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A COMPARISON BETWEEN THE USE OF STRAIGHT AND CURVED BEAM ELEMENTS FOR MODELLING CURVED RAILWAY TRACKS

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Abstract. *A major environmental concern related to railway traffic is vibration. A lot of research has been carried out to understand vibration of straight tracks, with less attention been paid to curved tracks. Modelling the dynamic behaviour of a curved railway track is important to understand the physics of generation and propagation of vibration from trains at non-straight sections of tracks. Modelling is also important to assess the current and any alternative track designs from an environmental point of view.*

In this paper a curved track is modelled and the effect of curvature is investigated. Two models have been developed and their results have been compared. In the first, the curved track is modelled using straight beam elements. In the second curved beam elements are used. For both, the Euler-Bernoulli beam theory has been adopted to describe their bending behaviour. The elements have 12 degrees of freedom accounting for displacements and rotations in the lateral, transverse and longitudinal directions. The excitation comes from an axle traversing the rails with subcritical velocity, accounting for the wheel-rail contact forces.

The described models are solved using the Finite Element Method. The time domain response of the versine of the curved track due to the passage of the axle is computed. A comparison is made on the efficiency of the two models for different curve radii and frequencies. The two models provide very similar results showing that the piecewise straight beam approximation represents the behaviour of the curved track accurately. Also the curved beam model used in this study shows some limitations for the specific application and therefore the straight element method is recommended.

1 INTRODUCTION

Vibration from trains is an environmental concern affecting residents and buildings near railway lines. Modelling the dynamic behaviour of tracks aids in the understanding of the physics which is essential for the provision of solutions. In the literature there is a large number of models reported on the dynamic behaviour of railway tracks. Most of these focus on straight sections of tracks and less attention has been paid to curved tracks. This is mainly attributed to the higher complexity associated with the modelling of curved tracks compared to straight ones. However, curved tracks are reported to have additional problems which increase the need for further research.

In the literature, the effect of curvature of beams has been subjected to limited study. Following are some of the studies found in the literature. In [1], the analytical solution was derived for the dynamic response of a horizontally curved beam subjected to vertical and horizontal moving loads. This was solved for a simply supported beam. In a series of papers [2–7], a new approach to derive the displacement functions of a curved beam was presented, based on which the shape functions and the mass and stiffness matrices for an arch element of constant radius were derived. The method was used to study the free vibration of a curved beam due to a moving load for both the in-plane and out-of-plane response. One may refer to the first sections of papers [1], [2] and [3] for a background on the study of curved beams for both analytical and numerical approaches. More recent approaches include that of [8] who used the moving element method to model a curved beam subjected to a moving load using straight beam elements. In [9] the forced vibration of curved beams on a two-parameter elastic foundation subjected to impulse loading is studied. In the work of [10] the general dynamic response induced by a moving load along a curved path on an elastic semi-infinite space was obtained and applied to the case of a periodic curved track structure. None of the literature cited above analysed curved beams under harmonic loads.

In this work a curved track is formulated as a discretely supported beam on elastic foundation and is analysed using the Finite Element Method. The model is discretised using both straight and curved beam elements which are formulated using the Euler-Bernoulli beam bending theory. Torsional effects are also taken into account. The paper presents results describing the dynamics of a curved track with focus on the displacement of the mid-span. Calculations are performed for vertical harmonic moving loads and the displacements are compared for the cases of straight and curved beam elements. The effects of curvature and loads frequency are also investigated.

2 MODEL DEVELOPMENT

For this model, the railway track has been modelled as a single rail traversed by a point load. In the following sections, details are given on the formulation of the model and the method for solution based on a matrix approach.

2.1 Form of the applied load

The excitation force is a moving harmonic load as shown in Figure 1 and has the following form:

$$F_{ap} = P \cos\left(\overline{\omega} t + \overline{\varphi}\right) \tag{1}$$

where P is the load amplitude (N), $\overline{\omega}$ is the radial frequency of the applied load (rad/sec), V is the speed of the moving load (m/s) and t is the time (s). $\overline{\varphi}$ is an initial phase applied to the load to ensure that for every frequency, it will arrive at the mid-span with maximum amplitude. The load is applied at the center-line of the rail at all times.

2.2 Track layout

The rail is modelled as a beam discretely supported on an elastic foundation. In order to define the geometry of the curved track the parameters needed are a) the radius of curvature R (m), b) the total length of the track L_t (m) and/or c) the total subtended angle θ (rad). Since these variables are related by the relationship $\theta = L_t/R$, one only needs two of the three parameters. Figure 1 depicts the layout of the track.

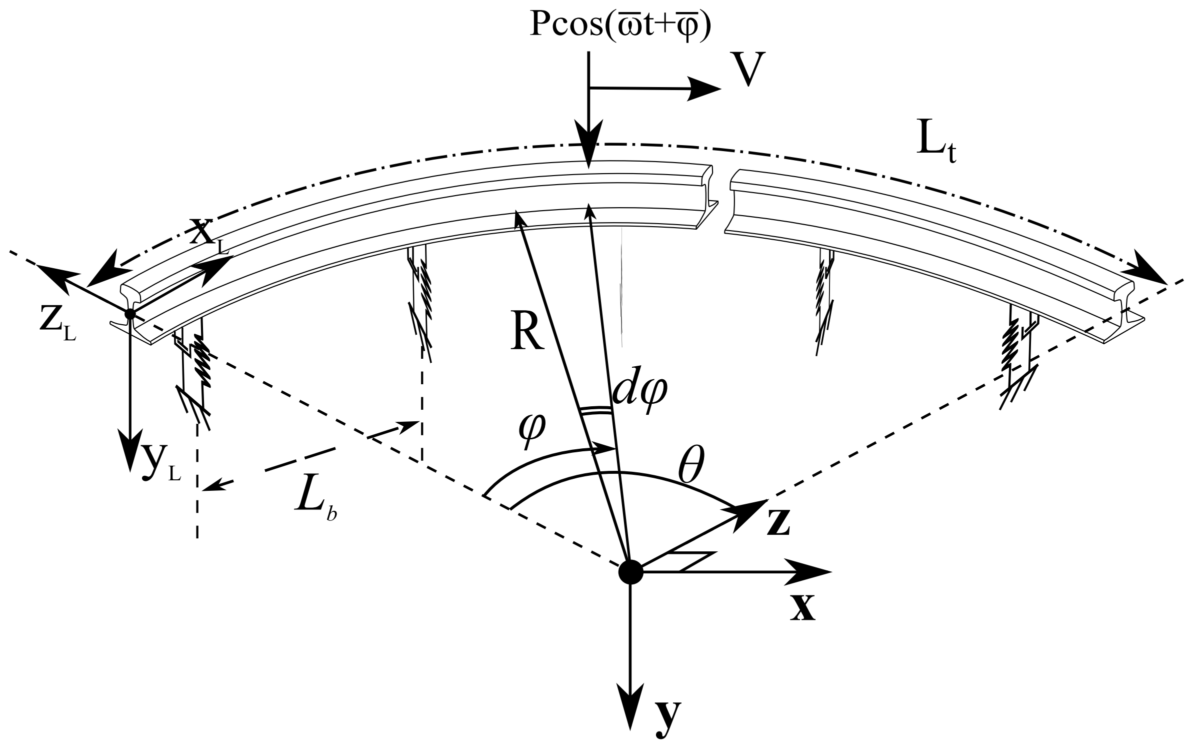

Figure 1: Curved beam on elastic foundation

In Figure 1, φ is the angle between the left support and the position of the load. The beam is discretely supported through rail-pads on a two-parameter visco-elastic foundation with pad stiffness k_p (N/m) and damping c_p (Ns/m). This is done through the use of a spring and a dashpot in parallel, also known as the Kelvin-Voigt model. The damping is applied as a percentage (ζ) of the critical damping (c_{cr}) of the system, $c_p = \zeta c_{cr}$. The distance between two successive supports/rail-pads is denoted as L_b. The foundation itself rests on a rigid base. In this paper, the track origin is located directly above the global origin at a distance R (i.e. $(0, R)$).

2.3 Discretisation of the model

In order to analyse and solve the above system using a matrix approach, the track will have to be discretised into a finite number of elements, n. Each of the individual elements will have length $L_{el} = L_t/n$. Then the local mass (M_L), stiffness (K_L) and damping (C_L) matrices will be derived, converted to the global mass (M_G), stiffness (K_G) and damping (C_G) matrices and the system will be solved in the time domain using direct time-step integration.

The interest of this work is to compare the use of straight and curved beam elements, thus two separate models were developed respectively. In the model with the piecewise straight approximation of the curved elements, the length of the element is approximated by that of the arc length as shown in Figure 2. For the curved elements the arc length is used.

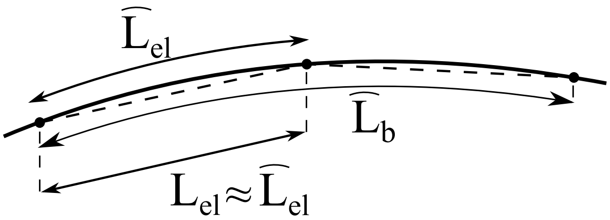

Figure 2: L_{el} approximation for straight beam elements.

The points where the elements connect with each other are called nodal points. Each node has six degrees of freedom (dof's) accounting for displacements (u) and rotations (θ) in the axial, vertical and horizontal (x, y, z) direction as shown in Figure 3.

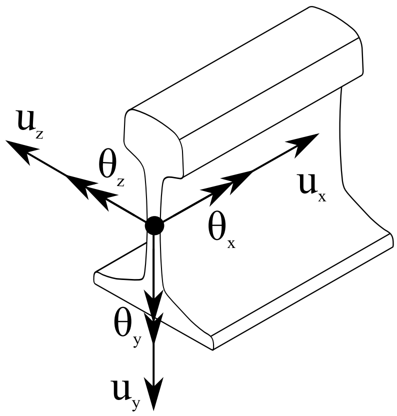

Figure 3: Degrees of freedom per node.

Each element for both models, has been modelled using the Euler-Bernoulli beam theory for bending, also accounting for torsional effects. For the straight beam elements, the elementary stiffness and mass matrices are readily available in the literature [11, 12]. For the curved elements, the approach presented in [2–7] was used. The in-plane responses (u_x,u_z,θ_y) are derived based on [5] while the out-of-plane responses (u_y,θ_x,θ_z) based on [7]. The local matrices will be of size $12 \ by \ 12$ for all the degrees of freedom of each element.

In order to assemble the global mass and stiffness matrices, the elementary matrices need to be converted from their local co-ordinates to the global ones. This means a transformation from the local Cartesian and curvilinear co-ordinates to the global Cartesian co-ordinates for the straight and curved beam element matrices respectively. This process is achieved by using the transfer matrix **T** as follows:

$$\begin{aligned} \mathbf{M_G} &= \mathbf{T}^T \mathbf{M_L} \mathbf{T} \\ \mathbf{K_G} &= \mathbf{T}^T \mathbf{K_L} \mathbf{T} \end{aligned} \tag{2}$$

where the superscript 'T' denotes the transpose of the matrix. The transfer matrix for both the straight and curved beam elements has the following format:

$$\mathbf{T} = \begin{bmatrix} \mathbf{D}_{\psi 1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{\psi 1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{\psi 2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{\psi 2} \end{bmatrix} \tag{3}$$

where $\mathbf{D}_\psi$ is:

$$\mathbf{D}_\psi = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix} \tag{4}$$

For the straight beam elements, ψ_1 is equal to ψ_2 and it is the angle measured between the tangent to the track origin and the parallel to the straight beam element. For the curved beam elements, ψ_1 and ψ_2 denote the angles between the tangent to the track origin and the tangents at nodes ① and ② respectively.

After the global mass and stiffness matrices have been derived, a Rayleigh damping matrix is used which is proportional to the mass and stiffness matrix in the following way:

$$\mathbf{C_G} = \alpha\,\mathbf{M_G} + \beta\,\mathbf{K_G} \tag{5}$$

Variables α and β depend on the mode shapes one wishes to apply the damping on, as well as the damping ratio of each mode. For more details one may refer to [13]. The size of the global matrices is $12(n+1)$ by $12(n+1)$. The stiffness and damping of the rail-pads can be added directly on the global stiffness and damping matrices since they only act on the vertical degree of freedom.

2.4 Equation of motion and solution of the system

After the global mass, stiffness and damping matrices have been derived, the equation of motion of the system can be written as:

$$\mathbf{M_G}\ddot{\mathbf{u}} + \mathbf{C_G}\dot{\mathbf{u}} + \mathbf{K_G}\mathbf{u} = \mathbf{F_{ext}} \tag{6}$$

where $\mathbf{u}$ contains the displacements and rotations of all the nodal points at each time and the dot and double dot notation show the first and second derivative with respect to time respectively. $\mathbf{F_{ext}}$ is a vector containing all the nodal forces at each time.

The above equation is typically solved using a direct time-step integration scheme. This procedure involves converting the forces at each time-step, to end node forces and moments using shape functions, otherwise called the equivalent load vector [13,14]. The shape functions used for the two models are the Hermitian shape functions for the straight beam elements and the implicit shape functions presented in [5,7] for the curved beam elements. The integration scheme used for this work is a composite implicit time integration procedure presented in [15].

3 MODEL VALIDATION

In order to validate the two models, a comparison was made between the numerical models and the analytical solution provided in [1]. In this comparison, the track consists of a simply supported curved track traversed by a moving load of constant amplitude. The vertical displacement and rotation around the x-axis of the track end nodes are fully restrained. The natural damping of the beam is neglected. The details for the comparison are listed in Table 1 and the track layout is presented in Figure 4.

Parameter	Value	Parameter	Value
Total length, L_t	$24\ m$	Speed of load, V	$40\ m/s$
Subtended angle, ϕ	$30°$	Load magnitude, P	$1\ N$
Cross-section area, A	$9\ m^2$	Young's modulus, E	$33.2\ GPa$
Moment of inertia, I_x	$2.43\ m^4$	Shear modulus, G	$G = 13.833\ GPa$
Moment of inertia, I_y	$18.75\ m^4$	Polar moment of inertia, J	$21.18\ m^4$

Table 1: Data for the comparison between analytical solution and numerical models.

It is noted here that there is a typographical error in [1] and one needs to interchange coefficients a_1 and a_2 for b_1 and b_2 respectively in order to obtain consistent results (see for example [16]). The details for the numerical models are shown in Table 2.

Straight elements		Curved Elements	
No. of element	10	No. of element	4
Element size	$2.4\ m$	Element size	$6\ m$

Table 2: Element size and number for the developed models.

Figure 5 shows the results of the comparison for the displacement of the mid–span. One can see that both models show a good agreement with the analytical solution. The time-step used in this comparison is $\Delta t = 0.06s$.

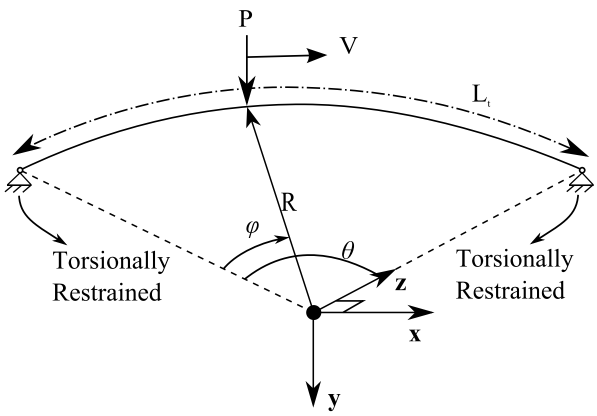

Figure 4: Simply supported track layout.

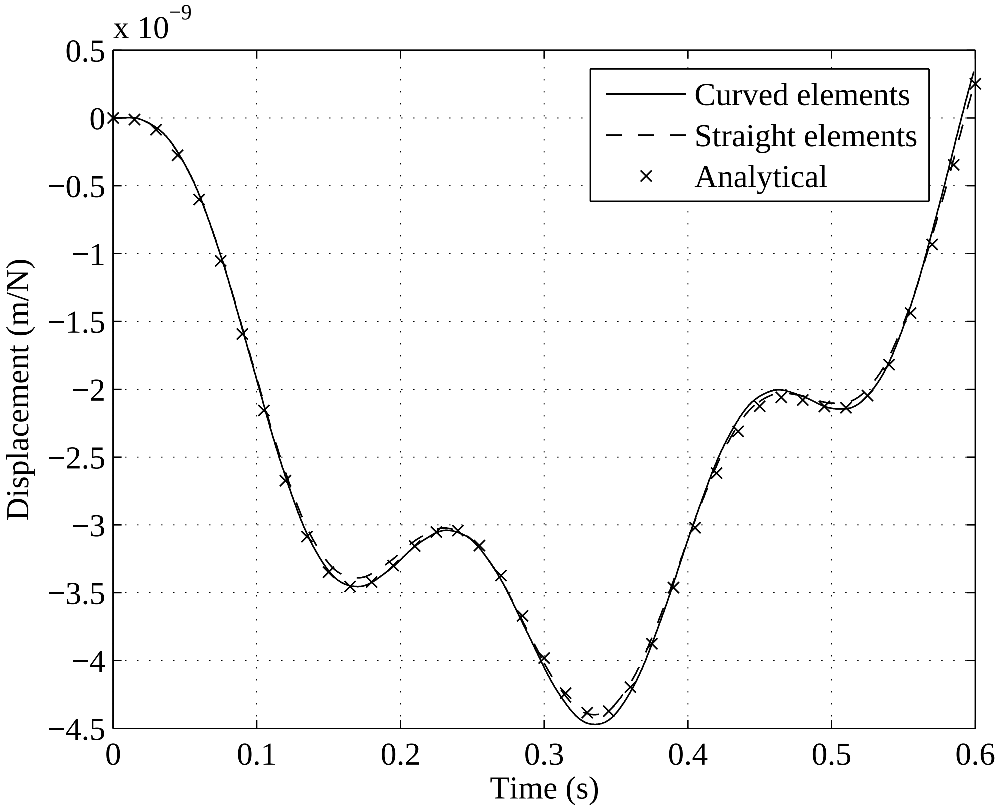

Figure 5: Transfer function for the comparison of the curved track using curved and straight beam elements against the analytical solution for a constant moving load with velocity $V = 40m/s$ on a track of length $L_t = 24m$ over an angle $\phi = 30°$

4 RESULTS AND DISCUSSION

The parameters used for the for the following simulations are summarised in Table 3. These parameters used for the rail are those for a 60-E1 rail specification. With the chosen foundation properties the cut-on frequency, f_{co} is at 118 Hz. This is based on the equation $f_{co} = \sqrt{k_{dist}/m}$ where $k_{dist} = k_p/L_b$.

In Figure 6 the transfer function for the vertical deflection of the mid-span is shown for two frequency cases. This simulation has been performed for both models using a single element between the beam supports. The results from the two models are almost identical showing a very good agreement for the frequency cases considered in this example. One will notice some fluctuations occurring for the case of $0Hz$ at the sleeper passing frequency. These are caused by the torsional forces created between the elements and if the radius is increased these fluctuations become smaller.

The percentage difference between using curved and straight elements for the given track are shown in Figure 7 for when the load is passing the two elements on either side of the mid-node. The difference between curved and straight elements fluctuates at about $1.5\ \%$ with only one peak exceeding $2\ \%$ for a radius of $R = 20\ m$. The differences become significantly smaller, fluctuating at $0.5\ \%$, for the case of $R = 50\ m$. This shows that the more straight the track becomes, the straight element approximation is becoming more accurate. If one considers that $50\ m$ radius is very small for a typical railway track and that the error of $0.5\ \%$ is expected to further reduce, then it can be seen how the straight element approximation is very good

Parameter	Value	Parameter	Value
Total length, L_t	$90\ m$	Speed of load, V	$20\ m/s$
Support distance, L_b	$0.6\ m$	Load magnitude, P	$1\ N$
Rail mass, m	$60.21\ kg/m$	Young's modulus, E	$210\ GPa$
Cross-section area, A	$7.672e^{-3}\ m^2$	Shear modulus, G	$79.56\ GPa$
Moment of inertia, I_x	$3038.3e^{-8}\ m^4$	Polar moment of inertia, J	$3550.6e^{-8}\ m^4$
Moment of inertia, I_y	$512.3e^{-8}\ m^4$	Beam damping, α, β	$10\ \&\ 5\ \%$
Railpad stiffness, k_p	$20\ MN/m$	Damping ratio, ζ	$10\ \%$

Table 3: Data used in numerical simulations.

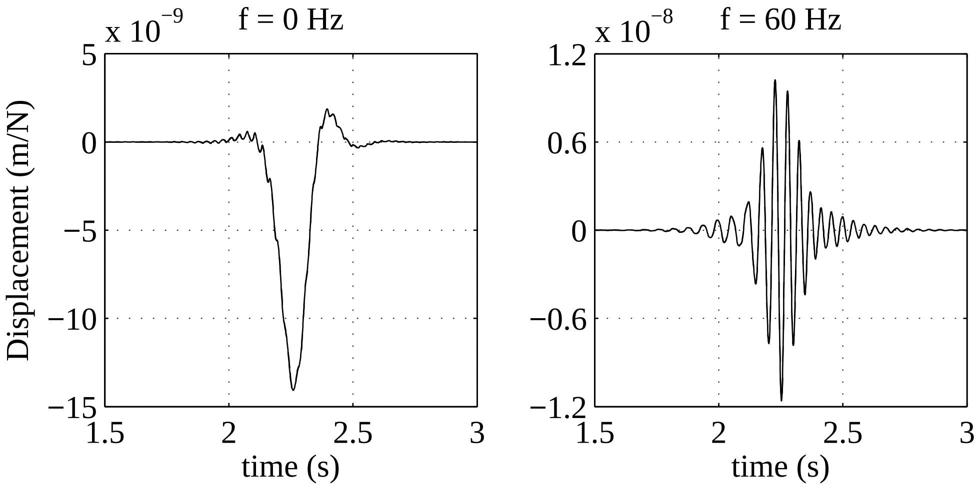

Figure 6: Transfer function for the vertical displacement of the mid-node of a track of length $L_t = 90\ m$ and radius $R = 20\ m$ subjected to a load with speed $V = 20\ m/s$ using —: curved and – –: straight elements.

compared to the curved element method.

The proposed theory in [2–7] seems to have a limitation as to the minimum curvature and length that the beam elements can have. A combination of short element length with big radius does not yield realistic results. Thus it has not been possible to test multiple curved beam elements between beam supports or bigger radii. Based on this, a restriction is also applied by the nature of the problem and the method of solution. In this problem a discretely supported track is being used, where rail pads are provided every L_b meters. Because the problem is solved using nodal points and the forces exerted on them, a node has to be provided every L_b meters to allow the pad stiffness and damping to be incorporated to the global system matrices. This in turn means that the length of the curved beams is limited to a maximum of L_b, not giving much space to the values of radius used.

Based on the above results, it is concluded that for this kind of problems it is not of great benefit to use the curved beam element approach presented in [5, 7]. Compared to the straight element theory, it does not provide significant improvements to the results for the vertical displacement of a discretely supported rail. The added complexity increases the risk of errors and there is a limitation to the parameters that can be investigated. Although this method is not recommended for the modelling of curved railway tracks subjected to vertical loads, it could be proven more beneficial to problems where greater spans of beams can be modelled. It is also necessary to investigate and compare the two methods for horizontal loads, as these are affected

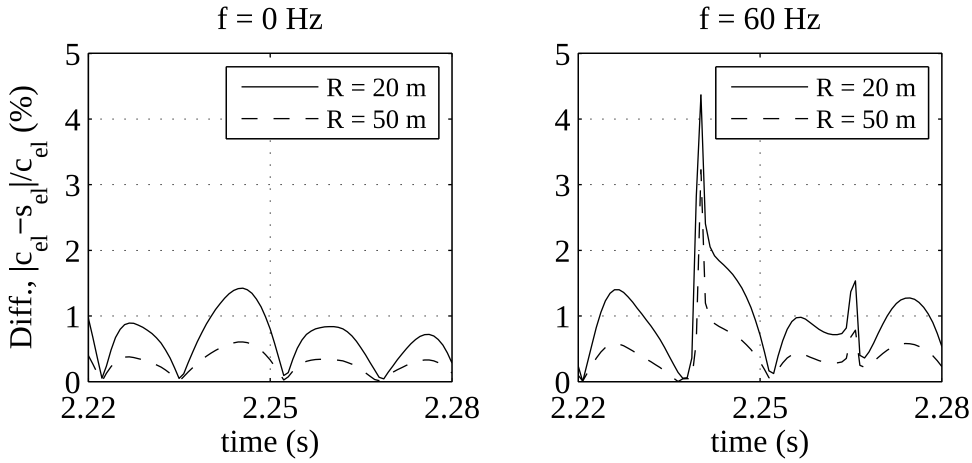

Figure 7: Percentage difference for the vertical displacement between using curved and straight elements for a track of length $L_t = 90\ m$ subjected to a load with speed $V = 20\ m/s$.

at a higher degree from the horizontal geometry.

5 CONCLUSION AND FUTURE WORK

In this paper, the dynamic response of a curved railway track discretely supported on an elastic foundation has been investigated. Two methods have been compared. The classical piecewise straight beam element method and the curved beam element method proposed by [5, 7]. Firstly the development of the two numerical models was described. Then a validation procedure was carried out for the case of a simply supported track for which the analytical solution is readily available in the literature [1]. In the final section, results where shown for a case of a curved rail supported on rail pads and excited by a harmonic moving load. Both methods show a good agreement. The curved element method has added complexity without giving substantial improvement and it is thus not recommended for this kind of problem/formulation.

Further research is on-going to improve the current model of the curved track. Plans for continuing this work comprise to:

- Investigate the effect of horizontal forces
- Investigate the effect of torsion
- Model different formulations for the curved beam element existent in the literature
- Develop the model further, to account for additional geometrical properties of the track such as cant, track twist and irregularities

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