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RECENT RESEARCH ADVANCES OF EXACT SOLUTIONS FOR FREE VIBRATIONS OF PLATES AND SHELLS

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Abstract. *This paper reviews the recent research advances of exact solutions for the transverse free vibrations of isotropic and orthotropic Kirchhoff and Mindlin plates and Donnell shells, and for the in-plane free vibrations of isotropic and orthotropic plates. Most new exact solutions were obtained by the authors of present paper using the direct separation of variable method.*

1 INTRODUCTION

The vibration problem of rectangular plates and circular cylinder shells, although more than some two hundred years old in its research account, continues to be of considerable research interest, for the reason that rectangular plates and circular cylinder shells are basic structural elements and hence practical applications may involve enormous parametric variations as loading, materials, aspect ratio and support conditions etc. Another reason is that there are some plate and shell configurations for which we have not solved the exact solutions until now.

The solution for free vibration of bar, beam, plate and shell in nature all are eigenvalue problems in mathematics and only in homogeneous boundary conditions one can obtain the exact solution. As for one-dimensional case it is not difficult to obtain the exact solution of free vibration for any homogeneous boundary conditions. However the free vibration problems of rectangular plates and circular cylinder shells are respective two and three dimensional problems, people have obtained exact solutions only for a limit of configurations.

In these years there are some essential advances in researching for the exact solutions of free vibration of thin and shear rectangular plates and circular cylinder Donnell shells. And this paper reviews these recent research advances. It is noteworthy that most new exact solutions were obtained by the authors of present paper using the direct separation of variable method.

2 FREE VIBRATION OF THIN PLATE

The analysis of free vibration of rectangular thin plate is of great interest all along in many important fields as civil, naval and aerospace engineering. In 1823, by using a double trigonometric series, Navier [1] obtained the exact solutions of rectangular thin plate with all edges simply supported (SS). In 1899, by using a single Fourier series, Levy [2] developed a method for solving the exact solutions of rectangular thin plate with two opposite edges simply supported and the remaining opposite edges with arbitrary boundary conditions. There, since then, was no essential progress being made pertaining to this problem. The Navier and Levy solutions are widely used in textbooks and monographs, and apparently Navier solution is a particular case of Levy solution.

Except for the simply supported, clamped (C) and free (F) edges, the guided (G) boundary condition was examined by Bert and Malik [3] up to 1994, and the used method was inverse method as Navier method and Levy method. All together there are 55 plate configurations with different combinations of SS, C, F and G conditions, for which there are three distinct categories:

(i) plates with all edges simply supported or/and guided (altogether 6 configurations);

(ii) plates with only a pair of opposite edges simply supported or/and guided (altogether 21 configurations);

(iii) plates which do not fall into any of the above categories (altogether 28 configurations).

Problems of the first and second categories are amenable to straightforward rigorous analysis in terms of the well-known Navier and Levy solutions [3]. However, owing to coupled multiple differential equations of high order, it was believed before 2009 that there were not exact solutions for the free vibration of the third category problems [3, 4, 5] or difficult to obtain the exact solutions [6]. For this reason many efforts were devoted to develop approximate methods. The Levy type of solutions for the problems of the first and the second categories are referred to reference [3] and [7]. Recently, Xiang and Wang [8] introduced the Levy solution method to the problem of thin stepped plates having *n*-step variations in one direction parallel to the plate edges while the thickness is constant in the other direction in conjunction with state space method.

More recently the present authors, by using the separation-of-variable method, obtained a few new exact solutions for the free vibration of isotropic [9, 10] and orthotropic [10, 11] rectangular thin plates with two adjacent edges clamped and remaining edges with SS, C and G boundaries for the first time. It is worthy stressing that exact solutions for the six configurations (SS-SS-C-C, SS-C-C-C, SS-G-C-C, G-G-C-C, C-C-C-G and C-C-C-C) are not achievable by the inverse method. And most researchers concluded that there were not exact solutions for these six configurations, and this conclusion is addressed in most textbooks and publications. A few words are presented below for the idea and method used in Refs. [9-11].

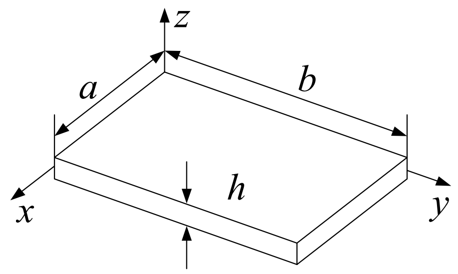

Figure 1: thin plate and coordinates.

Consider a rectangular plate of thickness *h*, as shown in Fig. 1. The solutions of its characteristic differential equation depend basically on the edge boundary conditions. With all possible combinations of SS, C, F and G conditions at four edges, 55 rectangular plate configurations are possible [3]. In reference [9-11], a separation of variables solution was assumed

$$W(X,Y)=\phi(X)\psi(Y) \tag{1}$$

where dimensionless coordinates $X=x/a$, $Y=y/b$ and aspect ratio $\theta=a/b$ are introduced, W is the natural mode function, ϕ and ψ are respective X-direction and Y-direction mode functions or eigenfunctions, and they have the forms as

$$\phi(X)=A\mathrm{e}^{\mu X},\quad \psi(Y)=B\mathrm{e}^{\lambda Y} \tag{2}$$

where μ and λ are respective x-direction and y-direction eigenvalues. In classical separation of variables method or the inverse method used by Levy and others, one of μ and λ and its corresponding eigenfunction is given or assumed for the pair of opposite simply supported or/and guided edges, then the plate problems become beamlike problems. On the contrary, the two spatial eigenvalues μ and λ and frequency are all unknowns and solved simultaneously using the separation of variables method [9-11], in which the explicit relationship among the two spatial eigenvalues and the temporal eigenvalue are derived from characteristic differential equation as

$$(\mu^2+\theta^2\lambda^2)^2=\gamma^4 \tag{3}$$

where $\theta = a/b$ is the aspect ratio, $\gamma^2=a^2\omega\sqrt{\rho h/D}$, $D=Eh^3/12(1-\upsilon^2)$ is the bending rigidity, ω is the circular frequency, ρ is the density, E is the Yang's modulus, υ is the Poison's Ratio. The roots of Eq. (3) are

$$\mu_{1,2}=\pm\mathrm{i}\alpha_1,\ \mu_{3,4}=\pm\beta_1,\ \lambda_{1,2}=\pm\mathrm{i}\alpha_2,\ \lambda_{3,4}=\pm\beta_2 \tag{4}$$

where

$$\alpha_1=\sqrt{\gamma^2-\theta^2\alpha_2^2},\ \beta_1=\sqrt{\gamma^2+\theta^2\alpha_2^2},\ \alpha_2=\frac{1}{\theta}\sqrt{\gamma^2-\alpha_1^2},\ \beta_2=\frac{1}{\theta}\sqrt{\gamma^2+\alpha_1^2} \tag{5}$$

And from Eq. (5) we have

$$\left(\alpha_1^2+\beta_1^2\right)^2=4\gamma^4$$
$$\left(\alpha_2^2+\beta_2^2\right)^2=4\frac{\gamma^4}{\theta^4} \tag{6a, b}$$

The Levy-type eigenfunctions $\phi(X)$ and $\psi(Y)$ can be expressed by the roots $\mu_{1,2}$, $\mu_{3,4}$ and $\lambda_{1,2}$, $\lambda_{3,4}$ as

$$\begin{aligned}\phi(X) &= A_1 \cos\alpha_1 X + B_1 \sin\alpha_1 X + C_1 \cosh\beta_1 X + H_1 \sinh\beta_1 X \\ \psi(Y) &= A_2 \cos\alpha_2 Y + B_2 \sin\alpha_2 Y + C_2 \cosh\beta_2 Y + H_2 \sinh\beta_2 Y\end{aligned} \quad (7a, b)$$

It follows from Eq. (5) that, if given α_1, β_1 and the frequency parameter γ, the eigenvalues α_2 and β_2 can be solved accordingly, which means α_1, β_1 and γ can be taken as independent variables. Certainly, α_2, β_2 and γ can also be the independent variables.

Additionally, one can obtain two transcendental eigenvalue equations by substituting expression (7) into boundary conditions of two pairs of opposite edges. Therefore, the independent variables α_1, β_1 and γ are solved from the two transcendental eigenvalue equations together with Eq. (6a). If α_2, β_2 and γ are taken as the independent variables, they should be solved from the two transcendental eigenvalue equations together with Eq. (6b). And it should be noted that the transcendental equations can be solved by the Newton-Raphson method without any difficulty, just like solving the free vibration of a beam.

3 FREE IN-PLANE VIBRATION OF THIN PLATE

The free in-plane vibration of rectangular plates involves two independent displacement variables, so it becomes more difficult to get its exact solution compared with thin plate vibration where there is only a single independent displacement variable. The pioneering work was done by Lord Rayleigh [12] in 1894 who dealt with what was referred to as 'simply-supported' plates. A hundred years later since then, Gorman [13] obtained the exact solutions for the rectangular plates with two opposite edges simply supported, and the remaining edges being both clamped or both free. In Gorman's work, a quarter of the rectangular plate was analyzed to avoid the calculation difficulties and missing problems of repeated eigenvalues and the interpretations of the mode shapes as well.

Recently, Xing and Liu obtained all possible exact solutions for isotropic [14] and orthotropic [15] rectangular plate via the direct separation of variables. The exact solutions for the isotropic plates with two opposite edges simply supported and the other two opposite edges being asymmetrical were not achieved before [14]. Moreover, there are no problems that Gorman tried to avoid in the present authors' paper [14].In addition, Liu and Xing [15] first obtained the exact solutions for free in-plane vibration of rectangular orthotropic plates with four edges not being simply supported.

Consider a rectangular plate as shown in Fig. 1. The direct separation of variables method [9-11] is also employed. The mode functions in terms of separation-of-variable form are

$$U(X,Y) = A\mathrm{e}^{\mu X}\mathrm{e}^{\lambda Y},\ V(X,Y) = B\mathrm{e}^{\mu X}\mathrm{e}^{\lambda Y} \quad (8)$$

where U and V are the in-plane displacements in x and y directions, respectively. The obtained relations of two spatial eigenvalues μ and λ and frequency ω are

$$\left[\theta^2\lambda^2 + \mu^2 + \left(a\frac{\omega}{c}\right)^2\right]\left[\theta^2\lambda^2 + \mu^2 + \upsilon_1\left(a\frac{\omega}{c}\right)^2\right] = 0 \quad (9)$$

where $c=(G/\rho)^{0.5}$ is the shear wave velocity, G is the shear rigidity, $\upsilon_1=(1-\upsilon)/2$. Ref. [14] shown that the exact solutions for in-plane free vibration are only available for the rectangular plates with a pair of simply supported opposite edges. And all possible exact solutions were obtained for the isotropic [14] and orthotropic [15] rectangular plate with a pair of opposite edges simply supported, and arbitrary combinations of SS, C and F supports for remaining edges.

The mode functions were presented for three different cases in Ref. [14]. The three cases were synthesized as a general form in Ref. [15].

4 FREE VIBRATIONS OF RECTANGULAR MINDLIN PLATES

The free vibrations of rectangular plates was first investigated by Mindlin et al. [16] in 1956, who gave the exact solution of simply supported plate, and investigated the frequency variations of plates, a pair of parallel edges of which are simply supported and the other pair are free. Srinivas [17] obtained the exact solutions for the homogeneous and laminated rectangular Mindlin plates with four simply supported edges. In 1986, Mindlin [18] obtained the exact solutions, upon two main restrictions: (i) the length/width must be a ratio of integers; (ii) for each value of Poisson's ratio, all the allowable modes must have the same frequency.

Xiang and Wei [19] and Xiang [20] employed the Levy solution approach in conjunction with the state space technique to derive the analytical eigensolutions of rectangular Mindlin plates with two opposite edges simply supported. Hashemi and Arsanjani [21] presented the exact characteristic equations for plates with one pair of opposite plate edges simply supported, for which, in point of fact, a more easy solution method was given by Brunelle [22] when he studied the buckling load of plate.

Using a refined theory [23], Shimpi and his co-worker [24] obtained the exact solutions of simply supported plate, and Thai and Kim [25] obtained the Levy-type exact solutions for plates with two opposite edges simply supported and the other two edges having arbitrary boundary conditions by applying the state space approach. Senjanović et al. [26] reduced a system of three equations to a single equation in terms of bending deflection only which was fundamental variable, and obtained closed-form solutions for simply supported plate and the characteristic equation for a plate with two opposite edges simply supported.

It was believed that more difficulties would be encountered in solving the exact solutions for the free vibrations of rectangular Mindlin plates without two simply supported opposite edges. From 2009, Xing and Liu obtained a few closed form solutions for isotropic plates with a pair of simply supported opposite edges, and remaining edges having any combinations of SS and C boundary conditions [27, 28]. The free edges can be dealt with if a pair of opposite edges is simply supported. The exact solutions for rectangular orthotropic plates with two opposite edges simply supported have also be obtained recently [29].

Assume the deflection W, the angles of rotations Ψ_x and Ψ_y in separation of variable form as

$$W(x,y)=p\mathrm{e}^{\mu x}\mathrm{e}^{\lambda y},\quad \Psi_x(x,y)=q\mathrm{e}^{\mu x}\mathrm{e}^{\lambda y},\quad \Psi_y(x,y)=r\mathrm{e}^{\mu x}\mathrm{e}^{\lambda y} \qquad \text{(10a, b, c)}$$

To eliminate W and Ψ_x or W and Ψ_y, we can obtain an eigenvalue equation as

$$\left[\left(\mu^2+\lambda^2\right)^2+\left(\frac{D}{C}+\frac{J}{h}\right)\frac{\omega^2\rho h}{D}\left(\mu^2+\lambda^2\right)-\left(1-\frac{\omega^2\rho J}{C}\right)\frac{\omega^2\rho h}{D}\right]\cdot\left[\frac{\upsilon_1 D}{C}\left(\mu^2+\lambda^2\right)-1+\frac{\omega^2\rho J}{C}\right]=0 \qquad (11)$$

where $C=\kappa Gh$ is the shear rigidity, κ is the shear correction factor. To eliminate Ψ_x and Ψ_y, we can obtain another eigenvalue equation of different order from (11) as

$$\left(\mu^2+\lambda^2\right)^2+\left(\frac{D}{C}+\frac{J}{h}\right)\frac{\omega^2\rho h}{D}\left(\mu^2+\lambda^2\right)-\left(1-\frac{\omega^2\rho J}{C}\right)\frac{\omega^2\rho h}{D}=0 \qquad (12)$$

It follows from Eqs. (11) and (12) that Ψ_x and Ψ_y should be expressed by the six roots of Eq. (11), and W by only four roots of Eq. (12) for isotropic rectangular plates. This is the particularity of Mindlin plate theory. But for orthotropic plate, different methods give the same eigenvalue equation [29], that is the deflection and two rotations must be expressed by six roots of algebraic eigenvalue equation.

In Eq. (11), factor $Dv_1(\lambda^2+\mu^2)/C+\omega^2\rho J/C-1=0$ is identical with the eigenvalue equation for the free vibration of membrane on elastic foundation. For the free vibrations of thick plate with four edges simply supported, the roots of equation $Dv_1(\lambda^2+\mu^2)/C+\omega^2\rho J/C-1=0$ have no contributions to rotations Ψ_x and Ψ_y [22], that means the defection and the rotations can be given by the same four eigenvalue roots.

In point of fact, accurate results can be obtained not only for rectangular plates with a pair of simply supported edges, but also for plates with other boundary combinations by expressing W, Ψ_x and Ψ_y with four roots of Eq.(12) [27, 28]. The approach will be briefly presented below. From Eq. (12) we have

$$\mu^2+\lambda^2=-R_1^2 \text{ or } \mu^2+\lambda^2=R_2^2 \tag{13}$$

where

$$R_1^2=\left(\sigma+\sqrt{1-\frac{\omega^2\rho J}{C}+\sigma^2}\right)\sqrt{\frac{\omega^2\rho h}{D}},\ R_2^2=\left(-\sigma+\sqrt{1-\frac{\omega^2\rho J}{C}+\sigma^2}\right)\sqrt{\frac{\omega^2\rho h}{D}} \tag{14}$$

$$\sigma=\frac{1}{2}\left(\frac{D}{C}+\frac{J}{h}\right)\sqrt{\frac{\omega^2\rho h}{D}} \tag{15}$$

The roots of Eq. (13) are

$$\mu_{1,2}=\pm \mathrm{i}\alpha_1,\ \mu_{3,4}=\pm\alpha_2,\ \lambda_{1,2}=\pm \mathrm{i}\beta_1,\ \lambda_{3,4}=\pm\beta_2 \tag{16a, b}$$

$$\alpha_1=\sqrt{R_1^2-\beta_1^2},\ \alpha_2=\sqrt{R_2^2+\beta_1^2},\ \beta_1=\sqrt{R_1^2-\alpha_1^2},\ \beta_2=\sqrt{R_2^2+\alpha_1^2} \tag{17}$$

The mode functions in separation of variables form are given as

$$W(x,y)=\phi(x)\psi(y),\quad \Psi_x(x,y)=g(x)\varphi(y),\quad \Psi_y(x,y)=\phi(x)h(y) \tag{18}$$

where

$$\begin{aligned}
\phi(x)&=A_1\sin\alpha_1x+A_2\cos\alpha_1x+A_3\sinh\alpha_2x+A_4\cosh\alpha_2x\\
\psi(y)&=B_1\sin\beta_1y+B_2\cos\beta_1y+B_3\sinh\beta_2y+B_4\cosh\beta_2y\\
g(x)&=g_1\alpha_1(A_1\cos\alpha_1x-A_2\sin\alpha_1x)+g_2\alpha_2(A_3\cosh\alpha_2x+A_4\sinh\alpha_2x)\\
h(y)&=h_1\beta_1(B_1\cos\beta_1y-B_2\sin\beta_1y)+h_2\beta_2(B_3\cosh\beta_2y+B_4\sinh\beta_2y)
\end{aligned} \tag{19a, b, c, d}$$

In Ref. [28], g_1, g_2, h_1 and h_2 were given by using two rotational differential equations

$$g_1=h_1=1-\frac{\omega^2\rho h}{CR_1^2},\quad g_2=h_2=1+\frac{\omega^2\rho h}{CR_2^2} \tag{20}$$

In Ref. [27], g_1, g_2, h_1 and h_2 were given by using translational differential equation

$$g_1=h_1=1-\frac{DR^4}{CR_1^2},\ g_2=1+\frac{\alpha_1^2}{\alpha_2^2}\frac{DR^4}{CR_1^2},\ h_2=1+\frac{\beta_1^2}{\beta_2^2}\frac{DR^4}{CR_1^2} \tag{21}$$

where $R^2=\omega\sqrt{\rho h/D}$. After simplifying the boundary conditions to be same as those of classical thin plate theory, the closed form eigensolutions can be obtained for any combinations of simply supported and clamped edges, the free edges can also be dealt with if a pair of opposite edges is simply supported.

5 FREE VIBRATIONS OF CIRCULAR CYLINDRICAL SHELLS

The problem of calculating the free vibration characteristics of circular cylindrical shells of finite length has been of interest to engineers and scientists for about one and half a century [30]. The equations of motion of cylindrical shells together with relevant boundary conditions are more complex than those of beams and plates, therefore only the exact solutions for shells with all four boundaries having the shear diaphragm (SD) or simply supported conditions were used in literature [32]. The complexity of the solutions resulting for two opposite edges having SD boundaries, with the others arbitrary, prevented most researchers from getting results [30-32].

If possible and practical, the equations describing the free vibration of a cylindrical shell should be solved exactly because it is difficult to assess the degree with which an approximate solution approximates the exact solution [33]. Forsberg [34] presented an exact solution of the basic differential equations for isotropic Flügge circular cylindrical shells. Although the method requires numerical computation, the results are exact in the same sense that the numerical solution to the transcendental frequency equation for a beam yields an exact solution. However, the method leads to an eighth-order algebraic equation and an eighth-order frequency determinant which are coupled together. The simultaneous solution of these two systems of equations involves extremely laborious computation [31].

The disadvantages in the method of Forsberg [34], and Smith and Haft [35] were overcome by Vronay and Smith [36] through carrying all complex quantities through the solution process as complex, which led to a more truly "general" solution. In 1999, Callahan and Baruh [37] presented a systematic procedure for obtaining the closed-form eigensolution for thin circular cylindrical shell, which used the software packages MAPLE® and MATLAB® to generate the required expressions and iterate through a natural frequency band to find all associated zeros of the frequency equation. Xiang et al [38] gave an exact method based on the state space technique.

Until now, most methods above needs too much numerical computation and have computational problems even if the relative thickness or length/radius ratios are not small. Therefore, the widely used exact solutions are only available for isotropic and orthotropic circular cylindrical shells with all edges simply supported shells [32, 39, 40] before Xing et al. [41, 42] presented an analytical procedure and exact solutions in compact and neat closed-form for open and closed isotropic circular cylindrical shells [42] and closed orthotropic circular cylindrical shells [41] having classical boundary conditions.

Consider a circular cylindrical shell element having a constant thickness h, an axial length a, a circumferential length b and a middle surface radius R as shown in Fig. 2. The mode functions on the middle surface in α-, β-, and z-directions are denoted by $U(\alpha, \beta)$, $V(\alpha, \beta)$ and $W(\alpha, \beta)$ respectively, where $\beta = R\phi$ and ϕ is the circumferential coordinate.

An open circular cylindrical shells with $\beta =$ 0 and b edges of shear diaphragm conditions is discussed below.

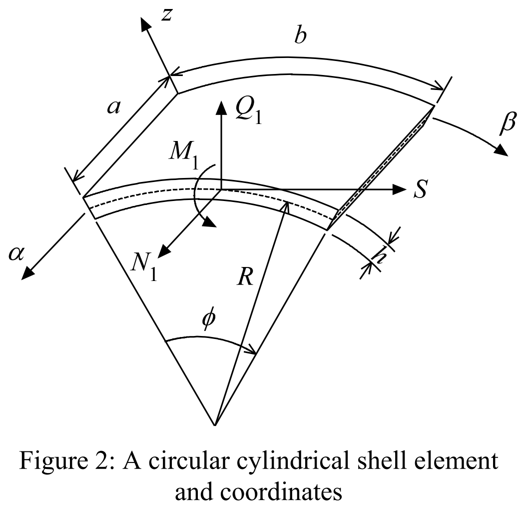

Figure 2: A circular cylindrical shell element and coordinates

Assume the separation-of-variable mode function $W = e^{\mu\alpha}e^{\lambda\beta}$, where $\lambda = i\Lambda$, $i^2 = -1$, $\Lambda = n\pi/b$, n is the number of half-waves in the β-direction. The resulted algebraic eigenvalue equations has the form

$$\tilde{a}\mu^8 + \tilde{b}\mu^6 + \tilde{c}\mu^4 + \tilde{d}\mu^2 + \tilde{e} = 0 \tag{22}$$

whose coefficients have explicit relations with Λ. Eq. (22) is a fourth-order polynomial in μ^2. Note that closed-form roots for third- and fourth-order polynomials are available in mathematic handbooks. When $R \to \infty$, Eq.(22) reduces to

$$\left[\mu^2-\Lambda^2+\left(\frac{\omega}{c}\right)\right]\left[\mu^2-\Lambda^2+\left(\frac{\omega}{c}\right)^2\upsilon_1\right]\left[D\left(\mu^2-\Lambda^2\right)^2-\rho h\omega^2\right]=0 \tag{23}$$

where $c=\sqrt{G/\rho}$ is the shear wave velocity, G is shear modulus. It follows from Eq. (23) that the in-plane and out-of-plane free vibration are decoupled when $R\to\infty$. The first two factors of Eq. (23) are the eigenvalue equation of in-plane free vibration of plate [14], and the last one is the eigenvalue equation of transverse free vibration of thin plate [9].

Assume the roots of Eq. (22) as

$$\mu_{1,2}=\pm\mathrm{i}\bar{\alpha}_1,\quad \mu_{3,4}=\pm\mathrm{i}\bar{\alpha}_2,\quad \mu_{5,6}=\pm\mathrm{i}\bar{\alpha}_3,\quad \mu_{7,8}=\pm\mathrm{i}\bar{\alpha}_4 \tag{24}$$

where $\bar{\alpha}_j$ (j = 1, 2, 3, 4) may be real, pure imaginary or complex. The mode functions U, V, and W in separation-of-variable form can be expressed in terms of four potentials w_j (j = 1, 2, 3, 4) as

$$U=\sum_{j=1}^{4}\bar{p}_j\frac{\partial\bar{w}_j}{\partial\alpha},\quad V=\sum_{j=1}^{4}\bar{q}_j\frac{\partial\bar{w}_j}{\partial\beta},\quad W=\sum_{j=1}^{4}\bar{w}_j \tag{25}$$

where

$$\bar{w}_j(\alpha,\beta)=\left(\bar{A}_j\sin\bar{\alpha}_j\alpha+\bar{B}_j\cos\bar{\alpha}_j\alpha\right)\sin\Lambda\beta \tag{26}$$

Using above method we have obtained the exact solutions [41, 42] for closed circular cylindrical Donnell shells with any classical boundary conditions, and open circular cylindrical Donnell shells with one pair of shear diaphragm edges while the other pair of opposite edges having arbitrary classical supports.

For open circular cylindrical shells with α= 0 and a edges having shear diaphragm condition while β = 0, b edges arbitrary, the exact solutions can be directly obtained from above solutions by a substitution of variables. For closed circular cylindrical shells with different supports, to substitute $\Lambda = n/R$ for the Λ in above solutions can produce all exact eigensolutions.

6 CONCLUSIONS

New analytical solutions obtained in recent years can be summarized as follows.

- For the free vibration of rectangular thin plate, the new exact solutions were obtained for isotropic and orthotropic thin plates with any combinations of simply supported, clamped and guided boundary conditions.
- For the free vibrations of rectangular Mindlin plate, the closed-form analytical solutions were obtained for plate with any combinations of simply supported and clamped edges; and exact closed-form solutions were obtained for orthotropic plates with two opposite edges simply supported.
- For the in-plane free vibration of rectangular plate, the exact solutions were obtained for isotropic and orthotropic rectangular plates with a pair of simply supported opposite edges and any classical boundary conditions at the other two edges.
- For the free vibrations of circular cylindrical Donnell-Mushtari shell, the closed form exact solutions were obtained for isotropic and orthotropic open and closed shells having classical boundary conditions.

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