11th International Conference on Vibration Problems
Z. Dimitrovová *et.al.* (eds.)
Lisbon, Portugal, 9–12 September 2013

A HOMOGENIZATION APPROACH USING EIGENFORMS FROM NUMERICAL MODAL ANALYSIS

D.Kreuter*[1], M. Beitelschmidt[1]

[1]Institute for Solid Mechanics, Dresden University of Technology, Germany
daniel.kreuter@tu-dresden.de
michael.beitelschmidt@tu-dresden.de

Keywords: Numerical Modal Analysis, Homogenization methods, Effective Material Parameters

Abstract. *The multiscale and scale–to–scale homogenization method for structural analysis are well known. This numerical approach, using FEM to calculate effective material parameter for periodic structures, can be used for any dimension [1]. But this method requires a unit cell (or RVE), that is a parallelepiped. So it is not recommended for cylindrical unit cells. We show a new approach, where we use eigenforms of a cylinder to calculate effective material parameters. The global eigenforms of a orthotropic cylinder can be analyzed analytically and depend on the boundary conditions, the geometry and the material. Therefore we can define the displacement field for a single unit cell and calculate with help of the strain energy the effective material parameters of a heterogeneous periodic cylindric structure.*

1 Introduction

A lot of homogenization methods are dealing with the energy formulation with help of a FEM-approch. These methods base on a periodic structure. But if the structure is cylindrical these technique can not be used. In this paper we show an approach which can handle this problem. First we retry the kinematics and kinetics of cylindrical structures. With help of this and with the knowledge of the eigenforms of an isotropic cylinder we can introduce an optimization tool to calculate effective material parameters. An example will show you the detailled approach.

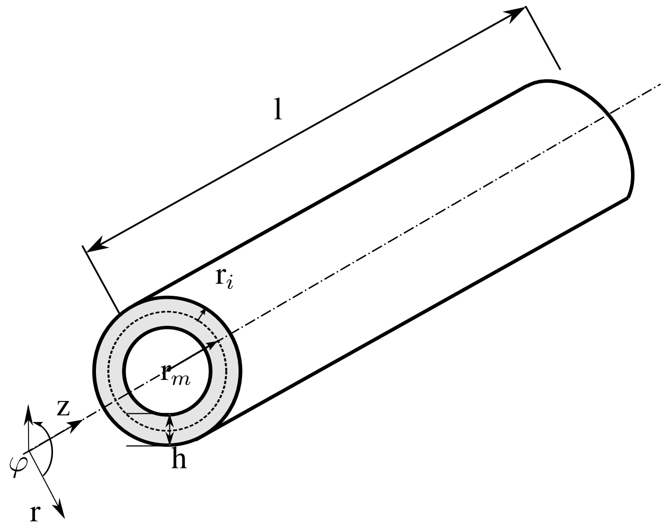

Figure 1: Geometry of cylinders

2 Kinematics and Kinetics of cylindrical shells

2.1 Kinematics of cylindrical shells

The displacement-strain behavior in cylindrical coordinates are given by [2]

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\varepsilon_{\varphi\varphi} &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
\varepsilon_{r\varphi} &= \frac{1}{2}\left(\frac{\partial u_r}{r\partial\varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) \\
\varepsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\
\varepsilon_{\varphi z} &= \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \varphi}\right).
\end{aligned} \tag{1}$$

How you can see in Figure 2, the axial displacement of a material point P, which is out of the midplace,

$$u_{zP} = u_z - r_i u_{r,r} \tag{2}$$

can be described with help of the axial displacemt of the midplace u_z and the radial rotation [3]. This assumption is equal to the cartesian deformation behavior of a Kirchhoff plate. The displacement in tangential direction

$$u_{\varphi P} = \frac{r_m + r_i}{r_m} u_\varphi - \frac{r_i}{r_m} u_{r,\varphi} \tag{3}$$

can be defined with help of the tangential displacement of the mdiplace and the derivation of the radial displacement in the tangential direction. Finally, the displacement in the radial direction

$$u_{rP} = u_r(\varphi, z) \tag{4}$$

is consistent with the radial displacement of the midplace w_r.

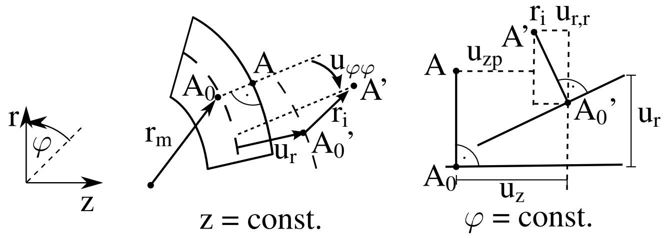

Figure 2: Slice plane of a cylnder

2.2 Strain-definition of cylindrical shells

Defining the displacements are not depending of the derivation in radial direction, with help of Equation 2 we can define the strain of a cylindrical shell with

$$\begin{aligned}
\varepsilon_{rr} &= u_{r,r} \approx 0 \\
\varepsilon_{\varphi\varphi} &= \frac{u_r}{r_m + r_i} + \frac{1}{r_m} u_{\varphi,\varphi} - \frac{r_i}{(r_m + r_i)\, r_m} u_{r,\varphi\varphi} \\
\varepsilon_{zz} &= u_{z,z} - r_i u_{r,zz} \\
\varepsilon_{r\varphi} &= \frac{1}{2}\left(\frac{r_m - r_i}{(r_m + r_i)\, r_m} u_{r,\varphi} - \frac{1}{r_m} w_\varphi\right) \approx 0 \\
\varepsilon_{rz} &= \frac{1}{2} u_{r,z} \approx 0 \\
\varepsilon_{\varphi z} &= \frac{1}{2}\left(\frac{r_m + r_i}{r_m} u_{\varphi,z} + \frac{1}{r_i + r_m} u_{z,\varphi} - \left(\frac{r_i}{r_m} + \frac{r_i}{r_m + r_i}\right) u_{r,z\varphi}\right).
\end{aligned} \tag{5}$$

In the special case of a cylindrical ring ($r \approx r_m, r_i \approx 0$) the strain definition can be simplified to

$$\begin{aligned}
\varepsilon_{rr} &= 0 \\
\varepsilon_{\varphi\varphi} &= \frac{1}{r}\left(u_r + u_{\varphi,\varphi}\right) \\
\varepsilon_{zz} &= u_{z,z} \\
\varepsilon_{r\varphi} &= 0 \\
\varepsilon_{rz} &= 0 \\
\varepsilon_{\varphi z} &= \frac{1}{2}\left(u_{\varphi,z} + \frac{1}{r} u_{z,\varphi}\right).
\end{aligned} \tag{6}$$

2.3 Energy formulation

The strain energy of a structure with elastic material parameters can be defined with

$$U = \frac{1}{2}\int_{(V)} \boldsymbol{\varepsilon}^T \mathbf{C} \boldsymbol{\varepsilon} \mathrm{d}V. \tag{7}$$

Introducing the strain-stress behavior for orthotropic materials

$$\begin{pmatrix} \sigma_{\varphi\varphi} \\ \sigma_{zz} \\ \sigma_{\varphi z} \end{pmatrix} = \begin{pmatrix} C_{22} & C_{32} & 0 \\ C_{32} & C_{33} & 0 \\ 0 & 0 & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{\varphi\varphi} \\ \varepsilon_{zz} \\ \varepsilon_{\varphi z} \end{pmatrix} \tag{8}$$

for thin cylindrical shells, the strain energy results to

$$\begin{aligned} U \quad = \quad & \frac{1}{2}\int_{(V)} \left[C_{22} \left(\frac{u_r}{r_m + r_i} + \frac{1}{r_m} u_{\varphi,\varphi} - \frac{r_i}{(r_m + r_i)\, r_m} u_{r,\varphi\varphi} \right)^2 \right. \\ \ldots \quad & + C_{33} \left(u_{z,z} - r_i u_{r,zz} \right)^2 \\ \ldots \quad & \left. + C_{66} \frac{1}{4} \left(\frac{r_m + r_i}{r_m} u_{\varphi,z} + \frac{1}{r_i + r_m} u_{z,\varphi} - \left(\frac{r_i}{r_m} + \frac{r_i}{r_m + r_i} \right) u_{r,z\varphi} \right)^2 \right] \mathrm{d}V \end{aligned} \tag{9}$$

with the infinitesimal volume $\mathrm{d}V = (r_m + r_i)\mathrm{d}r_i \mathrm{d}\varphi \mathrm{d}z$.

3 Effective Material paramters

Goal of a homogeniziation procedure is the determination of effective material parameters. Starting with a detailed structure with different materials we can define a representative volume element (RVE) which contains all geometric and material information of the global structure. With help of periodic boundary conditions and defined displacement functions we get, with help of FEM, as a result of the homogenization the strain energy of the RVE. With help of this strain energy we can determine the effective material parameters $\overline{C}$ for a equivalent homogenious continuum (EHC). This procedure, particular the definition of the displacement functions and

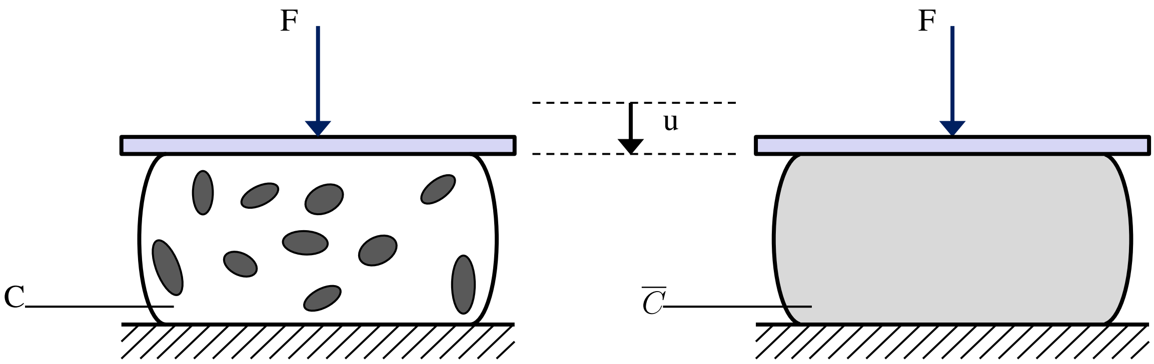

Figure 3: Original structure and homogenized structure with effective Materials

the periodic boundary conditions are available only by cartesian RVEs by implication. If we have a cylindrical periodical structure we need a different approach. In this paper, we use the eigenforms of the structure, which are a typical indicator of the structure.

3.1 Modal ansatz functions

The displacement components resulting from an eigenform of a cylinder for any point of the structure can be defined with [4]

$$\begin{aligned} u_r &= Ag(z)\cos(k\varphi) \\ u_\varphi &= Bg(z)\sin(k\varphi) \\ u_z &= CAg(z)\cos(k\varphi)\,. \end{aligned} \tag{10}$$

The function $g(z)$ depends on the boundary conditions. The parameters A, B, C are the amplitudes of the eigenform. If we have a free support at all boundaries, we can use the ansatz of a Bernoulli-beam [5]. Regarding periodic boundary conditions we can define according to [5] the first part of the eigen displacement function

$$g = e^{\alpha_i\left(\zeta^2-\zeta\right)}\left[\cos\left(\lambda_m\left(\zeta+\zeta_i\right)\right)\right] \tag{11}$$

with $\zeta = z/l$ and $i = r, \varphi, z$. The fading term $e^{\alpha_i\left(\zeta^2-\zeta\right)}$ ist negligible in the most cases. The parameter λ_m can be calculated with

$$1 = \cos\lambda_m \cosh\lambda_m \tag{12}$$

for every order. The strain energy can be calculated with

$$\begin{aligned} U \;=\; & \frac{1}{2}\int_{(V)}\Bigg[C_{22}\left(A\frac{1}{r_m+r_i}\left(1+\frac{r_i}{r_m}k^2\right)+\frac{1}{r_m}Bk\right)^2\cos^2\left(\lambda_m\zeta\right)\cos^2\left(k\varphi\right) \\ \ldots\; & +C_{33}\left(C\frac{\lambda}{l}+Ar_i\frac{\lambda^2}{l^2}\right)^2\cos^2\left(\lambda_m\zeta\right)\cos^2\left(k\varphi\right) \\ \ldots\; & +C_{66}\frac{1}{4}\left(-\frac{r_m+r_i}{r_m}B\frac{\lambda}{l}-\frac{1}{r_i+r_m}kC-\left(\frac{r_i}{r_m}+\frac{r_i}{r_m+r_i}\right)\frac{\lambda}{l}kA\right)^2\sin^2\left(\lambda_m\zeta\right)\sin^2\left(k\varphi\right)\Bigg]\mathrm{d}V, \end{aligned} \tag{13}$$

in which the fading term is not considered.

4 Example

As an example we use a thin perforated cylinder as you can see in Figure 4 which has isotropc material parameters ($E = 5700$ MPa; $\nu = 0.3$; $\rho = 871$ kg/m^3). We use the displacement functions 11 to define a displacement field in a FEM structural analys. As a result we get the strain energy for every loadcase of the RVE. With help of this strain energy we can calculate by optimization with help of Equation 14 the effective material parameters

$$\mathbf{C}_{EHC} = \begin{pmatrix} 1615 & 196 & 0 \\ 196 & 4417 & 0 \\ 0 & 0 & 1256 \end{pmatrix}\text{MPa} \tag{14}$$

as components of the stiffness matrix. We can now use this effective material parameters to calculate the eigenvalues of this structure with a FEM modal analysis with a coarser mesh. Figure 5 shows the results compared with an experimental modal analysis.

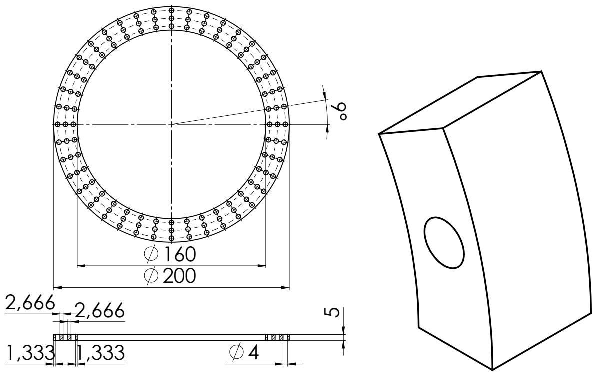

Figure 4: Perforated Cylinder and RVE of the structure

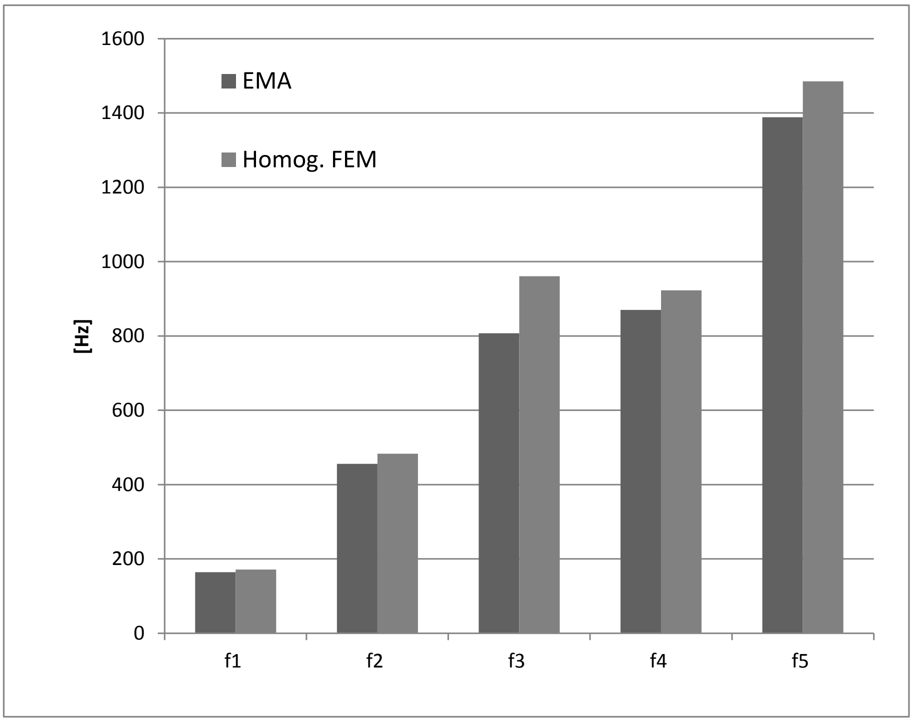

Figure 5: Results of experiental and numerical modal analysis

REFERENCES

[1] M.G.D. Geers, V.G. Kouznetsova, W.A.M. Brekelmans, Multi-scale computational homogenization: Trends and challenges, *Journal of Computational and Applied Mechanics*, 234: 2175-2182, 2009.

[2] Betten, Josef, *Kontinuumsmechanik / elastisches und inelastisches Verhalten isotroper und anisotroper Stoffe ; mit 20 Tabellen und 250 Übungsaufgaben*, Springer Berlin, 2001

[3] Wauer, Jörg, *Kontinuumsschwingungen/vom einfachen Strukturmodell zum komplexen Mehrfeldsystem; mit 74 Aufgaben und 41 Beispielen*, Vieweg + Teubner Wiesbaden, 2008

[4] M. Darvizeh and C.B. Sharma, Natural frequencies of laminated orthotropic thin circular cylinders, *Thin-Walled Structures*, 2, 207 - 217, 1984

[5] Ernest L. Akselrad, *Schalentheorie*, Teubner, Stuttgart, 1983