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NONLINEAR MODEL OF BELT-TENSIONER SYSTEM WITH EXPERIMENTAL VALIDATION

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Abstract. *In this study it is implemented a mathematical model of a belt transmission system with automatic tensioner (spring loaded), considering the rotating degrees of freedom of the accessory and drive pulleys and the transverse vibration of the belt, similar to the system studied by Beikmann et al. [1]. Nonlinear effects are introduced by consideration the elastic behavior of the belt when subjected to large deformations, and geometrical coupling introduced by tensioner on belt span endings.*

Theoretical modal analysis is performed on the linear portion of equations to determine natural frequencies and modeshapes, so the system can be discretized through a modal superposition and the nonlinear terms are treated as external loads. A two mode expansion was used, and nonlinear terms involving cubic restoring forces were found. Experimental setup is used to validate some parameters of the model, such as natural frequencies and belt tension for several driving speeds. These results were useful to feed the equations with data necessary for time response simulations for both linear and nonlinear cases.

Nonlinear effects due to initial conditions or torque fluctuations on accessory drive pulleys, such as energy exchange between modes, internal resonances, and multiple harmonic responses are studied on theory and on the test setup, as well.

1 INTRODUCTION

A recurrent problem in automotive belt drive systems is the occurrence of high levels of rotational vibration produced by torque fluctuation on pulleys. In automotive applications, particularly, this phenomenon is very usual due to engine crankshaft cyclic motion that produces speed and torque variations, known as engine irregularity. Besides rotational vibration, tension variation is also produced, and this can cause loss of tensioning and consequent belt slippage or increase of dynamic tension, accelerating belt wear by fatigue process.

Several authors studied the vibration problem on belt drives. Ulsoy *et al.* [2] described actuating torques on tensioner arm due to belt tension components and possible Mathieu instabilities caused by parametric excitation of the system. Beikmann *et al.* [3] proposed a method to determine natural frequencies of a prototypical system based on Holzer's Method. Nonlinear responses are also evaluated by means of modal superposition [1]. Damping effects produced by belt viscoelastic properties are studied by Zhang and Zu [4], who proposes an analytical solution to the problem and identifies nonlinear effects through Perturbation Methods.

In this study, we propose the application of Beikmann's model to a real engine front end accessory drive. Firstly, Experimental Modal Analysis is performed to determine modal parameters, such as natural frequencies, damping factors and modeshapes for test subject. Different types of modeshapes are observed, and some effects of belt bending stiffness, which are not considered on model, as well.

Natural frequencies present dependence on operational speed and belt tension. To verify speed influence, engine acceleration was performed while measurements were taken at tensioner arm, as system has natural excitation due to belt frequency harmonics, natural frequencies are excited at several speeds and a RPM map is produced.

Time responses to torque fluctuation excitations and initial conditions are evaluated by theoretical model, which is feed by data obtained in Experimental Modal Analysis. Nonlinear effects are verified to torque conditions that could only be evaluated within a dynamometer.

Finally, nonlinearity of the system is observed through Frequency Response Function variation due to different excitation amplitude applied.

2 MATHEMATICAL MODEL

2.1 Equationing

In order to describe belt drive dynamic behavior, it was implemented a mathematical model, similar to the proposed by Beikmann *et. al* [1]. It is considered a system with three pulleys, as represented on Figure 1, where pulley 1 correspond to crankshaft pulley, pulley 2 is the idler pulley from tensioner, and pulley 4 is accessory pulley. There is a spring loaded tensioner that applies a force normal to the belt.

Some considerations made are: $1)$ Belt properties, such as elastic modulus E, transverse section A and linear density m are time invariant; $2)$ Belt bending stiffness is negligible; $3)$ Damping effects are initially neglected; $4)$ System has constant operational speed c.

Degrees of freedom $\theta_i(t)$ represent rotational vibration on pulleys and tensioner arm and $w_i(x,t)$ and $u_i(x,t)$ represent belt spans transverse and longitudinal vibration, respectively.

System equations considering assumptions made by Beikmann *et al.* [1] and eliminating static equilibrium terms become

$$mw_{i,tt} + 2mcw_{i,xt} - P_{ti}w_{i,xx} = P_{di}w_{i,xx} \qquad i = 1,2,3 \tag{1}$$

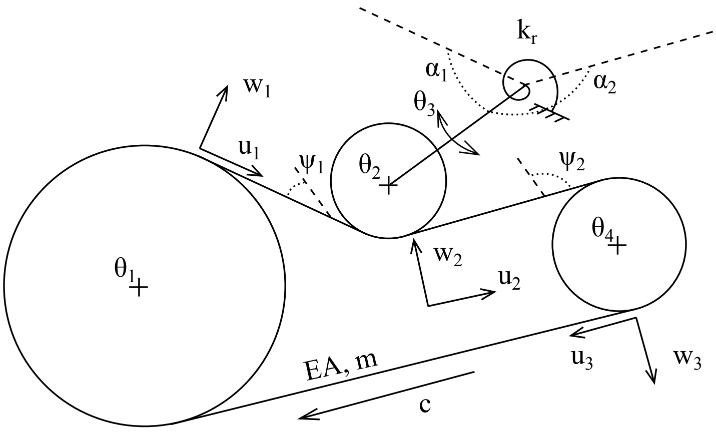

Figure 1: Geometrical model of the system.

$$m_1\chi_{1,tt} = k_1\left(\chi_3\cos\psi_1 + \chi_2 - \chi_1\right) - k_3\left(\chi_1 - \chi_4\right) + F_{d1} + P_{d1NL} - P_{d3NL} \tag{2}$$

$$m_2\chi_{2,tt} = k_2\left(\chi_3\cos\psi_2 + \chi_2 - \chi_1\right) - k_1\left(\chi_3\cos\psi_1 + \chi_2 - \chi_1\right) + F_{d2} + P_{d2NL} - P_{d1NL} \tag{3}$$

$$\begin{aligned} m_3\chi_{3,tt} = &\left[-P_{t1}w_{1,x}(l_1,t) + mcw_{1,t}(l_1,t)\right]\sin\psi_1 + \left[P_{t2}w_{2,x}(0,t) - mcw_{2,t}(0,t)\right]\sin\psi_2 + \\ &- k_1(\chi_3\cos\psi_1 + \chi_2 - \chi_1)\cos\psi_1 - k_2(\chi_3\cos\psi_1 + \chi_2 - \chi_1)\cos\psi_2 + \\ &+ F_{d3} - k_4\chi_3 - P_{d1NL}\cos\psi_1 + P_{d2NL}\cos\psi_2 \end{aligned} \tag{4}$$

$$m_4\chi_{4,tt} = k_3\left(\chi_1 - \chi_4\right) - k_2\left(\chi_3\cos\psi_1 + \chi_2 - \chi_1\right) + F_{d4} + P_{d3NL} - P_{d2NL} \tag{5}$$

where $k_4 = k_s + k_{gr}$, and

$$k_s = \frac{k_r}{r_3^2} \qquad k_{gr} = \frac{1}{r_3}\left(P_{t1}\sin\psi_1 - P_{t2}\sin\psi_2\right)$$

Subscripts '$_{,k}$' represent partial derivative with respect to variable k.

Substitutions $F_{di} = M_{di}/r_i$, $m_i = J_i/r_{i,}^2$, $\chi_i = r_i\theta_i$ and $k_i = EA/l_i$ are considered, where M_{di} is the dynamic torque applied externally on pulleys and tensioner, J_i are discrete elements moment of inertia. Belt operational belt tension is given by $P_{oi} = P_{ti} + mc^2$, where P_{ti} are tractive tension and mc^2 is the centrifugal tension. There is no equation for longitudinal displacement coordinate u_i because its motion is uncoupled of the remaining of the system.

Dynamic tension is given by $P_{di} = P_{diL} + P_{diNL}$, where its linear portion is

$$P_{diL} = k_i\left[u_i(l_i,t) - u_i(0,t)\right] \tag{6}$$

and it is due to belt finite elongation, while nonlinear portion is due to infinitesimal deformation and it is given by

$$P_{diNL} = \frac{k_i}{2}\int_0^{l_i} w_{i,x}^2 dx \tag{7}$$

System equations can be written in matrix form

$$[M]\left\{\ddot{W}\right\}+[G]\left\{\dot{W}\right\}+[K]\{W\}=\{Q\} \tag{8}$$

where $\{W\}=\left\{\begin{array}{ccccccc} w_1 & w_2 & w_3 & \chi_1 & \chi_2 & \chi_3 & \chi_4 \end{array}\right\}^T$ is the displacement vector and the excitation vector, which contains external excitation terms and nonlinear terms is

$$\{Q\}=\left\{\begin{array}{c} P_{d1}w_{i,xx} \\ P_{d2}w_{2,xx} \\ P_{d3}w_{3,xx} \\ F_{d1}+P_{d1NL}-P_{d3NL} \\ F_{d2}+P_{d2NL}-P_{d1NL} \\ +F_{d3}-P_{d1NL}\cos\psi_1+P_{d2NL}\cos\psi_2 \\ F_{d4}+P_{d3NL}-P_{d2NL} \end{array}\right\} \tag{9}$$

2.2 Modal Superposition

As it can be seen from Eqs. (1-5), system is defined by partial differential equations and direct methods cannot be applied for numerical integration. To overcome this problem, Beikmann *et al.* [1] and Moon and Wickert [5] use modal superposition based on linear portion of equations so the spatial dependence is written as a combination of modeshapes, while nonlinear terms are expressed as excitation terms. Then time response can be obtained by numerical integration of ordinary differential equation system. First, it is necessary to express the matricial equation system in state space form, once system is gyroscopic and cannot be properly evaluated in configuration space. System become

$$[A]\left\{\dot{U}\right\}+[B]\{U\}=\{X\} \tag{10}$$

$$[A]=\left[\begin{array}{cc} M & 0 \\ 0 & K \end{array}\right] \quad \text{e} \quad [B]=\left[\begin{array}{cc} G & K \\ -K & 0 \end{array}\right] \tag{11}$$

with $\{U\}=\left\{\begin{array}{cc} \dot{W} & W \end{array}\right\}^T$ and $\{X\}=\left\{\begin{array}{cc} Q & 0 \end{array}\right\}^T$.

Using a solution $\{U\}=\left\{\overline{U}\right\}e^{\lambda t}$ on homogeneous part of Eq. 10 we will get a complex conjugate eigenvalue pair $\lambda=\pm i\omega_r$, with corresponding complex eigenvector $\left\{\overline{U}_r\right\}$ [6]. Eigenvector can be expressed in terms of real and imaginary parts

$$\left\{\overline{U}_r\right\}=\frac{1}{\sqrt{2}}\left(\left\{\overline{Y}_r\right\}+i\left\{\overline{Z}_r\right\}\right) \tag{12}$$

Due to symmetry properties of generalized mass matrix $[A]$ and generalized stiffness matrix $[B]$, according to [1] and [7], orthogonality properties are

$$\begin{array}{ccc} \left\langle \overline{Y}_r, A\overline{Y}_s\right\rangle=\delta_{rs} & \left\langle \overline{Z}_r, A\overline{Z}_s\right\rangle=\delta_{rs} & \left\langle \overline{Z}_r, A\overline{Y}_s\right\rangle=0 \\ \left\langle \overline{Y}_r, B\overline{Y}_s\right\rangle=0 & \left\langle \overline{Z}_r, B\overline{Z}_s\right\rangle=0 & \left\langle \overline{Z}_r, B\overline{Y}_s\right\rangle=\omega_s\delta_{rs} \end{array} \tag{13}$$

where it is necessary to use inner product to overcome spatial dependence of eigenfunctions that express belt spans behavior.

Based on Modal Superposition principle, system's response can be expressed as a finite expansion of eigenvectos.

$$\{U(t)\} \approx \sum_{r=1}^{n} \left\{\xi_r(t)\overline{Y}_r + \eta_r(t)\overline{Z}_r\right\} = [P]\,\{V(t)\} \tag{14}$$

where $[P] = \left[\begin{array}{ccccc} \overline{Y}_1 & \overline{Z}_1 & \overline{Y}_2 & \overline{Z}_2 & \dots \end{array}\right]$ is the modal matrix with the modeshapes considered on the expansion and vector $\{V(t)\}$ contains the generalized modal coordinates for real (ξ_r) and imaginary (η_r) parts of eigenvectors.

Substituting Eq. 14 in Eq. 10 and making the inner product of the result with modal matrix $[P]$, we have

$$\langle P, AP\rangle \left\{\dot{V}(t)\right\} + \langle P, BP\rangle \{V(t)\} = \langle P, X(t)\rangle \tag{15}$$

$$[I]\left\{\dot{V}(t)\right\} + [H]\{V(t)\} = \{R(t)\} \tag{16}$$

where matrix $[I]$ is the identity matrix and $[H]$ is a block-diagonal matrix with system's natural frequencies.

Each couple of equations from Eq. 16 describes time behavior of a mode r. It is possible include now damping to the system by assuming equivalent viscous damping factor ζ_r to each mode r.

$$\dot{\xi}_r(t) - \omega_r\eta_r(t) = \left\langle \overline{Y}_r, X(t)\right\rangle - 2\zeta_r\omega_r\xi_r(t) \tag{17}$$

$$\dot{\eta}_r(t) + \omega_r\xi_r(t) = \left\langle \overline{Z}_r, X(t)\right\rangle \tag{18}$$

Vector $\{X\}$ presents dependence with transverse displacement coordinates $w_i(x,t)$ through dynamic tension terms. This variables should be expressed in terms of eigenvector expansion.

$$w_i(x,t) \approx \sum_{r=1}^{n} \left(\xi_r\overline{v}_{ir}^{R} + \eta_r\overline{v}_{ir}^{I}\right) \tag{19}$$

When Eq. 19 is substituted on dynamic tension terms in excitation vector, nonlinear terms in the form of cubic restoring forces appear, such as $\xi_r\xi_h\xi_k$, $\xi_r\xi_h\eta_k$, $\xi_r\eta_h\xi_k$, and other possible combinations of generalized modal coordinates.

3 EXPERIMENTAL AND NUMERICAL RESULTS

3.1 Experimental Modal Analysis

A belt drive system of a real engine was studied on experimental analysis. For this, an engine front end was mounted on a inertial test bench and powered by an electric motor. In a first step, it was performed a modal analysis of the system, through random excitation applied tangentially on pulley 1 by an electromagnetic shaker and responses were measured by accelerometers fixed tangentially to pulleys and tensioner arm. Transverse belt span vibration was measured using proximity probes placed next to belt. Figure 2 shows test setup used.

Modal parameters such as natural frequencies and damping factors were obtained by Least Squares Complex Exponential Method [8]. Table 1 presents first natural frequencies identified. As one can see, in contrast to theory in mathematical model, vibration modeshapes due

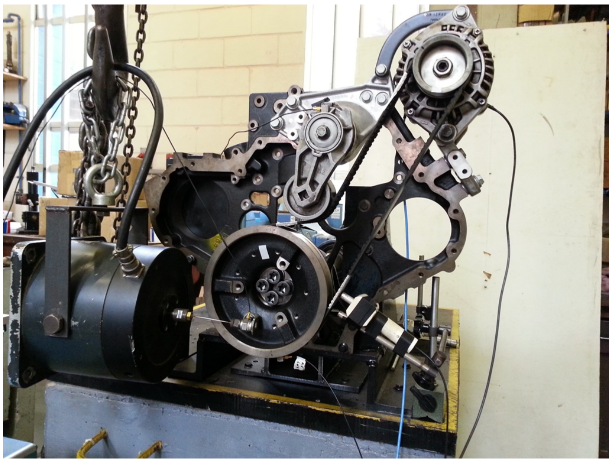

Figure 2: Experimental Setup.

Table 1: Natural Frequencies and Damping Factors estimated.

Mode	Frequency [Hz]	Damping Factor [%]	Description
1	58.95	0.66	First rotational mode
2	71.83	0.20	First transverse mode of span 3
3	109.58	4.53	Rotational + bending mode
4	121.56	0.30	First transverse mode of span 2
5	124.79	2.60	Rotational + bending mode
6	143.96	0.23	Second transverse mode span 3

to belt bending stiffness are verified. This modes are characterized by coupling between rotational modes and span 3 modes (which are considered uncoupled by its contour conditions on theory[3]).

Modeshapes obtained in experimental modal analysis are presented in Figure 3, for first and second natural frequencies estimated.

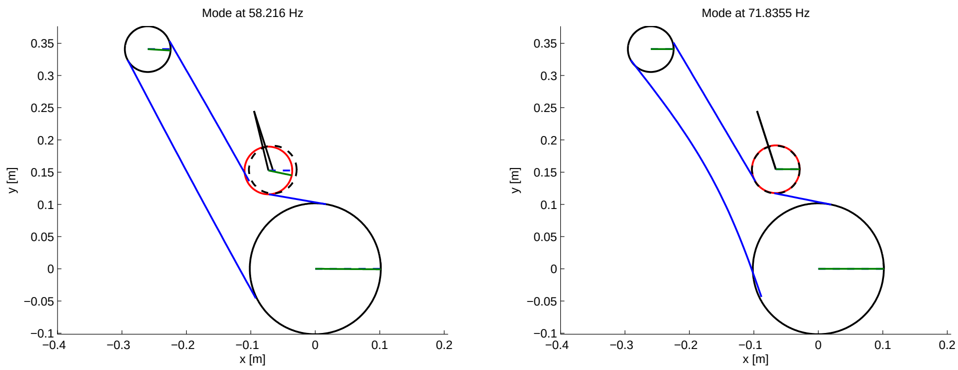

Figure 3: Modeshapes obtained through modal estimation.

3.2 Speed Dependence of Natural Frequencies

As the system equations present gyroscopic terms, it is natural to observe influences on modal parameters depending on operating speed. To verify this, experimentally, it was performed measurements tangentially on tensioner arm while system was accelerated producing a run-up curve. With vibration data acquired, it was possible to plot vibration response as a function of operating speed as a RPM map, presented on Figure 4.

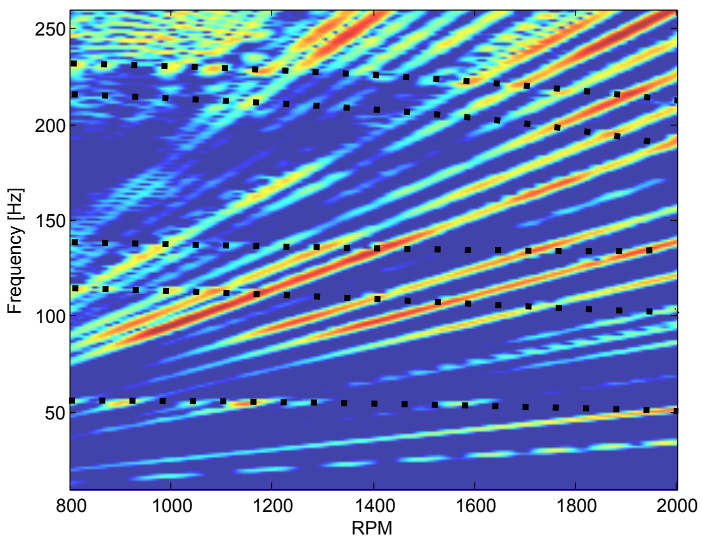

Figure 4: RPM map for run up of operating speed excitation and identified natural frequencies decay.

Rotational mode next to 56Hz at 900RPM is the same obtained by modal analysis, but operational speed effects cause natural frequency to decrease, as expected[7, 3]. First and second mode of span 2 are also present. Modes at 141Hz and 247Hz next to 800RPM region indicates possible rotational or bending modes. Modes at 71.83Hz in zero operational speed and 109.58Hz cannot be identified since they are not observable from tensioner degree of freedom.

3.3 Theoretical Time Response

Theoretical model can be used now by considering finite modal expansion. It was used a two mode expansion to describe tensioner coupled subsystem (modes at 58.95Hz and 109.58Hz) and a single mode to describe span 3 behavior (71.83Hz mode), which is linearly uncoupled. Model was implemented on Matlab®version 7.1 R14. Once inertial properties of the system are unknown, modal properties obtained by Experimental Modal Analysis were used to feed theoretical model.

Nonlinear effects can only be observed if belt and tensioner spring elastic properties are supplied. They are: $EA = 177000$N and $k_r = 53.4$Nm/rad.

As a first test case, we consider torque fluctuation excitation of amplitudes of 20Nm and 40Nm applied on pulley 1 at frequency of 71.83Hz and initial condition $\xi_3 = 0.05$ and $\xi_3 = 0.1$. Initial conditions must be considered for mode 3, otherwise it will remain uncoupled and cannot be excited by external loads.

Figure 5 shows time response of modal generalized coordinates for both excitation conditions. It would be expected that generalized coordinate 3, which correspondes to 71.83Hz mode, should respond as a resonance, but it doesn't occur because it was not directly coupled to linear portion of the system, even with initial condition applied, as it can be seem for 20Nm excitation case.

Response for 40Nm excitation case present resonant behavior though. Coupling of mode 3 is produced by nonlinear effects achieved by higher amplitude of excitation.

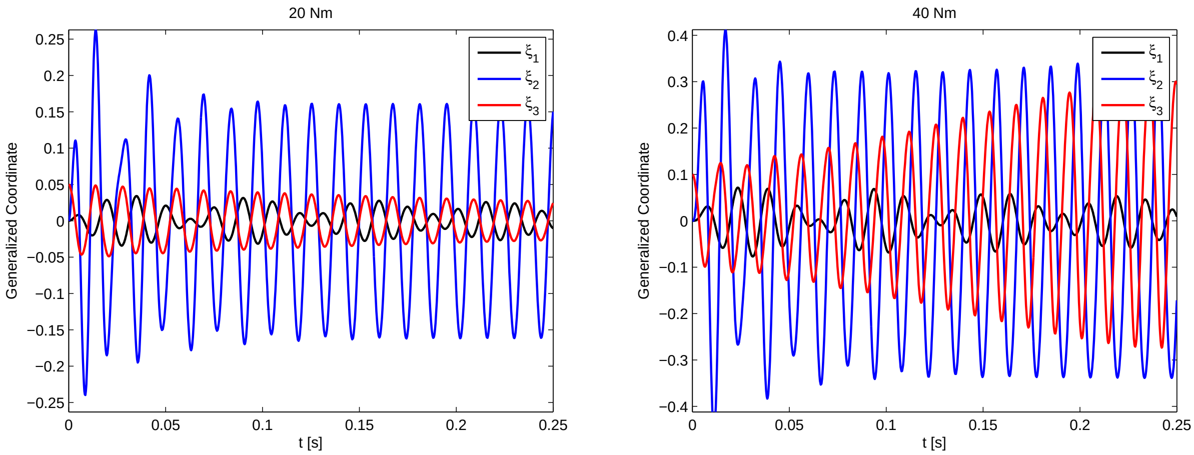

Figure 5: Generalized coordinate responses to torque fluctuation excitation applied on pulley 1 at 71.83 Hz.

In a second test case, system was excited by initial condition $\xi_1 = 0.4$ while ξ_2 and ξ_3 were kept null. Figure 6 shows results for system with regular damping as obtained from experimental modal analysis and time response for same excitation condition but system is considered undamped.

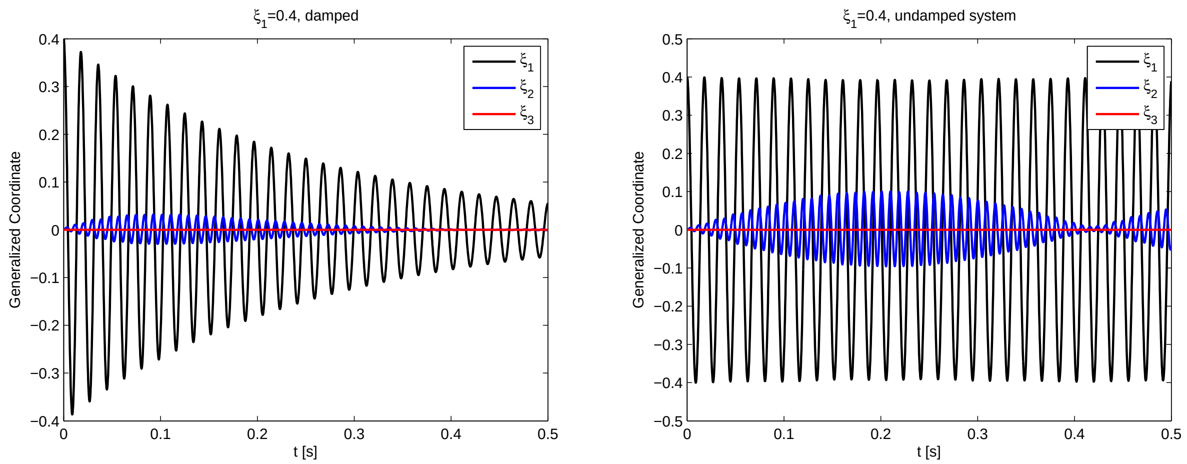

Figure 6: Generalized coordinate responses to initial condition $\xi_1 = 0.4$ for damped and undamped cases

It is observed from Figure 6 that for damped case that generalized coordinate ξ_2 is slightly excited, due to nonlinear terms, once linear theory predicts uncoupled response for generalized modal coordinates. Nonlinear effect is highlighted when damping factors are neglected, ξ_2 has its response amplified and beating behavior is observed pointing out to an Internal Resonance characteristic.

Coupling due to nonlinear terms $\xi_1^2\xi_2$ and $\xi_1\xi_2^2$ promote exchange of energy between modes when fraction ω_2/ω_1 is approximately a rational number and this can produce resonances on non-excited modes through this energy exchange[1]. If exists a detuning parameter $\sigma = 2\omega_1 - \omega_2$, nonlinear theory predicts that beat phenomenon or resonance will occur as consequence to internal exchange of energy. As σ is smaller, beat modulation tends to become faster until it reach resonant behavior when σ is zero [9].

3.4 Nonlinearity Identification On Test Setup

To investigate nonlinear effects on experimental setup it was applied a sine sweep excitation tangentially on pulley 1, with different amplitudes. Plots of obtained Inertances are presented on Figure 7 for frequency range tested. According to linear theory, system Frequency Response Functions should be independent of excitation amplitude, and that was not observed.

FRFs show clear dependence on excitation force magnitude, and as pointed by [8], inertance amplitude decrease as excitation amplitude increase and natural frequencies of response present slight change.

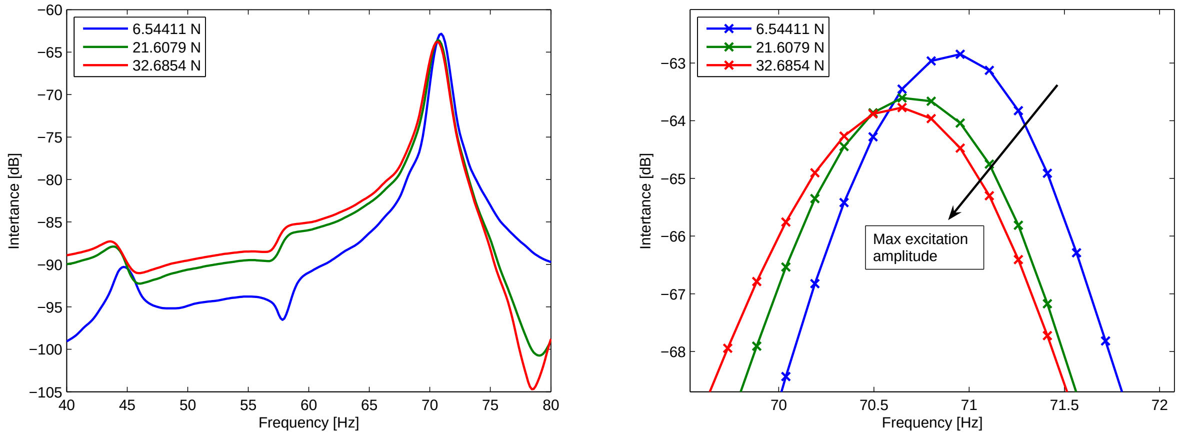

Figure 7: Inertance for experimental system subjected to different excitation amplitudes and detailed view next to third natural frequency.

4 CONCLUSIONS

As proposed, a real front end accessory drive with spring loaded tensioner was tested and had some of its dynamic characteristics identified. Theoretical model was implemented to determine time response of the system to different excitation conditions based on modal superpotion and considering nonlinear terms produced by infinitesimal elongation of belt spans.

To feed the theoretical model, Experimental Modal Analysis were performed with random excitation and modal parameters were estimated by means of Least Squares Complex Exponential Technique. System presented, besides expected transverse and rotational modes, coupling of free span modes and rotational modes, which indicates belt bending stiffness influence on the real system.

Natural frequencies dependence with operating speed were evaluated through vibration measurement to engine acceleration response. RPM map was generated and some natural frequencies and its variations according to operating speed were observed, proving that gyroscopic effects play an important role on system description.

Once Modal Analysis were performed, data obtained were used to supply theoretical model, and within elastic properties of belt and tensioner, and system geometry, time responses to external and nonlinear excitations were obtained. It was observed strong dependence of nonlinear behavior to amplitude of excitation which can produce coupling between modes of vibration (which are uncoupled, according to linear theory).

Internal resonance were also verified to numerically tested system. Since rate between natural frequencies ω_2 and ω_1 is approximately 2, exchange of energy occurs and the slight detuning produces beats on mode 2 response, as predicted by nonlinear theory [9]. This is an important effect, once the kind of response produced by Internal Resonance could generate large responses at frequency ranges where it would not be expected. And it could be used as an design parameter for belt drives subjected to great dynamic loads.

Finally, nonlinear behavior on test setup was observed by means of its FRFs. Linear systems have its FRFs invariant in relation to excitation amplitude, and tested system showed opposite behavior, with FRF amplitude and natural frequencies presenting variations according to excitation amplitude.

5 ACKNOWLEDGEMENTS

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