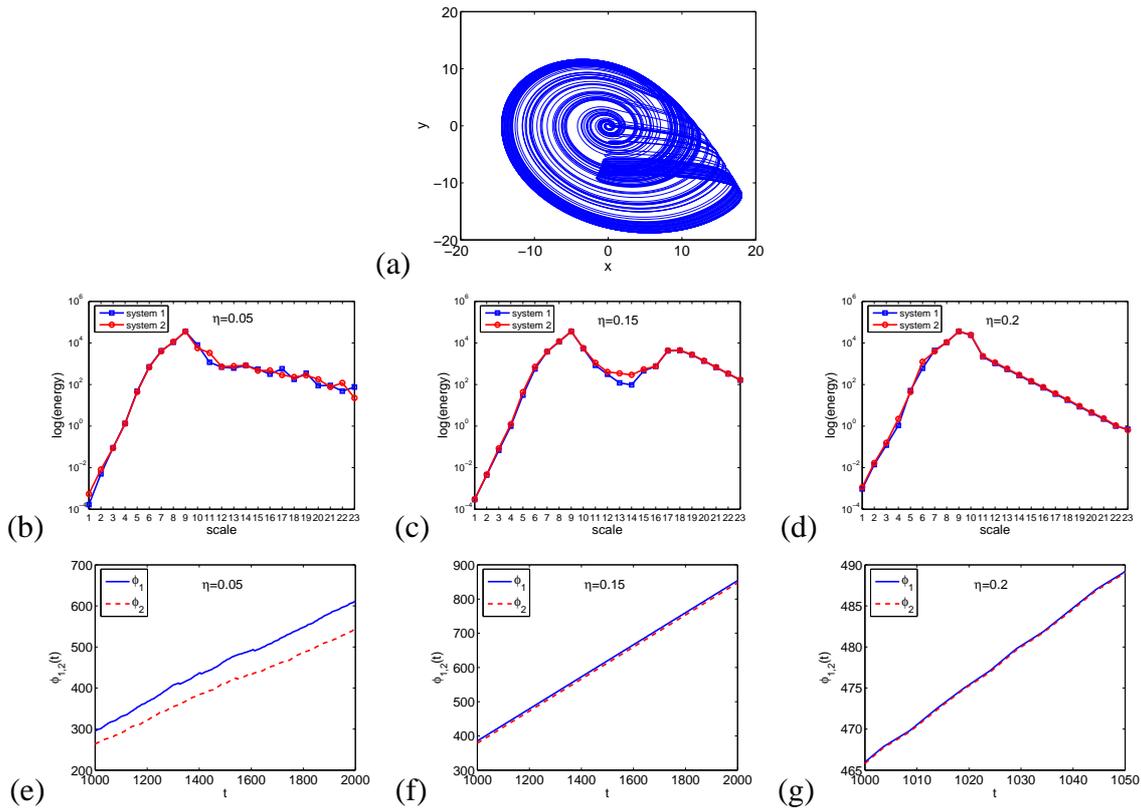


Figure 3: In (a) a projection of the attractor in the xy plane; scale versus the maximum energy in each scale considering a coupling strength of (b) 0.05, (c) 0.15, (d) 0.2; phase at $t = [1000, 2000]$ in the scale of maximum energy considering a coupling strength of (e) 0.05 and (f) 0.15 and (g) phase at $t = [1000, 1050]$ in the scale of maximum energy considering a coupling strength of 0.2.

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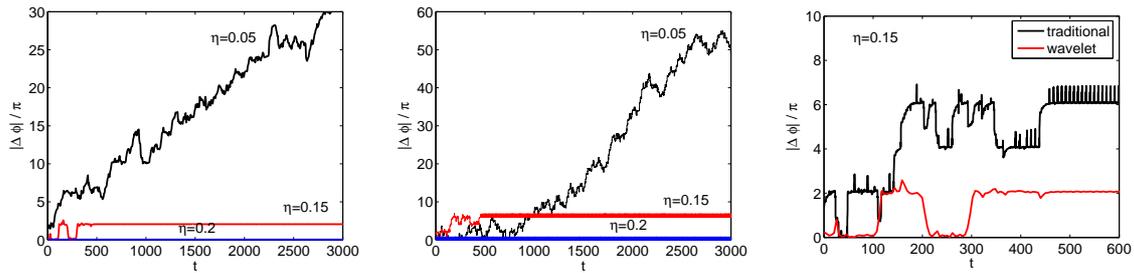


Figure 4: In (a) the wavelet method using the x variable; (b) the traditional method and (c) is shown a zoom, $t = [0, 600]$ of the phase difference between systems considering $\eta = 0.15$ using the traditional method for calculate the phase [8], and the wavelet method proposed.

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