

STABILITY OF THE ACTIVE VIBRATION CONTROL OF CANTILEVER BEAMS

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Keywords: Active vibration control, Cantilever beam, Stability, Non-collocated control, All pass filter.

Abstract. *The paper deals with analysis of a cantilever beam equipped by the active vibration control system. The cantilever beam as a continuum is approximated by the lumped-parameter system. The lumped-parameter model enables to derive the transfer functions relating forces to displacement, velocity or acceleration to determine the appropriate controller type and to determine under what conditions the system is stable having regard to the transfer function as a sum of partial fractions, that represent the second-order systems. The mathematical model of the cantilever beam represents a parallel connection of slightly damped second order systems with gain, whose polarity varies for each of these sub-systems at the non-collocated control. These sub-systems correspond to the various modes of vibration. Feedback as a part of an active vibration damping is negative for some of these elementary systems, and for the others, feedback is positive, i.e. causing instability. The paper deals with the stabilization of the active vibration control systems.*

1 INTRODUCTION

The paper deals with simulation of the active vibration control of a cantilever beam. For analysing vibration of the cantilever beam a lumped parameter model is created. This approach is due to the methods which are used for modelling control systems. The active vibration control is usually collocated what means that the sensor and actuator are located in the same place [1]. This paper is intended to study behaviour of a non-collocated control system of the cantilever beam. The transducer senses vibration of the free beam end while the actuator force acts close to the clamped end of the beam.

The mathematical model of the cantilever beam represents a parallel connection of slightly damped second order systems with gain, whose polarity varies for each of these sub-systems at the non-collocated control. These sub-systems correspond to the various modes of vibration. Feedback as a part of an active vibration damping is negative for some of these elementary systems, and for the others, feedback is positive, i.e. causing instability. The paper deals with the stabilization of the active vibration control systems.

2 LUMPED PARAMETER MODEL

A cantilever beam of the length L as a continuum can be divided into discrete elements of the same length ΔL that are modelled with the use of rigid-body dynamics. How to create the lumped parameter model of the cantilever beam of the rectangular cross section and to associate this multibody system with the Cartesian coordinates x, y, z is shown in Figure 1. The cantilever beam is clamped at the xy -plane and its centreline is parallel to the z -axis. It is assumed only planar motion of the cantilever beam in the yz -plane. Let N be the number of flexible links in the model. Linking of a pair of adjacent beam elements is considered in the mentioned plane as free with a torsion spring. To avoid the additional set of constrains for linking of the individual beam elements in one point the coordinate system is chosen in such a way that describes motion of the meeting points of two adjacent elementary beams. The vertical coordinates of these points are designated by y_1, y_2, \dots, y_N . The angle of rotation with respect to the horizontal axis can be designated by $\delta_1, \delta_2, \dots, \delta_N$.

The deflection Δy of the beam may be expressed as a function of the beam length and the difference $\Delta\delta$ of the adjacent elements $\Delta y = \Delta\delta \Delta L$. The bending stiffness K_δ of the elementary cantilever beam relates the applied bending moment T to the resulting relative rotation $\Delta\delta$ of the elementary beam

$$K_\delta = \frac{T}{\Delta\delta} = \frac{3EI_x}{\Delta L}, \quad (1)$$

where E is Young's modulus of the beam material [N/m^2], $I_x = bh^3/12$ is the area moment of inertia of the beam cross-section [m^4] about the horizontal x -axis, b is the beam width and h is the beam height.

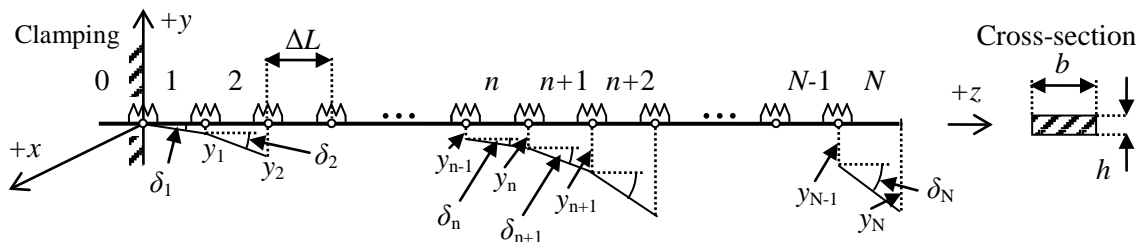


Figure 1: Coordinates and elements of a cantilever beam

The coordinates of the beam equidistant points in the Cartesian coordinates and the independent generalized coordinates for Lagrangian equations of motion are identical. For further derivation it makes sense only motion in the direction of the y -axis. Because they are assumed small deformations, the shifts of the meeting points in the direction of the z -axis are neglected. If all angles are small enough, then their measure in radians is given by the formula $\delta_n = (y_n - y_{n-1})/\Delta L$. The coordinates of the gravity centre of the elementary beams are as follows $(y_n + y_{n-1})/2$.

The potential V and kinetic T energy of the cantilever beam in the horizontal position as a continuum replaced by its lumped parameter model is as follows

$$\begin{aligned} V &= \sum_{n=0}^{N-1} \frac{1}{2} K_\delta (\delta_{n+1} - \delta_n)^2 + \sum_{n=1}^N \Delta m g (y_n - y_{n-1})/2, \\ T &= \sum_{n=1}^N \left(\frac{1}{4} \Delta m \left(\frac{d(y_n - y_{n-1})}{dt} \right)^2 + \frac{1}{2} J_x \left(\frac{d\delta_n}{dt} \right)^2 \right), \end{aligned} \quad (2)$$

where J_x is the moment of inertia [kg m^2] about the horizontal x -axis and perpendicular to the centreline of the elementary beam. For the beam element of the solid brick shape with height h , length ΔL and mass Δm the moment of inertia of such element is given by the formula $J_x = \Delta m (\Delta L^2 + h^2)/12$, The cantilever beam is a conservative system. Lagrange's equations of motion of such a system are as follows

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{y}_n} \right) - \frac{\partial T}{\partial y_n} + \frac{\partial V}{\partial y_n} = 0, \quad n = 1, 2, \dots, N \quad (3)$$

After introduction symbols \mathbf{M} for a mass square matrix and \mathbf{K} for a stiffness square matrix and \mathbf{G} for a gravity force column vector and \mathbf{y} for a coordinate column vector into the matrix equation of motion we obtain

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \mathbf{G} = \mathbf{0} \quad (4)$$

where $\mathbf{y} = [y_1, y_2, \dots, y_N]^T$ (the upper index T designates transposition of a vector) and $\mathbf{G} = \Delta m g [1, 1, \dots, 1, 1/2]^T$ are column vectors and

$$\mathbf{M} = \begin{bmatrix} B & A & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & A & B & A & \\ \vdots & \vdots & \vdots & \vdots & \\ & & & A & B/2 \end{bmatrix}, \quad \mathbf{K} = \frac{K_\delta}{\Delta L^2} \begin{bmatrix} 6 & \dots & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & 1 & -4 & 6 & -4 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & \dots & 1 \end{bmatrix} \quad (5)$$

$$A = \frac{\Delta m}{4} \left(1 - \frac{1}{3} \left(1 + \left(\frac{h}{\Delta L} \right)^2 \right) \right), \quad B = \frac{\Delta m}{2} \left(1 + \frac{1}{3} \left(1 + \left(\frac{h}{\Delta L} \right)^2 \right) \right) \quad (6)$$

are the mass and stiffness matrices of the $N \times N$ size.

Steady state deformation of the beam in the horizontal position due to the gravity force is resulting from solution of the equation $\mathbf{K}\mathbf{y} + \mathbf{G} = \mathbf{0}$. If the cantilever beam is in vertical position then $\mathbf{G} = \mathbf{0}$ and $\mathbf{y}_0 = \mathbf{0}$. For testing a beam with the following parameters is prepared a

specimen of the following parameters: $L = 0.5$ [m], $b = 0.04$ [m], $h = 0.005$ [m]. The beam is divided into $N = 10$ elements. Deflection of the beam's own self-weight is shown in Figure 2A.

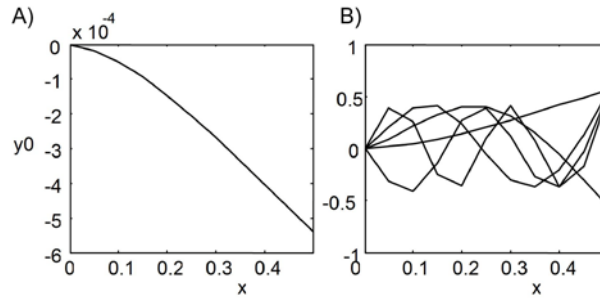


Figure 2: A) Deflection of the cantilever beam's own self-weight, B) The first 5 of 10 modal shapes of the cantilever beam

2.1 Modal analysis

For vertical position of the cantilever beam the governing equation of free vibration is as follows

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{0} \quad (7)$$

The solution of this equation of the homogenous type is assumed to be in the form of $\mathbf{y} = \mathbf{u}\exp(j\omega t)$, where \mathbf{u} is an N -dimensional vector of the oscillation amplitudes and ω is an angular frequency. After substitution into Eq. (7) we obtain a homogenous equation

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{u} = \mathbf{0}. \quad (8)$$

In Eq. (8) we may use a substitution $\lambda = \omega^2$. For the nonzero vector \mathbf{u} the determinant of the matrix $(\mathbf{K} - \lambda\mathbf{M})$ has to be zero. Because this determinant is an N -degree characteristic polynomial of the variable λ , the number of roots λ_n , called the eigenvalues, is equal to the degree of the polynomial. The corresponding nonzero solution of the homogenous system of equations is called an eigenvector. We form a spectral matrix $\mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ and an eigenvector matrix $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N]$.

The beam in the numeric example is divided into 10 elements; it is possible to calculate 10 modal frequencies and 10 modal shapes of vibration. Only 5 out of the modal shapes, identifiable by the number of nodes, are shown in Figure 2B. The modal frequencies are shown in Table 1 as well as modal constants $v_{n,N}v_{1,n}$ whose meaning is explained later.

Mode n	1	2	3	4	5	6	7	8	9	10
Freq Hz	26.6	167	470	922	1522	2256	3093	3965	4755	5309
$v_{n,N}v_{1,n}$	0.146	0.669	1.339	-1.775	1.866	1.628	1.176	0.670	-0.264	0.049

Table 1: Modal parameters for $N = 10$.

2.2 Excited vibration

The excited vibration of the cantilever beam in the vertical position describes the equation of motion with the external forces p_1, p_2, \dots, p_N assembled into a vector $\mathbf{p} = [p_1, p_2, \dots, p_N]^T$ on the right side and acting at the gravity centre of the beam elements

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{p} \quad (9)$$

The presence of viscous damping, such as a dissipative force, extends the left side of the equation of motion by an additional term which is proportional to velocity

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{p}, \quad \mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K}, \quad (10)$$

where the matrix \mathbf{C} of proportionality for Rayleigh damping is a linear combination of the mass and stiffness matrices, and α, β are constants of proportionality. The dependence of the damping ratio ζ on frequency f_0 in herz can be seen from the formula $\zeta = \pi(\alpha/f_0 + \beta f)$ [3].

2.3 Transfer function

Vibration of mechanical structures is dampened very slightly. It's only a few percent of critical damping. The purpose of active vibration control is increase ability of structures to absorb vibration by adding an artificial electronic feedback. To analyze the effect of active vibration damping we assume that the system is not damped at all.

If \mathbf{y} and \mathbf{p} are complex harmonic functions of time ($\exp(j\omega t)$) and \mathbf{Y} and \mathbf{P} are their complex magnitudes, respectively, then the transfer function in the form of a squared matrix \mathbf{H} , relating the displacement $y_r, r=1, \dots, N$ to the force $p_q, q=1, \dots, N$ is given by the formula

$$\mathbf{Y} = (\mathbf{K} - \lambda \mathbf{M})^{-1} \mathbf{P} = \mathbf{H} \mathbf{P} \quad (11)$$

A modal transform $\mathbf{y} = \mathbf{V} \mathbf{q}$ is the basis for the derivation of the transfer function. The coordinates \mathbf{y} are transformed into generalized coordinates \mathbf{q} by using the matrix \mathbf{V} [3,4]. The relationship of the transfer function to the modal properties of the structure can be defined if the modal transformation matrix \mathbf{V} has the following property $\mathbf{V}^T \mathbf{M} \mathbf{V} = \mathbf{I}$. It can be proved that the orthonormal eigenvectors $\mathbf{v}_n, n=1, \dots, N$ arranged in the matrix $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N]$ may be calculated from a set of the eigenvectors $\mathbf{u}_n, n=1, \dots, N$ for not generalized coordinates \mathbf{y} by the following formula

$$\mathbf{v}_n = \mathbf{u}_n / \sqrt{\mathbf{u}_n^T \mathbf{M} \mathbf{u}_n}, \quad n=1, \dots, N. \quad (12)$$

The transfer function matrix \mathbf{H} , called the receptance, as a function of $\lambda = \omega^2$ depends on the eigenvectors and eigenvalues according to the formulas

$$\begin{aligned} \mathbf{H} &= (\mathbf{K} - \lambda \mathbf{M})^{-1} = \mathbf{V} \mathbf{D} \mathbf{V}^T, \\ \mathbf{D} &= \text{diag}(1/(\lambda_1 - \lambda), 1/(\lambda_2 - \lambda), \dots, 1/(\lambda_N - \lambda)), \end{aligned} \quad (13)$$

where \mathbf{D} is a diagonal matrix. The matrix \mathbf{H} relates the force acting at the q -th lumped mass, to the displacement y_r of the r -th lumped mass, where is measured. The individual elements of the matrix \mathbf{H} are as follows

$$H_{r,q}(j\omega) = \sum_{n=1}^N \frac{v_{n,r} v_{q,n}}{\omega_n^2 - \omega^2}, \quad r, q = 1, 2, \dots, N, \quad (14)$$

where $v_{q,r}, q, r=1, \dots, N$ is the r -th element of the q -th normalized eigenvector. The poles of the transfer function lie on the imaginary axis. The system is on the stability margin, not stable and simultaneously not unstable. The parameter $k_n = v_{n,r} v_{q,n}$ is called a modal constant.

3 ACTIVE VIBRATION CONTROL

3.1 Configuration of AVC

The purpose of the system for the active vibration control (AVC) is to compensate the effect of a disturbing external force on the vibration of the beam. It is desirable to relocate the poles of the transfer function of the controlled system from the imaginary axis in the left half-plane of the complex plane. The cantilever beam is considered as an MIMO system composed of the lumped masses whose count is equal to N . Vibration of all these masses can be controlled by forces acting at all of them.

The location of the sensor and actuator which is shown in the left panel of Figure 3 is more practical configuration of the active vibration control than the mentioned MIMO system. We assume that the system is of the SISO type with a controller with a transfer function $R(s)$. For the non-collocated system it is assumed that the correcting force p_1 acts at the lumped mass indexed by $q=1$ and the vibrations are sensed at the lumped mass indexed by $r=N$. A block diagram of the closed loop system is shown in the right panel of Figure 3. The transfer function of the vibration sensor is assumed to be s^k . The output of the controlled system y_N is measured by either a displacement sensor ($k=0$) or a velocity sensor ($k=1$) or an acceleration sensor ($k=2$). Consequently the physical quantity for the set point of the controller is either position or velocity or acceleration. Zero value of the controller set point means reaching a target state, when the motion of the free end of the beam is stopped.

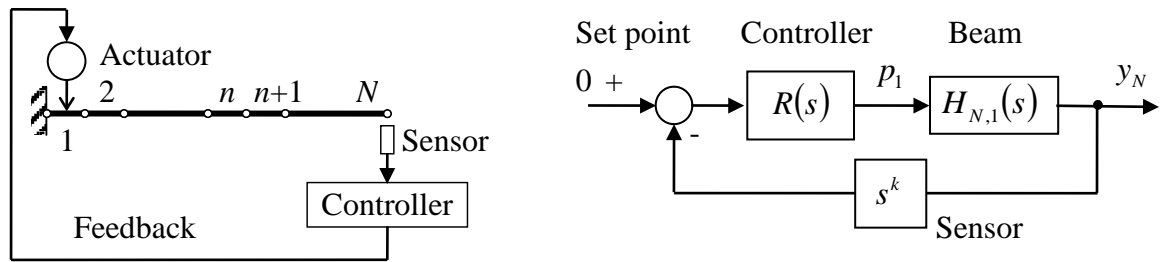


Figure 3: Non-collocated system of active vibration control.

There is a transfer function of the closed-loop system $\tilde{H}_{N,SP}$, relating the displacement y_N of N -th lumped mass to the set point y_{SP} , and the function $H_{N,1}$, relating the displacement y_N to the feedback force p_1 acting at the first lumped mass

$$\tilde{H}_{N,SP}(s) = \frac{Y_N(s)}{Y_{SP}(s)} = \frac{R(s)s^k H_{N,1}(s)}{1 + R(s)s^k H_{N,1}(s)} = \frac{R(s)s^k \sum_{n=1}^N \frac{v_{n,1}v_{N,n}}{\omega_n^2 + s^2}}{1 + R(s)s^k \sum_{n=1}^N \frac{v_{n,1}v_{N,n}}{\omega_n^2 + s^2}} = \frac{M(s)}{N(s)} \quad (15)$$

The denominator of the transfer function of the closed loop is as follows

$$N(s) = \prod_{n=1}^N (\omega_n^2 + s^2) + R(s)s^k \sum_{n=1}^N v_{n,N}v_{1,n} \prod_{\substack{k=1 \\ k \neq n}}^N (\omega_k^2 + s^2) \quad (16)$$

With the exception of the term $R(s)s^k$ both polynomials are a function of the squared variable s and thus there are missing the odd powers of this variable s . For a structurally stable system, the variable s raised to odd powers has to be added

$$s^k R(s) \sum_{n=1}^N v_{n,N} v_{1,n} \prod_{\substack{k=1 \\ k \neq n}}^N (\omega_k^2 + s^2) = \sum_{n=1}^N T_{D,n} s^{2n-1} \Rightarrow R(s) = \frac{\sum_{n=1}^N T_{D,n} s^{2n-1-k}}{\sum_{n=1}^N v_{n,N} v_{1,n} \prod_{\substack{k=1 \\ k \neq n}}^N (\omega_k^2 + s^2)} \quad (17)$$

where $T_{D,n}$, $n=1, \dots, N$ are the mentioned positive coefficients.

3.2 Controller type

The following choice of the controller with a time constant T_D allows easy tuning

$$s^k R(s) = T_D s \Rightarrow R(s) = T_D s^{1-k} \quad (18)$$

Suitable controllers for all relevant three types of vibration sensors are in Table 2.

Sensor	Controller
Displacement ($k = 0$)	$R(s) = T_D s$
Velocity ($k = 1$)	$R(s) = T_D$
Acceleration ($k = 2$)	$R(s) = T_D / s$

Table 2: Types of the controller saving the structural stability.

The disturbance force can be of the broad or narrow frequency spectrum. Suppose that the frequency spectrum of disturbance affects only the d -th mode of vibration.

$$H_{N,1}(s) = \frac{v_{d,N} v_{1,d}}{\omega_d^2 + s^2} = \frac{K_d}{\omega_d^2 + s^2} \quad (19)$$

The transfer function of the close loop with the controller described by the transfer function (20) is as follows

$$\tilde{H}_{N,SP}(s) = \frac{Y_N(s)}{Y_{SP}(s)} \approx \frac{T_D s K_d / (\omega_d^2 + s^2)}{1 + T_D s K_d / (\omega_d^2 + s^2)} = \frac{K_d T_D s}{s^2 + K_d T_D s + \omega_d^2} \quad (21)$$

The damping ratio ξ and the decay constant σ

$$\xi = \frac{1}{2} \frac{K_d T_D}{\omega_d}, \quad \sigma = \xi \omega_d = \frac{K_d T_D}{2} \quad (22)$$

The polarity of the feedback is negative for a positive value of the modal constant. If the modal constant is negative, then the feedback is positive and the system of the active vibration control is unstable.

The actual cantilever beam is damped by the natural damping, which is modelled for instance by Rayleigh. Positive feedback can be so weak that the poles of the transfer function cannot get to the instability area, i.e. to the right side of the imaginary axis of the complex plane for a small gain of a controller. A tool for analysis of the influence of any parameter of a controller on the position of the transfer function poles of the closed loop is root locus.

For a given beam, which is divided into 5 elements, and the assumption of the Rayleigh's damping with the parameters $\alpha = 0.159$ and $\beta = 0.0000411$ are the root locus shown in Figure 4. The time constant T_D of the controller ranges from 0 to one thousand. The poles are calculated as the roots of the following polynomial

$$0 = 1 + T_D s H_{N,1}(s) \Rightarrow 0 = \prod_{n=1}^N (\omega_n^2 + 2\xi_n \omega_n s + s^2) + T_D s \sum_{n=1}^N v_{n,N} v_{1,n} \prod_{\substack{k=1 \\ k \neq n}}^N (\omega_k^2 + 2\xi_k \omega_k s + s^2) \quad (23)$$

The stability margin crosses the pole for the mode $n = 2$ and pole for the mode $n = 4$ approaches this margin.

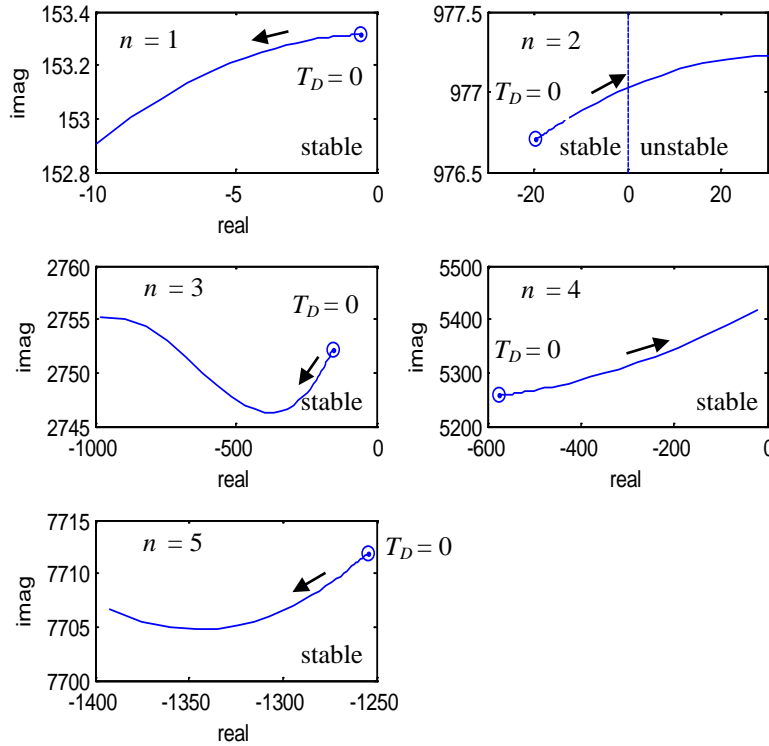


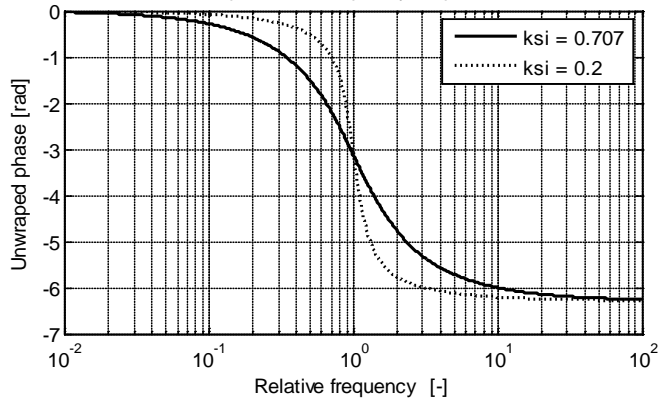
Figure 4: Root locus for the time constant T_D ($N = 5$).

3.3 Converting positive feedback to negative

There is a frequency filter which modifies the phase of the harmonic signal at the output compared to the input without changing the signal magnitude. This filter is called an all-pass filter. The filter of this type of the first-order changes the phase from 0 to π radians when changing the frequency from 0 to infinity. For the unstable modes it is necessary to change the phase by 2π , while the phase equal to π matches the frequency of the vibration mode whose modal constant is to be changed from negative to positive. The advantage of this filter is controllable rate of change of phase with respect to change of the frequency by setting the value of the parameter ξ_{APF}

$$G_{APF,n}(s) = \frac{s^2 - 2\xi_{APF} \omega_n s + \omega_n^2}{s^2 + 2\xi_{APF} \omega_n s + \omega_n^2} \quad (24)$$

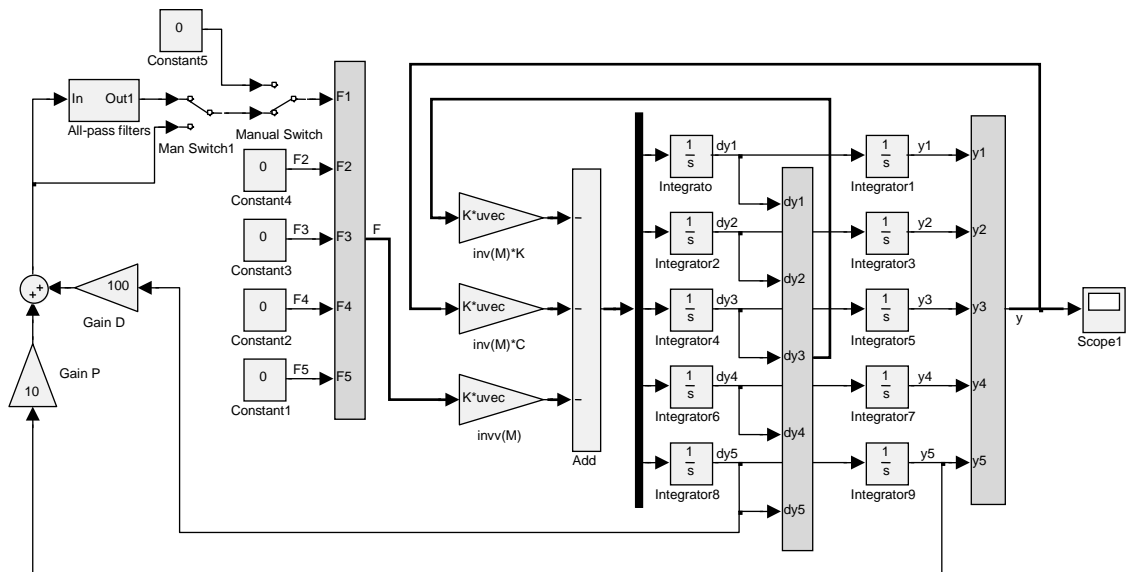
Since $|G_{APF,n}(j\omega)| = 1$ only a phase response for two values of ξ_{APF} is shown in Figure 5.


 Figure 5: All-pass filter frequency response ($\xi_{APF} = \xi$).

All-pass filters are connected in series with the controller. The count of these filters is as many as the count of the negative modal constants.

4 SIMULATION RESULTS

Simulink model of the cantilever beam for the five elements is shown in figure 6. The circuit configuration also includes a PD controller (Gain P and gain D) and the all-pass filter of the second order (All-pass filter). The effect of the active vibration control is often demonstrates on the vibration decay of the beam which is bent into a deflected position by acting force 10 N and suddenly released. In this case, only the lowest modes of vibration are excited.


 Figure 6: Simulink model of the cantilever beam for five elements ($N = 5$).

The effect of ACV on the decay of vibration is shown in figure 7. The time constant of the derivative component influences the damping effect strongly. The margin value of the derivative time constant T_D is slightly greater than 15. The slowest decay of the beam vibration is reached if AVC is switched off. Without the use of the all-pass filter and for the limited value of the time constant ($T_D = 15$) the decay of vibration is faster than in the case when AVC is switched off, but the fastest decay is achieved using two all-pass filters of the second order which cancel two unstable vibration modes.

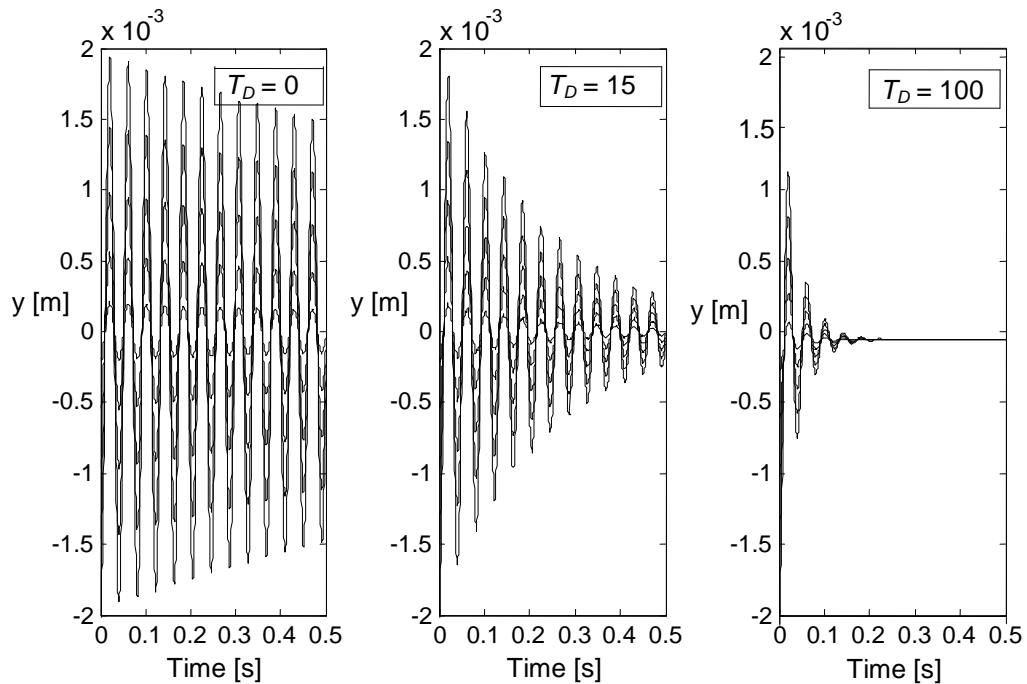


Figure 7: The effect of ACV on the decay of vibration for $T_D = 0$ (AVC OFF), $T_D = 15$ (AVC ON) and $T_D = 100$ (AVC ON and using two all pass filters) ($N = 5$).

5 CONCLUSIONS

The lumped-parameters model of the cantilever beam was designed using the method based on the modal analysis. It was proved that the cantilever beam can be actively damped only by a force which is controlled by the derivative controller if the displacement is measured. The feedback of the D type is sufficient for damping of the slightly damped systems. The all pass filter of the second order removes the problem with positive feedback for some modes of vibration and increase the efficiency of damping.

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ACKNOWLEDGMENT

This research has been supported by the Czech Grant Agency project No. P101/12/2520 “Active vibration damping of rotor with the use of parametric excitation of journal bearings” and elaborated in the framework of the project Opportunity for young researchers, reg. no. CZ.1.07/2.3.00/30.0016, supported by Operational Programme Education for Competitiveness and co-financed by the European Social Fund and the state budget of the Czech Republic.