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# SYSTEM ORDER REDUCTION FOR FLEXIBLE-LINK MANIPULATORS CONTROL PROBLEMS

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**Abstract.** *The order reduction method for singularly perturbed optimal filtering problem and linear-quadratic control problem is used for a single flexible-link manipulator model. We investigate these problems in the case when weak dissipative forces like viscous friction forces are present. The slow integral manifolds for the matrix Riccati equation of Kalman-Bucy filter and linear-quadratic control problem are constructed and it is shown that the method of integral manifolds allows us to reduce the dimension of filtering and control problems.*

## 1 INTRODUCTION

It is common knowledge that a wide range of processes in various aspects of nature are characterized by extreme differences in the rates of change of variables, so singularly perturbed ordinary differential systems are used as models of such processes.

Consider the ordinary differential system

$$\frac{dx}{dt} = f(x, y, t, \varepsilon), \;\; \varepsilon \frac{dy}{dt} = g(x, y, t, \varepsilon), \tag{1}$$

with vector variables $x$ and $y$, and a small positive parameter $\varepsilon$. The usual approach in the qualitative study of (1a) is to consider first the degenerate system

$$\frac{dx}{dt} = f(x, y, t, 0), \;\; 0 = g(x, y, t, 0),$$

and then to draw conclusions about the qualitative behavior of the full system (1) for sufficiently small $\varepsilon$. In the present paper we use a method for the qualitative asymptotic analysis of differential equations with singular perturbations. The method relies on the theory of integral manifolds, which essentially replaces the original system by another system on an integral manifold with dimension equal to that of the slow subsystem. Recall, that a smooth surface $S \in R^m \times R^n \times R$ is called an integral manifold of the system (1) if any trajectory of the system that has at least one point in common with $S$ lies entirely in $S$. Formally, if $\left(x(t_0), y(t_0), t_0\right) \in S$ then the trajectory $\left(x(t, \varepsilon), y(t, \varepsilon), t\right)$ lies entirely in $S$. The integral manifolds of system (1a) can be represented as the graphs of vector-valued functions

$$y = h(x, t, \varepsilon).$$

We also stipulate that $h(x, t, 0) = h^{(0)}(x, t)$, where $h^{(0)}(x, t)$ is a function whose graph is a sheet of the slow surface, and we assume that $h(x, t, \varepsilon)$ is a sufficiently smooth function of $\varepsilon$.

The motion along an integral manifold is governed by the equation

$$\dot{x} = f\left(x, h(x, t, \varepsilon), t, \varepsilon\right).$$

If $x(t, \varepsilon)$ is a solution of this equation, then the pair $\left(x(t, \varepsilon), y(t, \varepsilon)\right)$, where $y = h\left(x(t, \varepsilon), t, \varepsilon\right)$, is a solution of the original system (1a), since it defines a trajectory on the integral manifold.

Consider the *associated* subsystem, that is,

$$\frac{dy}{d\tau} = g\left(x, y, t, 0\right), \; \tau = t / \varepsilon,$$

treating $x$ and $t$ as parameters. We shall assume that some of the steady states $y^0 = y^0(x, t)$ of this subsystem are asymptotically stable and that a trajectory starting at any point of the domain approaches one of these states as closely as desired as $t \to \infty$. This assumption will hold, for example, if the matrix

$$(\partial g / \partial y)\left(x, h^0(x, t), t, 0\right)$$

is stable for $y^0 = h^0(x, t)$. In this case the slow integral manifold exists and can be found as an asymptotic expansion in powers of $\varepsilon$ [1,2].

The case in which the last assumption is violated is called critical. We distinguish the following subcases:

- The Jacobian matrix $g_y\left(x,h^0\left(x,t\right),t,0\right)$ is singular on some subspace of $R^m \times R^n \times R$. In that case, system (1) is referred to as a singular singularly perturbed system. In this case it is possible to introduce new variables in such a way that the transformed differential system has a structurally hyperbolic fast subsystem of lower dimension. This means that the original differential system has a slow integral manifold of higher dimension [4].
- The Jacobian matrix $g_y\left(x,h^0\left(x,t\right),t,0\right)$ has eigenvalues on the imaginary axis with nonvanishing imaginary parts. If this part of the eigenvalues is pure imaginary but, after taking into account the perturbations of higher order, they move to the complex left half-plane, then the system under consideration has stable slow integral manifolds. Such a situation is typical for some problems of mechanics of gyroscopes and manipulators with high-frequency and weakly damped transient regimes [3-4].

In the paper we consider problems which are characterized by both these subcases together.

## 2 FLEXIBLE-JOINT MANIPULATOR

Consider a model of a rigid-link flexible joint manipulator. The equations of motions have the form:

$$\begin{aligned} &J_1\ddot{q}_1 + Mgl\sin q_1 + c\left(\dot{q}_1 - \dot{q}_m\right) + k\left(q_1 - q_m\right) = 0, \\ &J_m\ddot{q}_m - c\left(\dot{q}_1 - \dot{q}_m\right) - k\left(q_1 - q_m\right) = u, \end{aligned} \tag{2}$$

where $J_m$ is inertia moment of a motor installed in the base, $J_1$ is inertia moment of a link, $M, l$ are mass and length of a link, $c$ is attenuation factor, $k$ is stiffness of a flexible joint. Besides, $q_1$ is rotation angle of a link, $q_m$ is rotation angle of a motor shaft. The manipulator consists of a drive connected with a link by means of a flexible joint.

We consider the motion of manipulator in the absence of high damping. The model also features weak dissipative forces of the viscous friction type. In this case, the model differs from the one explained in the paper [5]. In actual practice, a flexible joint usually can be a metal spring or an elastic joint with a sufficiently high stiffness. The differential system (2) may be considered as a singularly perturbed one with a small positive parameter $\varepsilon^2 = 1/k$. Introduce new variables $x = Dq$, where

$$D = \begin{pmatrix} J_1/\left(J_1+J_m\right) & 0 & J_m/\left(J_1+J_m\right) & 0 \\ 0 & J_1/\left(J_1+J_m\right) & 0 & J_m/\left(J_1+J_m\right) \\ 1 & 0 & -1 & 0 \\ 0 & \varepsilon & 0 & \varepsilon \end{pmatrix}, \tag{3}$$

and the following values to simplify the subsequent calculations:

$$\alpha = \frac{1}{J_1+J_m},\ \beta = Mgl\alpha,\ \gamma = \frac{1}{\alpha J_1 J_m},\ I = \frac{1}{J_1},\ I_m = \frac{1}{J_m},\ \varphi = \beta J_m,\ \mu = \frac{1}{\alpha}, \tag{4}$$

If we consider the motions of the system (2) about the equilibrium position then the use of new variables (3), (4) yields the system:

$$\dot{x} = Ax + Bu, \tag{5}$$

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\beta & 0 & -\varphi\alpha & 0 \\ 0 & 0 & 0 & \dfrac{1}{\varepsilon} \\ -\varepsilon I\beta\mu & 0 & -\dfrac{1}{\varepsilon}\gamma-\varepsilon I\varphi & -c\gamma \end{pmatrix}, B=\begin{pmatrix} 0 \\ \alpha \\ 0 \\ -\varepsilon I_m \end{pmatrix}, x=\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Note that $x_0$, $x_1$ are slow variables and $x_2$, $x_3$ are fast variables.

## 3 MATRIX RICCATI EQUATION

Consider the linear-quadratic control problem:

$$\varepsilon\dot{x}=A(t,\varepsilon)x+B(t,\varepsilon)u,\ x\in R^{n+m},\ x(0)=x_0, \tag{6}$$

$$J=\frac{1}{2}x(1)^T F_x x(1)+\frac{1}{2}\int_0^1\left(x(t)^T Q_x x(t)+u(t)^T Ru(t)\right)dt. \tag{7}$$

with:

$$A=\begin{pmatrix} \varepsilon A_1 & \varepsilon A_2 \\ A_3 & A_4 \end{pmatrix}, B=\begin{pmatrix} \varepsilon B_1 \\ B_2 \end{pmatrix}, F=\begin{pmatrix} F_1 & F_2 \\ F_2^T & F_3 \end{pmatrix}, Q=\begin{pmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{pmatrix},$$

where $A_1, F_1, Q_1\in R^{m\times m}$, $A_2, F_2, Q_2\in R^{m\times n}$, $A_4, F_3, Q_3\in R^{n\times n}$, $B_1\in R^{m\times r}$, $R\in R^{r\times r}$. We suppose that all matrices can be found as asymptotic expansion in integer powers of the small parameter $\varepsilon$

$$A_i(t,\varepsilon)=\sum_{j\ge 0}\varepsilon^j A_{i1}(t),\ i=\overline{1,4};\ B_i(t,\varepsilon)=\sum_{j\ge 0}\varepsilon^j B_{ij}(t),\ i=1,2;\ Q_i(t,\varepsilon)=\sum_{j\ge 0}\varepsilon^j Q_{ij}(t),\ i=\overline{1,3};$$

$$R(t,\varepsilon)=\sum_{j\ge 0}\varepsilon^j R_j(t);\ F_i(t,\varepsilon)=\sum_{j\ge 0}\varepsilon^j F_{ij}(t),\ i=\overline{1,3}.$$

First $m$ variables of the $x$ vector are slow, and the other $n$ variables are fast. The solution of this problem is given by the formula:

$$u=-R^T\begin{pmatrix} B_1^T & \varepsilon^{-1}B_2^T \end{pmatrix}\begin{pmatrix} P_1 & \varepsilon P_2 \\ \varepsilon P_2^T & \varepsilon P_3 \end{pmatrix}x.$$

Matrices $P_1, P_2, P_3$ satisfies the following singularly perturbed matrix Riccati differential system of equations

$$\begin{aligned}
\dot{P}_1&=-P_1A_1-A_1^TP_1-P_2A_3-A_3^TP_2^T+P_1S_1P_1+P_1S_2P_2^T+P_2S_2^TP_1+P_2S_3P_2^T-Q_1,\\
\varepsilon\dot{P}_2&=-P_1A_2-P_2A_4-\varepsilon A_1^TP_2-A_3^TP_3+P_1S_2P_3+P_2S_3P_3+\varepsilon\left(P_1S_1P_2+P_2S_2^TP_2\right)-Q_2,\\
\varepsilon\dot{P}_3&=-P_3A_4-A_4^TP_3+P_3S_3P_3+\varepsilon\left(-P_2^TA_2-A_2^TP_2+\varepsilon P_2^TS_1P_2+P_2^TS_2P_3+P_3S_2^TP_2\right)-Q_3,
\end{aligned} \tag{8}$$

where $S_1 = B_1 R^{-1} B_1^T$, $S_2 = B_1 R^{-1} B_2^T$, $S_3 = B_2 R^{-1} B_2^T$, with boundary conditions $P_1(1,\varepsilon) = F_1$, $P_2(1,\varepsilon) = \varepsilon^{-1} F_2$, $P_3(1,\varepsilon) = \varepsilon^{-1} F_3$. The corresponding Lurie equation for the last equation in the system (8) has the form $-MA_{40} - A_{40}^T M + MS_{30}M - Q_{30} = 0$, where $S_{30} = B_{20} R_0^{-1} B_{20}^T$.

Usually, Lurie equation has positive-definite solution $M(t)$ and eigenvalues of the $D_{40} = A_{40} - S_{30}M$ matrix have negative real parts for $t \in [0,1]$. In this case, the boundary functions method or the integral manifolds method can be applied. System (8) has the stable integral manifold of slow motions. In the linear-quadratic control problem for a system with low dissipation the main role is played by the linear operator $LX = XA_{40} + A_{40}^T X$, where eigenvalues of the $A_{40}$ matrix are pure imaginary. The linear operator has a nontrivial kernel, since differences $\left(\lambda_i(t) - \lambda_j(t)\right), i, j = 1, \ldots, n$ form its spectrum [7]. Thus, the dimension of the slow integral manifold of (8) is greater than the amount of variables in the $P_1$ matrix.

Note that finding of a numerical solution of the system (8) is connected with significant amount of calculations because of singular perturbations. To find a sufficiently precise numerical solution of such a system it will be required to divide a time interval into a big number of points due to presence of fast variables. In practice, use of large computation facilities is impossible in some cases, therefore it is critical to reduce the amount of computation operations required to solve these tasks with desired accuracy.

## 4 LINEAR-QUADRATIC PROLEM

### 4.1 Cost functional

We investigate the linear-control problem with the cost functional:

$$J = \frac{1}{2} q(1)^T F q(1) + \frac{1}{2} \int_0^1 \left( q(t)^T Q q(t) + u(t)^T R u(t) \right) dt, \tag{9}$$

where $F = diag\{f_0, f_1, 0, 0\}$, $Q = diag\{d_0, d_1, d_2, d_3\}$, $R = r$. Note that $d_1, d_3$ in the $Q$ matrix are of the order $O(\varepsilon)$. Using the change of variables $x = Dq$ we obtain

$$J = \frac{1}{2} x(1)^T F_x x(1) + \frac{1}{2} \int_0^1 x(t)^T Q_x x(t) + u(t)^T R u(t),$$

where $F_x = \left(D^{-1}\right)^T F D^{-1}$, $Q_x = \left(D^{-1}\right)^T Q D^{-1}$.

### 4.2 Integral manifold

We designate the elements of the covariance matrix $P$ as follows:

$$P = \begin{pmatrix} p_0 & p_1 & p_3 & p_6 \\ p_1 & p_2 & p_4 & p_7 \\ p_3 & p_4 & p_5 & p_8 \\ p_6 & p_7 & p_8 & p_9 \end{pmatrix} \tag{10}$$

There are four slow variables in the Riccati differential system for a linear-quadratic regulator. The fourth variable is expressed as a linear combination of variables $p_{13} = \gamma p_{10} + p_{12}$, where $p_{10} = \varepsilon p_9$, $p_{11} = \varepsilon p_8$, $p_{12} = \varepsilon p_5$. Then we calculate the terms of asymptotic expansion and obtain equation for slow variables on invariant manifold

$$\dot{\xi} = \begin{pmatrix} v_0 - 2\beta\xi_1 - \frac{1}{r}\alpha^2\xi_1^2 + \varepsilon^2 \frac{1}{\gamma}\left(\frac{2}{r}I_m\alpha v_2\xi_1 - \frac{2}{r}I_m\alpha^2\varphi\xi_1^2 - 2I\mu\beta v_2 + 2I\beta\varphi\xi_1\right) \\ \xi_0 - \beta\xi_2 - \frac{1}{r}\alpha^2\xi_1\xi_2 + \varepsilon^2 \frac{1}{\gamma}\left(\frac{1}{r}I_m\alpha v_2\xi_2 + I\beta\varphi\xi_2 - \frac{2}{r}I_m\alpha^2\varphi\xi_1\xi_2\right) \\ 2\xi_1 - \frac{1}{r}\alpha^2\xi_2^2 + \varepsilon v_1 - \varepsilon^2 \frac{2}{\gamma}I_m\alpha^2\varphi\xi_2^2 \\ \gamma v_5 - \gamma c\xi_3 + \varepsilon\left(v_3 - \frac{I_m^2\xi_3^2}{4\gamma r}\right) + \varepsilon^2\left(\frac{1}{2}I\varphi v_5 + 2\alpha\varphi v_4\right) \end{pmatrix} + O\left(\varepsilon^3\right).$$

Note that differential equation for $\xi_3$ is isolated and its solution can be found precisely. The solution of $\xi_3$ on the integral manifold has the form:

$$\xi_3 = \frac{\sigma}{a}\left(1 - \frac{2c_1e^{2\sigma t}}{1 + 2c_1e^{2\sigma t}}\right) - \frac{b}{2a}, \tag{11}$$

where

$$a = -\frac{\varepsilon I_m^2}{4\gamma r},\ b = -\gamma c,\ c_0 = \gamma v_5 + \varepsilon v_3 + \varepsilon^2\left(\frac{1}{2}I\varphi v_5 + 2\alpha\varphi v_4\right), \sigma = \sqrt{b^2/4 - ac_0},\ c_1 = \frac{2\sigma - b}{2\sigma + b}.$$

### 4.3 Numerical simulation of control actions

A MATLAB program has been written for a numerical simulation of the system's motion in presence of control actions. The complete control action is shown in black in figure 1-a; a control action calculated on the integral manifold shown in grey. As shown in the figure, control actions differ insignificantly. Figure 1-b shows the difference of system's flow in presence of the complete control and the control on the manifold. Actually, these trajectories coincide with a high degree of accuracy.

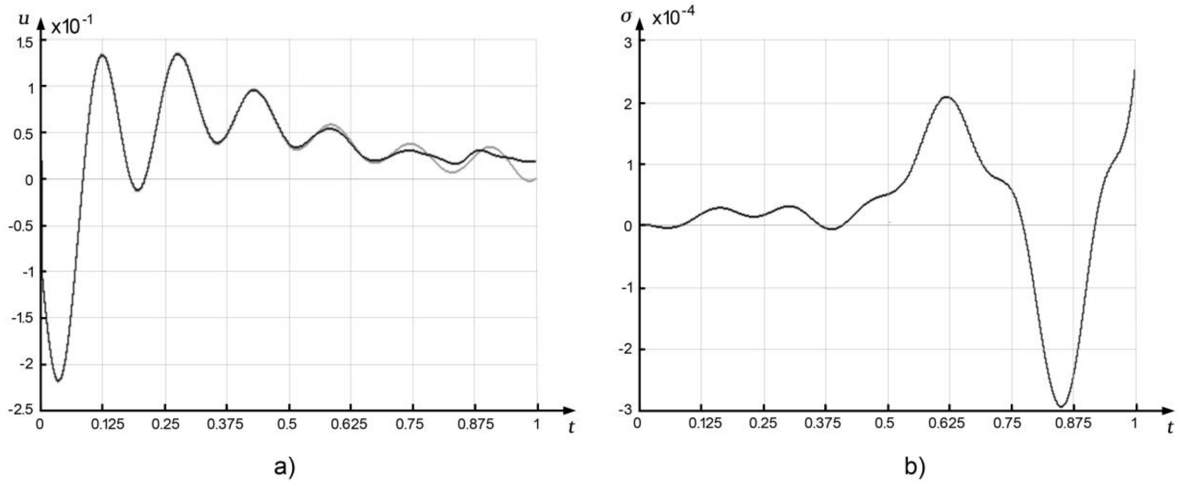


Figure 1: a) the complete control action (black) and control on the manifold (grey); b) the difference in system's trajectories.

## 5 OPTIMAL FILTERING PROBLEM

### 5.1 Statement of the problem

Consider the motion of the rigid-body flexible joint manipulator driven by white noise. The equations of motion take the form

$$\begin{aligned} &J_1\ddot{q}_1 + Mgl\sin q_1 + c\left(\dot{q}_1 - \dot{q}_m\right) + k\left(q_1 - q_m\right) = 0, \\ &J_m\ddot{q}_m - c\left(\dot{q}_1 - \dot{q}_m\right) - k\left(q_1 - q_m\right) = \dot{w}, \end{aligned} \tag{12}$$

where $\dot{w}$ is zero-mean stationary Gaussian white noise with covariance coefficient $q$. The measurement equation is $z = q_1 + \dot{v}$, where $\dot{v}$ is zero-mean stationary Gaussian white noise with covariance coefficient $r$. We suppose that $\dot{w}$ and $\dot{v}$ are independent. The problem is to estimate the state of the system by using measurements $z$.

We use the same change of variables $x = Dq$ (3) and investigate the motions of the system (12) about the equilibrium position. Actually, we obtain the same system (5), where the control action $u$ is replaced with a random action $\dot{w}$. The measurement equation takes the form $z = Cx + \dot{v}$, $C = \begin{pmatrix}1 & 0 & aJ_m & 0\end{pmatrix}$. To solve the problem we will use the Kalman-Bucy filter. Note that all the conclusions for matrix Riccati equations for linear-quadratic problem are the same for the optimal estimation problem. The only difference is that boundary conditions at the right end for $P$ matrix are replaced with boundary conditions at the left end.

### 5.2 Integral manifold

We designate the elements of the covariance matrix $P$ of the Kalman-Bucy filter according to (10). In this case the dimension of a slow subsystem equals to four because of the presence of the fourth slow variable $p_{10} = \gamma p_5 + p_9$. We construct the slow movements integral manifold with the accuracy of $O\left(\varepsilon^2\right)$. The motions along the invariant manifold are described by the following equation

$$\dot{\xi} = \begin{pmatrix} 2\xi_1 - 2\frac{1}{r}\alpha J_m h_0 \xi_0 - \frac{1}{r} h_0^2 \alpha^2 J_m^2 - \frac{1}{r}\xi_0^2 \\ \xi_2 - \beta\xi_0 - \alpha\varphi h_0 - \frac{1}{r}\xi_0\xi_1 - \frac{1}{r}\alpha J_m h_0 \xi_1 - \frac{1}{r} h_0 h_1 \alpha^2 J_m^2 - \frac{1}{r}\alpha J_m h_1 \xi_0 \\ -2\beta\xi_1 - 2\alpha\varphi h_1 - \frac{1}{r}\alpha^2 J_m^2 h_1^2 - \frac{2}{r}\alpha J_m h_1 \xi_1 - \frac{1}{r}\xi_1^2 + \alpha^2 q \\ -2\gamma c \xi_3 + 2\gamma^2 c h_2 - 2\varepsilon I \varphi h_5 - \varepsilon \frac{2}{\alpha} I \beta h_3 - \frac{1}{r}\upsilon + \varepsilon^2 \frac{q}{J_m^2} \end{pmatrix} + O\left(\varepsilon^3\right), \quad (13)$$

where $\upsilon = \left(\gamma h_0^2 + h_3^2 + \alpha^2 J_m^2 \left(h_5 + h_2^2 \gamma\right) + 2\alpha J_m \left(h_3 h_5 + \gamma h_0 h_2\right)\right)$ , and $h_0 - h_5$ are fast variables on the integral manifold. The equations for $h_0 - h_5$ are sufficiently bulky and we don't provide them here.

### 5.3 Numerical simulation of Kalman-Bucy filters

We compare the results by using a MATLAB program that allows to provide numerical simulation of system motion. Figure 2-a shows the trajectory of a manipulator link in presence of a random external action. At zero time a manipulator link is turned at some angle, due to this fact it starts rotating. The result of solution modeling of the stochastic differential equations of the movement was used as an input signal for filters. Figure 2-b show error charts for Kalman-Bucy filters. The black chart is the complete filter error, the grey chart is the manifold filter error.

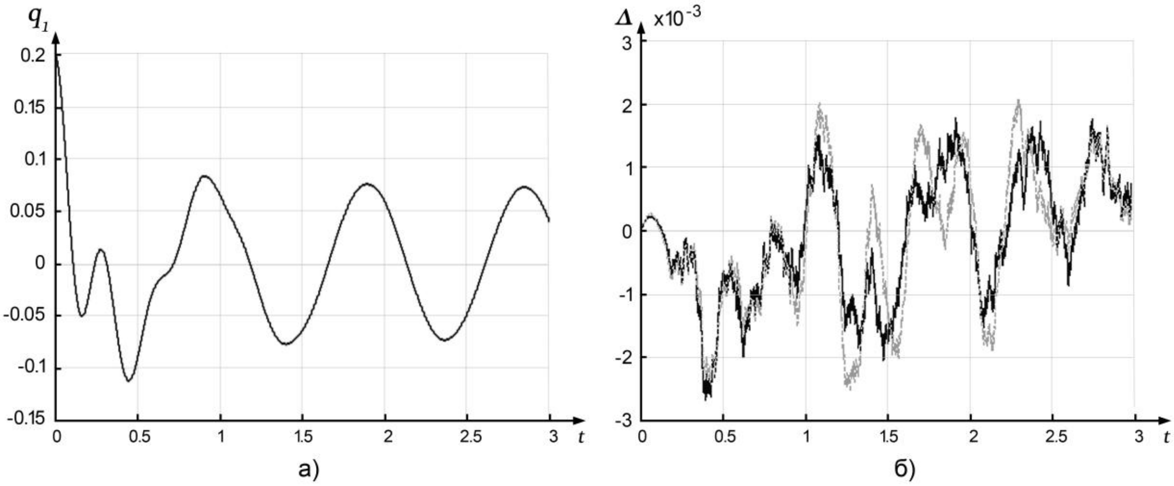


Figure 2: a) the result of a numerical simulation of a manipulator motion in presence of an external random action;b) the complete Kalman-Bucy filter error (black) and the manifold filter error (gray).

The accuracy of the manifold filter is comparable to the accuracy of the complete filter, but the calculation of the manifold filter requires significantly less number of algebraic operations.

## 6 CONCLUSION

We have investigated linear-quadratic control and optimal estimation problems for the rigid-link flexible-joint manipulator. It has been shown that the reduction of dimensions of these problems can be done by means of the integral manifold method.

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