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ON THE APPEARANCE OF NEIMARK-SACKER BIFURCATION IN A NON IDEAL SYSTEM

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Abstract. *In this paper, by using the Averaging Method, is proved, for the first time, the existence of Neimark-Sacker Bifurcation in a class of Non Ideal mechanical systems. In fact such problem drops in a more general research program on quenching for these mechanical systems*

1 INTRODUCTION

Mathematical models of real systems are usually idealized by prescribing the forcing term as a known function. In reality, for a great number of structures this is not the case, and such structures are called non-ideal. A number of authors studied this class of mechanical systems. We can mention some of them Kononenko [3], and Balthazar et al. [4], [5].

In this paper, it is proved, for the first time, that a class of non ideal systems exhibit a Neimark-Sacker Bifurcation. In fact such problem drops in a more general research program on quenching for this class of mechanical systems and this paper can be considered a natural follow-up of [1]. The basic technique used along this paper is the Averaging Theorem, see for example [2]. And, as the reader will see, the amount of algebraic computation is huge.

2 A NON IDEAL PROBLEM WITH THREE DEGREES OF FREEDOM

In this section let us consider a mechanical system compound of a damped linear spring linked to a wall and to a principal body. On this principal body there is a *DC* motor, with limited supply power, which rotates a small mass m_0 . Both, the spring and the body-motor set, are at the same height from the ground. In this system, M_0 denotes the mass of the fixed part of the mechanism such as its base. In the left part of the Figure 1 this mass is indicated by the region in gray. The mass of the rotating parts of the motor is denoted by M_1 and its moment of inertia by J. Thus M, given by M_0+M_1, means the mass of the body-motor set. The constant k_1 is the stiffness of the spring. The resistance of the oscillatory motion is a linear force $c_1 x_1'$. Let r be the distance between the mass m_0 and the axis of rotation of the *DC* motor. It is assumed that $-\pi \leq \theta < \infty$. Moreover, on the right, there is another body, the Secondary one, appended to the Principal Body by a linear spring. This Secondary Body is linked to another vertical wall, parallel to first wall, by a damped nonlinear spring. The non-linearity of this spring is cubic. All components of this mechanical system are in line. Of course this system has three degrees of freedom and is a non-ideal one.

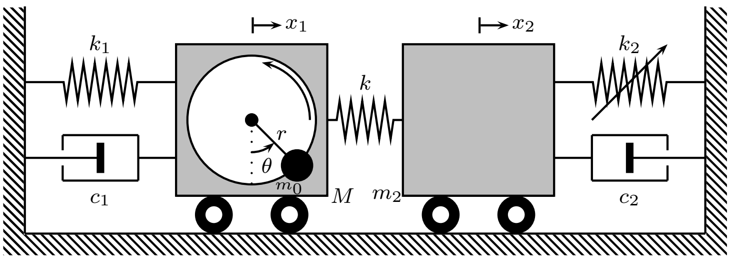

Figure 1: A Non-Ideal System with Three Degree of Freedom.

The equations of motion of the mechanical system given in Figure 1, in its dimensionless form and after the introduction of a small parameter ϵ are given by

$$x_1''(t) = -x_1(t) + \epsilon_1 \left(x_2(t) - x_1(t)\right) - \epsilon\, \lambda_1 x_1'(t) + \epsilon\, q_1 \Big(\sin(\theta(t)) \left(\theta'(t)\right)^2 - \cos(\theta(t))\, \theta''(t) \Big), \tag{1}$$

$$x_2''(t) = \epsilon_2 \left(x_1(t) - x_2(t)\right) - \epsilon\, \lambda_2\, x_2'(t) - \epsilon\, \gamma\, x_2(t)^3, \tag{2}$$

$$\theta''(t) = \epsilon^2 \Big(\Gamma(\theta'(t)) - q_2 \cos(\theta(t)) \, (x_1''(t)) + q_3 \sin(\theta(t)) \Big). \tag{3}$$

Of course the parameters in Eqs.(1 - 3) depend on the physical constants k_1, k_2, k, M, m_0, m_2, r as well as of the moment of inertia of rotating parts and acceleration of gravity. The function Γ is the difference between the driving torque of the source energy, in this case a motor, and the resistive torque applied to the rotor. Again this function is given in its dimensionless form. The Eqs.(1 - 2) will be written as a pair of perturbed harmonic oscillators coupled with Eq.3.

Writing Eqs.(1 - 3) as a first-order system, taking

$$\begin{cases} r_1 = \dfrac{\sqrt{\sqrt{{\epsilon_2}^2 + 2\,\epsilon_1\,\epsilon_2 - 2\,\epsilon_2 + {\epsilon_1}^2 + 2\,\epsilon_1 + 1} + \epsilon_2 + \epsilon_1 + 1}}{\sqrt{2}}, \\ r_2 = \dfrac{\sqrt{-\sqrt{{\epsilon_2}^2 + 2\,\epsilon_1\,\epsilon_2 - 2\,\epsilon_2 + {\epsilon_1}^2 + 2\,\epsilon_1 + 1} + \epsilon_2 + \epsilon_1 + 1}}{\sqrt{2}} \end{cases} \tag{4}$$

and using the change of variables given in equation (24) of [1] one gets

$$x_{11}'(t) = u_1(t), \tag{5}$$

$$u_1'(t) = -r_2^2\, x_{11}(t) + \epsilon \frac{r_2}{c_3} \Big(c_1 \Big(q_1\, p(t)^2 \sin(\theta(t)) - \lambda_1 (v_1(t) + u_1(t)) \Big) + c_2 \Big(-\gamma (c_7\, x_{21}(t) + c_8\, x_{11}(t))^3 - \lambda_2 \, (c_7\, v_1(t) + c_8\, u_1(t)) \Big) \Big) + O(\epsilon^2), \tag{6}$$

$$x_{21}'(t) = v_1(t), \tag{7}$$

$$v_1'(t) = -r_1^2\, x_{21}(t) + \epsilon \frac{r_1}{c_3} \Big(c_4 \Big(q_1\, p(t)^2 \sin(\theta(t)) - \lambda_1 \, (v_1(t) + u_1(t)) \Big) + c_5 \Big(-\gamma (c_7\, x_{21}(t) + c_8\, x_{11}(t))^3 - \lambda_2 (c_7\, v_1(t) + c_8\, u_1(t)) \Big) \Big) + O(\epsilon^2), \tag{8}$$

$$\theta'(t) = p(t), \tag{9}$$

$$p'(t) = \epsilon^2 \Big(\Gamma(p(t)) + q_3 \sin(\theta(t)) + (c_9 \, (x_{21}(t) + x_{11}(t)) - c_6\, q_2 \, (c_7\, x_{21}(t) + c_8\, x_{11}(t))) \cos(\theta(t)) \Big), \tag{10}$$

where $c_1, \ldots, c_9$ are adequate rational functions of r_1, r_2. Assuming now the *ansatz*

$$p(t) = r_1 + \epsilon\, p_1(t), \tag{11}$$

it follows from Eq.9 that t can be written as a function of θ. Therefore, a non autonomous system in the θ variable with five equations is obtained. By using the change of variables given in equation (27) of [1], where $\omega_0 = r_1$, and assuming the following *resonance condition*

$$r_1 = \frac{3}{2} r_2, \tag{12}$$

one obtains a θ–dependent system with period equal to $6\,\pi$. Computing the average of this last system , one has

$$\begin{aligned} \widehat{z}_{11}' &= F_1\left(\widehat{z}_{11},\widehat{z}_1,\widehat{z}_{21},\widehat{z}_2,\widehat{p}_1\right),\\ \widehat{z}_1' &= F_2\left(\widehat{z}_{11},\widehat{z}_1,\widehat{z}_{21},\widehat{z}_2,\widehat{p}_1\right),\\ \widehat{z}_{21}' &= F_3\left(\widehat{z}_{11},\widehat{z}_1,\widehat{z}_{21},\widehat{z}_2,\widehat{p}_1\right),\\ \widehat{z}_2' &= F_4\left(\widehat{z}_{11},\widehat{z}_1,\widehat{z}_{21},\widehat{z}_2,\widehat{p}_1\right),\\ \widehat{p}_1' &= F_5\left(\widehat{z}_{11},\widehat{z}_1,\widehat{z}_{21},\widehat{z}_2,\widehat{p}_1\right) \end{aligned} \tag{13}$$

where $F_i,\ i=1,\ldots,5$ are those functions given in [1, pg.279]. An equilibrium point of Eq.13 is given by

$$\begin{gathered} \widehat{z}_{11}^s=0,\ \widehat{z}_1^s=0,\ \widehat{z}_{21}^s=-\frac{8\,\Gamma\left(\frac{3\,r_2}{2}\right)}{9\,q_2\,{r_2}^2},\ \widehat{z}_2^s=\frac{1}{3\,q_2\,r_2}\sqrt{\frac{R_1}{R_2}\,\Gamma\left(\frac{3\,r_2}{2}\right)},\\ \widehat{p}_1^s=-\frac{\left(\begin{array}{c} 24\,\gamma\,q_1\,r_2^2\,\left(9\,r_2^2-4\right)\,\sqrt{R_2}\,\Gamma\left(\frac{3\,r_2}{2}\right)^2\\ +\sqrt{R_1}\,q_2\,\left(r_2^2-1\right)\,R_2^2\,\sqrt{\Gamma\left(\frac{3\,r_2}{2}\right)}\end{array}\right)}{40\,q_2\,\left(r_2^2-1\right)\,R_2^{\frac{3}{2}}\,\Gamma\left(\frac{3\,r_2}{2}\right)}. \end{gathered} \tag{14}$$

Of course, in Eq.14 the parameters c_i were substituted by the original ones. It is assumed that the following inequalities hold:

$$\Gamma\left(\frac{3\,r_2}{2}\right)>0, \tag{15}$$

$$R_1=243\,q_1\,q_2\,\left(1-r_2^2\right)\,{r_2}^3-16\,R_2\,\Gamma\left(\frac{3\,r_2}{2}\right)>0, \tag{16}$$

$$R_2=\left(9\,\lambda_2-9\,\lambda_1\right)\,{r_2}^2-4\,\lambda_2+9\,\lambda_1>0 \tag{17}$$

and

$$\left(9\,\lambda_2-9\,\lambda_1\right)\,{r_2}^2-9\,\lambda_2+4\,\lambda_1<0. \tag{18}$$

Hence, by computing the jacobian of Eq.13 at the equilibrium point Eq.14 one gets the matrix $M_1\oplus M_2$ where M_1 is 2×2 matrix and M_2 is a 3×3 one. Their expressions are given by the equations (36), (37) respectively, in [1]. It is worth to note, and this fact will be used in the next section, that in view of Eq.18, the eigenvalues of M_1 always have negative real parts and this fact is independent of the value of γ, the coefficient of the non-linear spring in the mechanical system.

3 SEARCHING FOR HOPF BIFURCATIONS IN THE AVERAGED SYSTEM

If $\gamma=\gamma_0$, where

$$\gamma_0=-\frac{81\,q_2^2\,r_2\,\left(r_2^2-1\right)^2\,R_2^{\frac{3}{2}}}{16\,\sqrt{R_1}\,\left(9\,r_2^2-4\right)\,\Gamma\left(\frac{3\,r_2}{2}\right)^{\frac{3}{2}}}, \tag{19}$$

thus the eigenvalues of M_2 are given by

$$\pm i\,Z,\ -\frac{2\,R_2}{15\,r_2} \tag{20}$$

where

$$Z = \frac{R_1^{\frac{1}{4}}\,\Gamma\left(\frac{3\,r_2}{2}\right)^{\frac{1}{4}}}{3\,r_2\,R_2^{\frac{1}{4}}}.$$

By translating the equilibrium point in Eq.14 to the origin, writing

$$\gamma = \delta + \gamma_0, \tag{21}$$

replacing the equations in Eq.13, and after a huge, but straightforward computation, one gets that

$$\begin{aligned} \widehat{z}_{21}' &= N_1\left(\widehat{z}_{11}, \widehat{z}_1, \widehat{z}_{21}, \widehat{z}_2, \widehat{p}_1, \delta\right), \\ \widehat{z}_2' &= N_2\left(\widehat{z}_{11}, \widehat{z}_1, \widehat{z}_{21}, \widehat{z}_2, \widehat{p}_1, \delta\right), \\ \widehat{p}_1' &= N_3\left(\widehat{z}_{11}, \widehat{z}_1, \widehat{z}_{21}, \widehat{z}_2, \widehat{p}_1, \delta\right), \\ \widehat{z}_{11}' &= N_4\left(\widehat{z}_{11}, \widehat{z}_1, \widehat{z}_{21}, \widehat{z}_2, \widehat{p}_1, \delta\right), \\ \widehat{z}_1' &= N_5\left(\widehat{z}_{11}, \widehat{z}_1, \widehat{z}_{21}, \widehat{z}_2, \widehat{p}_1, \delta\right) \end{aligned} \tag{22}$$

where

$$\begin{aligned} N_1 - & \frac{z_2\,R_2\,z_{21}\left(144\,k_2\,r_2^5\,\delta\,Z^6 - k_1^2\,R_1\,q_2^2\right)}{45\,k_1^2\,R_1\,q_2\,r_2^2\,Z^2} + \frac{z_{21}^2\left(144\,k_2\,r_2^5\,\delta\,Z^6 - k_1^2\,R_1\,q_2^2\right)}{2160\,k_1^2\,q_2\,r_2^3\,Z^4} \\ & + \frac{3\,z_{11}^2\left(144\,k_2\,r_2^5\,\delta\,Z^6 - k_1^2\,R_1\,q_2^2\right)}{40\,k_2^2\,q_2\,r_2^3\,Z^4} + \frac{z_2^2\left(144\,k_2\,r_2^5\,\delta\,Z^6 - k_1^2\,R_1\,q_2^2\right)}{1620\,k_1^2\,q_2\,r_2^5\,Z^4} \\ & + \frac{3\,z_1^2\left(144\,k_2\,r_2^5\,\delta\,Z^6 - k_1^2\,R_1\,q_2^2\right)}{40\,k_2^2\,q_2\,r_2^5\,Z^4} + \frac{z_2\,z_{21}^2\left(144\,k_2\,r_2^5\,\delta\,Z^6 - k_1^2\,R_1\,q_2^2\right)}{6480\,k_1^2\,r_2^4\,Z^6} \\ & + \frac{z_2\,z_{11}^2\left(144\,k_2\,r_2^5\,\delta\,Z^6 - k_1^2\,R_1\,q_2^2\right)}{40\,k_2^2\,r_2^4\,Z^6} + \frac{z_2^3\left(144\,k_2\,r_2^5\,\delta\,Z^6 - k_1^2\,R_1\,q_2^2\right)}{14580\,k_1^2\,r_2^6\,Z^6} \\ & + \frac{z_1^2\,z_2\left(144\,k_2\,r_2^5\,\delta\,Z^6 - k_1^2\,R_1\,q_2^2\right)}{40\,k_2^2\,r_2^6\,Z^6} - \frac{48\,k_2\,r_2^4\,R_2\,z_{21}\,\delta\,Z^6}{5\,k_1^2\,R_1\,q_2^2} \\ & + \frac{8\,k_2\,r_2\,z_2\,\delta\,Z^4}{45\,k_1^2\,q_2^2} - \frac{4\,p_1\,Z^2}{3\,q_2\,r_2} - \frac{4\,p_1\,z_2}{9\,r_2^2}, \end{aligned}$$

$$\begin{aligned} N_2 = & -\frac{72\,p_1\,r_2^2\,R_2\,Z^4}{R_1\,q_2} + \frac{z_{21}\left(36\,r_2^2\,R_2\,Z^2 - R_1\right)\left(36\,r_2^2\,R_2\,Z^2 + R_1\right)}{360\,R_1\,r_2^2\,Z^2} \\ & - \frac{3\,q_2\,R_2\,z_{21}^2}{40\,Z^2} - \frac{81\,k_1^2\,q_2\,R_2\,z_{11}^2}{20\,k_2^2\,Z^2} - \frac{q_2\,z_2^2\,R_2}{90\,r_2^2\,Z^2} \\ & - \frac{81\,k_1^2\,z_1^2\,q_2\,R_2}{20\,k_2^2\,r_2^2\,Z^2} + \frac{R_1\,q_2\,z_2\,z_{21}}{1080\,r_2^3\,Z^4} + \frac{R_1\,q_2^2\,z_{21}^3}{2880\,r_2^2\,Z^6} \\ & + \frac{9\,k_1^2\,R_1\,q_2^2\,z_{11}^2\,z_{21}}{160\,k_2^2\,r_2^2\,Z^6} + \frac{R_1\,q_2^2\,z_2^2\,z_{21}}{6480\,r_2^4\,Z^6} \\ & + \frac{9\,k_1^2\,z_1^2\,R_1\,q_2^2\,z_{21}}{160\,k_2^2\,r_2^4\,Z^6} + p_1\,z_{21} - \frac{2\,z_2\,R_2}{15\,r_2}, \end{aligned}$$

$$N_3 = \frac{3\,q_2\,r_2\,z_{21}}{4},$$

$$N_4 = \frac{9\,z_1\,R_2\,z_{21}\left(144\,k_2\,r_2^5\,\delta\,Z^6-k_1^2\,R_1\,q_2^2\right)}{10\,k_1\,R_1\,k_2\,q_2\,r_2^2\,Z^2} - \frac{z_1\,z_2\left(144\,k_2\,r_2^5\,\delta\,Z^6-k_1^2\,R_1\,q_2^2\right)}{60\,k_1\,k_2\,q_2\,r_2^5\,Z^4}$$
$$-\frac{z_1\,z_{21}^2\left(144\,k_2\,r_2^5\,\delta\,Z^6-k_1^2\,R_1\,q_2^2\right)}{160\,k_1\,k_2\,r_2^4\,Z^6} - \frac{81\,k_1\,z_1\,z_{11}^2\left(144\,k_2\,r_2^5\,\delta\,Z^6-k_1^2\,R_1\,q_2^2\right)}{320\,k_2^3\,r_2^4\,Z^6}$$
$$-\frac{z_1\,z_2^2\left(144\,k_2\,r_2^5\,\delta\,Z^6-k_1^2\,R_1\,q_2^2\right)}{360\,k_1\,k_2\,r_2^6\,Z^6} - \frac{81\,k_1\,z_1^3\left(144\,k_2\,r_2^5\,\delta\,Z^6-k_1^2\,R_1\,q_2^2\right)}{320\,k_2^3\,r_2^6\,Z^6}$$
$$+z_{11}\,A + z_1\,B\left(\delta\right) - \frac{4\,p_1\,z_1}{9\,r_2^2},$$

$$N_5 = -\frac{9\,R_2\,z_{11}\,z_{21}\left(144\,k_2\,r_2^5\,\delta\,Z^6-k_1^2\,R_1\,q_2^2\right)}{10\,k_1\,R_1\,k_2\,q_2\,Z^2} + \frac{z_2\,z_{11}\left(144\,k_2\,r_2^5\,\delta\,Z^6-k_1^2\,R_1\,q_2^2\right)}{60\,k_1\,k_2\,q_2\,r_2^3\,Z^4}$$
$$+\frac{z_{11}\,z_{21}^2\left(144\,k_2\,r_2^5\,\delta\,Z^6-k_1^2\,R_1\,q_2^2\right)}{160\,k_1\,k_2\,r_2^2\,Z^6} + \frac{81\,k_1\,z_{11}^3\left(144\,k_2\,r_2^5\,\delta\,Z^6-k_1^2\,R_1\,q_2^2\right)}{320\,k_2^3\,r_2^2\,Z^6}$$
$$+\frac{z_2^2\,z_{11}\left(144\,k_2\,r_2^5\,\delta\,Z^6-k_1^2\,R_1\,q_2^2\right)}{360\,k_1\,k_2\,r_2^4\,Z^6} + \frac{81\,k_1\,z_1^2\,z_{11}\left(144\,k_2\,r_2^5\,\delta\,Z^6-k_1^2\,R_1\,q_2^2\right)}{320\,k_2^3\,r_2^4\,Z^6}$$
$$+z_1\,A - r_2^2\,z_{11}\,B\left(\delta\right) + \frac{2^2\,p_1\,z_{11}}{3^2},$$

$$B\left(\delta\right) = -\Bigg(\Big(\sqrt{R_1}\,\left(4272\,q_1\,k_2\,r_2^2 - 4752\,q_1\,k_2\,r_2^4\right)\,\delta\,G^3$$
$$+\left(24057\,k_1^2\,q_1\,q_2^2\,r_2^5 - 21627\,k_1^2\,q_1\,q_2^2\,r_2^3 - 4\,k_1\,R_1\,k_2\,q_2\right)\,R_2^{\frac{3}{2}}\Big)$$
$$\Big/\Big(360\,k_1\,\sqrt{R_1}\,k_2\,q_2\,r_2^2\,R_2\,G\Big)\Bigg),$$

$$k_1 = r_2^2 - 1,\;\; k_2 = 9\,r_2^2 - 4,\;\; G = 9\,r_2^2\,\sqrt{\frac{R_2}{R_1}}\,Z^2.$$

Consider the following change of variables

$$\begin{aligned}
\widehat{z}_{21} &= \overline{z}_{21}\\
\widehat{z}_{2} &= \frac{R_2\,\overline{z}_{21}\left(145800\,r_2^6\,Z^6+1296\,r_2^4\,R_2^2\,Z^4-R_1^2\right)}{12\,R_1\,r_2\,Z^2\left(225\,r_2^2\,Z^2+4\,R_2^2\right)} - \frac{5\,\overline{z}_2\left(1296\,r_2^4\,R_2^2\,Z^4+R_1^2\right)}{8\,R_1\,Z\left(225\,r_2^2\,Z^2+4\,R_2^2\right)} + \overline{p}_1\\
\widehat{p}_1 &= \frac{3\,q_2\,r_2\,\overline{z}_2}{4\,Z}\\
\widehat{z}_{11} &= \overline{z}_{11}\\
\widehat{z}_{1} &= \overline{z}_{1}
\end{aligned} \tag{23}$$

By using Eq.23 into Eq.22 one gets

$$\begin{aligned}
\overline{z}_{21}' &= P_1\left(\overline{z}_{11}, \overline{z}_{1}, \overline{z}_{21}, \overline{z}_{2}, \overline{p}_1, \delta\right),\\
\overline{z}_{2}' &= P_2\left(\overline{z}_{11}, \overline{z}_{1}, \overline{z}_{21}, \overline{z}_{2}, \overline{p}_1, \delta\right),\\
\overline{p}_{1}' &= P_3\left(\overline{z}_{11}, \overline{z}_{1}, \overline{z}_{21}, \overline{z}_{2}, \overline{p}_1, \delta\right),\\
\overline{z}_{11}' &= P_4\left(\overline{z}_{11}, \overline{z}_{1}, \overline{z}_{21}, \overline{z}_{2}, \overline{p}_1, \delta\right),\\
\overline{z}_{1}' &= P_5\left(\overline{z}_{11}, \overline{z}_{1}, \overline{z}_{21}, \overline{z}_{2}, \overline{p}_1, \delta\right)
\end{aligned} \tag{24}$$

where $P_i,\ i = 1, \dots, 6$ are polynomials, with degree at most three, in the variables $\overline{z}_{21}$, $\overline{z}_2$, $\overline{p}_1$, $\overline{z}_{11}$, $\overline{z}_1$. Their coefficients will not be written in this note. Taking $\delta = 0$ in Eq.24, the jacobian of this system computed at the equilibrium point $(0, 0, 0, 0, 0)$ is given by

$$\begin{pmatrix} 0 & -Z & 0 & 0 & 0 \\ Z & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2\,R_2}{15\,r_2} & 0 & 0 \\ 0 & 0 & 0 & A & B\,(0) \\ 0 & 0 & 0 & -B\,(0)\ r_2^2 & A \end{pmatrix} \tag{25}$$

Note that the characteristic polynomial of the jacobian of (24) computed at $(0, 0, 0, 0, 0)$ is the product of two polynomials, one of them has two degree and another has three degree. Let us denote them as $P_2\,(\lambda)$ and $P_3\,(\lambda)$ respectively. In view of the comment given at the end of the foregoing section, one knows the real parts of the roots of $P_2\,(\lambda)$ are always negative. So the roots of the characteristic polynomial that can change the signal of the real part are those ones of $P_3\,(\lambda)$. This last polynomial is given by

$$\begin{aligned} P_3\,(\lambda) = \lambda^3 &+ \frac{\left(144\,k_2\,r_2^5\,R_2\,\delta\,Z^6 + 2\,k_1^2\,R_1\,q_2^2\,R_2\right)}{15\,k_1^2\,R_1\,q_2^2\,r_2}\,\lambda^2 \\ &+ \frac{\left(1296\,k_2\,r_2^4\,R_2^2\,\delta\,Z^6 + \left(R_1^2\,k_2\,\delta + 2025\,k_1^2\,R_1\,q_2^2\,r_2\right)Z^2\right)}{2025\,k_1^2\,R_1\,q_2^2\,r_2}\,\lambda \\ &+ \frac{144\,k_2\,r_2^5\,R_2\,\delta\,Z^8 + 2\,k_1^2\,R_1\,q_2^2\,R_2\,Z^2}{15\,k_1^2\,R_1\,q_2^2\,r_2}. \end{aligned} \tag{26}$$

Since the coefficients of the above polynomial depend on δ, the left hand side of Eq.26 will be written from now on as $P_3\,(\delta,\ \lambda)$. But $P_3\,(0,\ i\,Z) = 0$ and

$$\frac{\partial\,P_3}{\partial\,\lambda}\,(0,\ i\,Z) = -\frac{30\,r_2\,Z^2 - 4\,i\,R_2\,Z}{15\,r_2} \neq 0, \tag{27}$$

so, it follows from Implicit Function Theorem that there are $\epsilon_0 > 0$ and a C^∞ mapping $\lambda : (-\epsilon_0,\ \epsilon_0) \to \mathbb{C}$ such that $P_3\,(\delta,\ \lambda\,(\delta)) = 0$ for all $s \in (-\epsilon_0,\ \epsilon_0)$ and $\lambda\,(0) = i\,Z$. Moreover, one has that

$$\mathrm{Re}\,(\lambda'\,(0)) = -\frac{k_2\,R_2\,Z^2\ \left(1296\,r_2^4\,R_2^2\,Z^4 + R_1^2\right)}{135\,k_1^2\,R_1\,q_2^2\ \left(225\,r_2^2\,Z^2 + 4\,R_2^2\right)} \neq 0. \tag{28}$$

Thus, from [2, Theorem 3.4.2, pg.151] there is a center manifold

$$\begin{aligned} \overline{p}_1 &= H_1\left(\overline{z}_{21}, \overline{z}_2, \delta\right), \\ \overline{z}_{11} &= H_2\left(\overline{z}_{21}, \overline{z}_2, \delta\right), \\ \overline{z}_1 &= H_3\left(\overline{z}_{21}, \overline{z}_2, \delta\right), \end{aligned} \tag{29}$$

such that Eq.24 on the graphic of Eq.29 reduces itself to a two-dimensional system

$$\begin{aligned} \overline{z}_{21}' &= P_1\left(H_2\left(\overline{z}_{21}, \overline{z}_2, \delta\right), H_3\left(\overline{z}_{21}, \overline{z}_2, \delta\right), \overline{z}_{21}, \overline{z}_2, H_1\left(\overline{z}_{21}, \overline{z}_2, \delta\right), \delta\right), \\ \overline{z}_2' &= P_2\left(H_2\left(\overline{z}_{21}, \overline{z}_2, \delta\right), H_3\left(\overline{z}_{21}, \overline{z}_2, \delta\right), \overline{z}_{21}, \overline{z}_2, H_1\left(\overline{z}_{21}, \overline{z}_2, \delta\right), \delta\right) \end{aligned} \tag{30}$$

which Taylor expansion of degree 3 is given by the right-hand side of the equation $(3.4.8)$, pg.151 of [2]. In order to prove the existence of Hopf Bifurcation for Eq.24 it is necessary to prove that the coefficient a in that equation of [2] is different from zero when $\delta = 0$. Such coefficient is known as the first Lyapunov coefficient.

3.1 Computing the first Lyapunov coefficient

Due to the lack of space, only a general description of the computation of first Lyapunov coefficient will be given in this subsection.

Take $\delta = 0$ in Eq.30. Since $H = (H_1, H_2, H_3)$ is a center manifold, the following condition must be hold:

$$
\begin{aligned}
H\,(0,\,0,\,0) &= 0,\\
H'\,(0,\,0,\,0) &= 0
\end{aligned}
\tag{31}
$$

where the derivative is taken at the variables $\overline{z}_{21}, \overline{z}_2$. From the coefficients in Eq.24, the system Eq.30 can be written as

$$
\begin{cases}
\overline{z}'_{21} = -Z\,\overline{z}_2 + f\left(\overline{z}_{21}, \overline{z}_2\right),\\
\overline{z}'_2 = Z\,\overline{z}_{21}
\end{cases}
\tag{32}
$$

where $|f\left(\overline{z}_{21}, \overline{z}_2\right)| \leq$ const. $\left(\overline{z}_{21}{}^2 + \overline{z}_2{}^2\right)$. Let us obtain the Taylor expansion of degree 3 of the right hand side of Eq.32. So, one has to get the Taylor expansion of degree 2 of H. It is known H must satisfy the equation given at [2, eq. 3.2.16. pg. 131]. Computing the derivatives $\frac{\partial^2}{\partial \hat{z}_{21}^2}$, $\frac{\partial^2}{\partial \hat{z}_{21}\,\partial \hat{z}_2}$, $\frac{\partial^2}{\partial \hat{z}_2^2}$ of the cited equation at the point $(0,0)$ and using Eq.31, one gets a linear system of nine equations with nine variables given by $\frac{\partial^2 H_k}{\partial \hat{z}_{21}^2}$, $\frac{\partial^2 H_k}{\partial \hat{z}_{21}\,\partial \hat{z}_2}$, $\frac{\partial^2 H_k}{\partial \hat{z}_2^2}$, $k = 1, 2, 3$. Solving it, one gets the partial derivatives involved in and

$$
\begin{aligned}
f\left(\overline{z}_{21}, \overline{z}_2\right) = {} & L_{[3,0]}\overline{z}_{21}^3 + L_{[2,1]}\overline{z}_{21}^2\overline{z}_2 + L_{[1,2]}\overline{z}_{21}\overline{z}_2^2 + L_{[0,3]}\overline{z}_2^3\\
& + L_{[1,1]}\overline{z}_{21}\overline{z}_2 + L_{[2,0]}\overline{z}_{21}^2 + L_{[0,2]}\overline{z}_2^2 + O\left(\left(\overline{z}_{21}^2 + \overline{z}_2^2\right)^2\right).
\end{aligned}
\tag{33}
$$

As earlier, the above coefficients are huge ones. From Eq.33 one has the first Lyapunov coefficient a, given in [2, pg. 152, eq. 3.4.11], is the following one

$$
\begin{aligned}
a = -\Bigg(& q_2^2\,R_2\left(1296\,r_2^4\,R_2^2\,Z^4 + R_1^2\right)\\
& \cdot\Big(36733201920000\,r_2^{16}\,R_2^2\,Z^{16} + 326517350400\,r_2^{14}\,R_2^4\,Z^{14}\\
& + 725594112\,r_2^{12}\,R_2^6\,Z^{12} - 28343520000\,R_1^2\,r_2^{12}\,Z^{12}\\
& - 167961600\,R_1^2\,r_2^{10}\,R_2^2\,Z^{10} - 186624\,R_1^2\,r_2^8\,R_2^4\,Z^8 + 583200\,R_1^4\,r_2^6\,Z^6\\
& + 3024\,R_1^4\,r_2^4\,R_2^2\,Z^4 + R_1^6\Big)\Bigg)\Big/\Bigg(232190115840\,R_1^2\,r_2^{13}\,Z^{16}\\
& \cdot\left(225\,r_2^2\,Z^2 + R_2^2\right)\left(225\,r_2^2\,Z^2 + 4\,R_2^2\right)^2\Bigg),
\end{aligned}
\tag{34}
$$

So from [2, pg. 151, Theorem 3.4.2] one has a Hopf Bifurcation in Eq.24. Note if $Z \ll 1$, thus from Eq.34, one gets $a < 0$, so from [2, pg. 151, Theorem 3.4.2] the periodic orbit that happens in that bifurcation is a stable limit cycle.

Note that Eq.4 is equivalent to

$$
\begin{aligned}
\epsilon_1 &= \left(r_1^2 - 1\right)\,\left(1 - r_2^2\right),\\
\epsilon_2 &= r_1^2\,r_2^2.
\end{aligned}
\tag{35}
$$

This will be used in the next computation. In order to perform computations for Eq.24 is necessary to prescribe the value of the parameters involved in as well as the initial conditions. Let us take

$$\begin{gathered}\Gamma\left(s\right)=a-b\,s, a=7, b=5, \epsilon=0.01, q_1=-2.0, q_2=-3.0, q_3=0.1,\\ \lambda_1=1.0, \lambda_2=1.0, r_2=\tfrac{5}{6}.\end{gathered} \tag{36}$$

It is worthy to note that the above $\Gamma\left(s\right)$ is the same one adopted in [7]. Moreover, all others parameters can be obtained from Eq.36 by using Eq.12, Eqs.(16 - 17),Eqs. (20) and (35). Thus by using Eq.36 one obtains from Eq.34 that

$$a=-0.2302273489204\neq 0. \tag{37}$$

So, for those parameters in (36) it follows from Eq.37 there is a Hopf Bifurcation for Eq.24.

Doing $\delta=0.1$, using Eq.36 in Eq.30 and taking into account only the terms until the third order and the initial conditions $\overline{z}_{21}\left(0\right)=0$, $\overline{z}_2\left(0\right)=0.1$, one gets the Figure 2. If one takes $\delta=-0.1$ in Eq.30 the Figure 3 is obtained, where $\overline{z}_{21}\left(0\right)=-0.37$, $\overline{z}_2\left(0\right)=0$ are the initial conditions for the orbit in black, $\overline{z}_{21}\left(0\right)=0.09$, $\overline{z}_2\left(0\right)=0$ are the initial conditions for the orbit in green and the limit cycle in red has initial conditions given by $\overline{z}_{21}\left(0\right)=0.1426281404224$, $\overline{z}_2\left(0\right)=-0.11873421602497$.

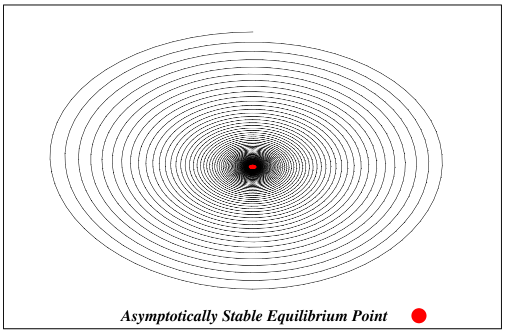

Figure 2: Case $\delta>0$.

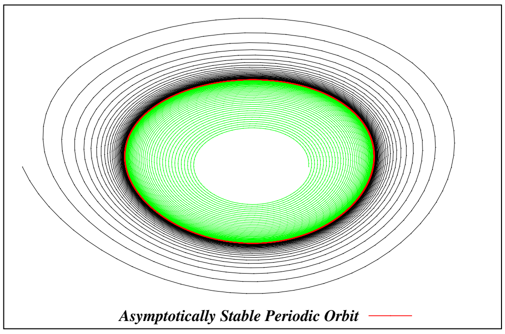

Figure 3: Case $\delta<0$.

4 A NEIMARK-SACKER BIFURCATION

The periodic orbit obtained in the Hopf Bifurcation, see Section 3 is an asymptotically stable one if $Z\ll 1$. From [6, pg.250, Theorem 7.1] this implies the original system, which is a $O\left(\epsilon^2\right)$ time-dependent perturbation of the averaged system Eq.13, has an invariant manifold (a torus), which is asymptotically stable. The equilibrium point obtained in the Hopf bifurcation is an asymptotically stable one. By using [6, pg.194, Theorem 3.2], one gets an asymptotically stable periodic orbit for the original system. So, a Neimark-Sacker Bifurcation is obtained in this case. Note that, in view of the changes of variables in [1] as well as of the properties of the final solution obtained from Averaging Method, corrections of the amplitude happens only in the first order aproximation.

It is worthy to emphasize that the same approach to find out a Neimark-Sacker Bifurcation has been used in [8].

5 CONCLUSIONS

The main contribution of this paper is the finding of Neimark-Sacker Bifurcation in a non-ideal mechanical system. And, to authors' knowledge, this is the first time that such result is obtained for this class of mechanical systems. Moreover, this outcome is obtained by using rigorous methods.

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