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ON THE MOTION OF A HEAVY DISK ON A VIBRATION RUBBED HORIZONTAL PLANE

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Abstract. *It is considered a free motion of the heavy disk on rubbed plane which is performing periodical oscillation in the vertical direction. Disk has form of the short circle cylinder, one of base of which is found on the plane, and, consequently, the normal contact stresses distribution is obeyed by Galin's law. It is supposed that disk is performing a free combined motion: simultaneously sliding and spinning. Arising dry friction effect are described with aid of theory of multi-component dry friction based on Pade approximants of the exact integral dry friction model constructed at supposition of the validity of the Coulomb law in the generalized differential form. It is found periodical motion of disk with taking in account the dynamics coupling of the components defined the force state.*

1 INTRODUCTION

A problem about the heavy disk dynamics on the horizontal rubbed plane is one of the classical problems of theoretical mechanics. It was investigated in many publications. Most effects which arise at moving the disk caused by the influence of the combined dry friction.

One of the first works, in which there was carried out a detailed study of the dynamics of a heavy circular cylinder on a rough plane under the assumption of uniform distribution of normal contact stress inside of the contact area, was made by a group of Russian scientists under under the leadership of A.Yu. Ishlinsky [1].

With that, a specialists in the field of theory of elasticity are well known that distribution of normal contact stress in the case of the indentation of cylindrical punch into an elastic half-space distribution of is obeys by Galin law [2]. However, the use of the real law of the normal contact stresses distribution in the dynamics problems was complicated by cumbersome methods of investigations. Progress was made only after the establishment of the theory of multicomponent friction.

Fact is that researchers in the field of dry friction has long been known that in case of combined of the kinematics, when the rubbed solids are participated, simultaneously, in the sliding, spinning and rolling motion, the use of the classical Coulomb's law is not correct, and the friction law is undergoing significant changes.

One of the first attempts to describe the relationship of friction and spinning in the case of non-point contact of moving solids was undertaken by Contensou The principle new development of the theory was given by Zhuravlev in [4]. Zhuravlev’s theory was used in [5] to investigate the dynamics of heavy disk on a rough plane in the case of combined kinematics in assumption that distribution of normal contact stresses is described by Galin law.

Just after the publication of [4-5], the theory of multicomponent dry friction had begun the development. The main distinguishing feature of this theory is that, at first, under the assumption of validity both the classical Coulomb's law in differential form for small element of the area inside the contact area and its generalized forms, there are constructed the exact coupled integral dry friction model, obtained by integrating the differentials of the principal vector and torque on the contact spot. In addition, in process of the exact integral models construction there are is used well known results from the theory of elasticity that tangent stresses lead to shift in the symmetric diagram of the normal contact stresses in the direction of the instantaneous sliding or rolling velocities [6]. To apply the theory of elasticity results in the dynamics problems, it is used the simple asymptotic representations for the contact stresses distributions based on their general properties known from the theoretical results of the theory of elasticity [7-8].

To this exact form, to facilitate its use in the dynamics problems, Pade expansions [9] or Zhuravlev’s approximations [7], based on the analytic properties of integral models, are applied. The approximate models preserve all properties of the models based on the exact integral expressions and correctly describe the behaviour of the net vector and torque of the friction forces and their first derivatives at zero and infinity. They are considered as phenomenological, as their coefficients are numbers that can be determined from the experiment that completely avoids the necessity to calculate the double integrals over the contact spot.

It is worth explain the used the "exact integral model" term because any model can not be exact, because it is only an approximation to a real phenomenon. This notion is used in the sense that, after the initial assumptions about the validity of Coulomb's law in classical and generalized differential form and general properties of the normal contact distributions inside of contact spot, all other computations, from a mathematical point of view, are being made exactly, without the use of approximate methods. Thus, after writing expressions for the differentials of the dry friction principal vector and torque, all subsequent transformations are exact results, reflecting the nature of the phenomenon.

Another important fact that affects on the construction of a friction model is a dynamic coupling of components that determine the force state of rubbed solids, appearing in case of combined kinematics due to skew in symmetric distribution chart of the normal contact stresses [10].

The investigation of the a heavy disk dynamics on the vibration rubbed plane, presented below, are took into account both the dynamic coupling between components that determine the force state of the rubbed solids, as well as a generalized differential form of Coulomb's law.

2 FRICTION MODELS OF THE HEAVY DISK FREE MOTION ON THE RUBBED HORIZONTAL PLANE

Friction models of the heavy disk free motion on the rubbed horizontal plane are constructed under the assumption that the Coulomb law in differential form holds for the small surface element dS in the interior of the contact spot, according to which the differentials of the resultant vector $d\mathbf{F}$ and the moment of friction dM_C with respect to the disk center are determined by the formulas:

$$d\mathbf{F} = -f\sigma\frac{\mathbf{V}}{|\mathbf{V}|}\left(1+\mu_1|\mathbf{V}|^3-\mu_2|\mathbf{V}|\right)dS,\ dM_C = -f\sigma\frac{\mathbf{r}\times\mathbf{V}}{|\mathbf{V}|}\left(1+\mu_1|\mathbf{V}|^3-\mu_2|\mathbf{V}|\right)dS,$$
$$\mathbf{V}=(v-\omega y,\omega x),\ \mathbf{r}=(x,y) \tag{1}$$

where f is the coefficient of friction, $\mathbf{r}=(x,y)$ is the position vector of an elemental area in the interior of the contact spot with respect to its center (Fig. 1), R - radius of contact spot, x - axe of the rectangular coordination systems with origin in the center of contact circle which is directed parallelly to velocity of instantaneous sliding v, ω is the angular velocity of rotation of the contact spot center, but μ_1 and μ_2 are the coefficients which can be defined in practice from experiments.

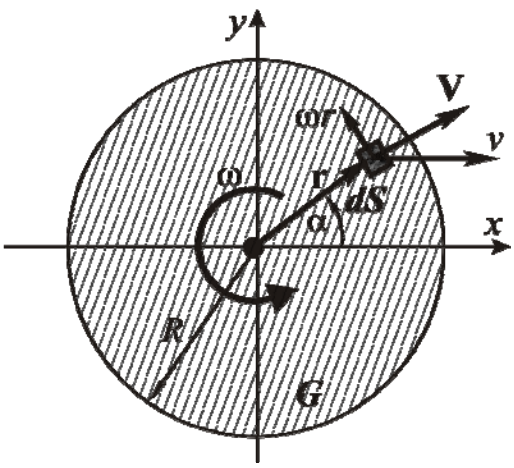

Figure 1: Kinematics inside the contact spot.

To use the theory of elasticity results in the dynamics problems, a simple linear approximation of the normal contact stresses distribution is proposed:

$$\sigma(x,y) = \sigma_0\left(1 + kx/R\right) \tag{2}$$

where $\sigma_0 = \sigma_0(r),\ r = |\mathbf{r}|$ - Galin distribution of normal contact stresses at absence of motion having the properties of central symmetry:

$$\sigma_0 = N\Big/\left(2\pi R^2\sqrt{1 - r^2/R^2}\right) \tag{3}$$

where N - normal reaction force.

To calculate coefficient k it is used the condition of equality of the external force F torque to the normal reaction force N torque which is appears from the shifting of the center of gravity of the contact spot in the direction of sliding on the value s:

$$Fh = Ns \tag{4}$$

where h - distance from the disk center mass to the plane of sliding.

On the other hand the shifting s of the gravity center relatively of the contact spot center can be defined by the following formula:

$$s = \iint_G x\sigma(x,y)dxdy \Big/ \iint_G \sigma(x,y)dxdy,\ \ G = \{(x,y) : x^2 + y^2 \le R^2\} \tag{5}$$

Substitution of the normal contact stresses distribution (3) to formula (4) yields:

$$s = \frac{\pi k}{R}\int_0^R \sigma_0(r) r^3 dr = \frac{R}{3} \tag{6}$$

Equalization values s calculated by the formulas (3) and (5) allows to define coefficient k which is characterized the dynamical coupling of the components defining the force state inside of contact spot.

2.1 Integral model

To obtain the resultant vector and the moment of friction, it is necessary to integrate the expressions (1) over the contact spot. The obtained dependencies, where $F_{||}$ and $F_{\perp}$ denote the respective components of the resultant vector directed along the tangent and the normal to the trajectory of motion, present an exact integral friction model of the disk motion on rubbed horizontal plane.

At the supposition that external forces are absence, the coefficient k in formula (1) is defined by the friction force component $F_{||}$ and the model has the following form:

$$\begin{gathered}
F_{||} = fN\int_0^{2\pi}\int_0^1 \frac{(v - ur\sin\alpha)r\sigma_0(r)}{\sqrt{u^2r^2 + v^2 - 2uvr\sin\alpha}}drd\alpha + 2\pi f\left((\mu_1 v^3 - \mu_2 v)I_1 + 2\mu_1 vu^2 I_3\right) \\
F_{\perp} = kfN\int_0^{2\pi}\int_0^1 \frac{ur^3\sigma_0(r)\cos^2\alpha}{\sqrt{u^2r^2 + v^2 - 2uvr\sin\alpha}}drd\alpha,\ k = \frac{fhR}{\pi I_3}\int_0^{2\pi}\int_0^1 \frac{(v - ur\sin\alpha)r\sigma_0(r)}{\sqrt{u^2r^2 + v^2 - 2uvr\sin\alpha}}drd\alpha \\
M_C = fRN\int_0^{2\pi}\int_0^1 \frac{(ur^2 - vr\sin\alpha)r\sigma_0(r)}{\sqrt{u^2r^2 + v^2 - 2uvr\sin\alpha}}drd\varphi + 2\pi f\left((2\mu_1 v^2 - \mu_2)uI_3 + \mu_1 u^3 I_5\right)
\end{gathered} \tag{7}$$

where coefficients of polynomials terms in formulas (7) are the first moments of the normal contact stresses distribution: $I_1 = \int_0^1 r\sigma_0(r)dr = \frac{1}{2\pi}$ - moment of the first order, $I_3 = \int_0^1 r^3\sigma_0(r)dr = \frac{1}{3\pi}$ - moment of the third order and $I_5 = \int_0^1 r^5\sigma_0(r)dr = \frac{4}{15\pi}$ - moment of the fifth order.

2.2 Models based on Pade expansions

Double integrals in formulas (7) can be calculated in elementary function [10-11], but much more convenient to use they direct approximations based on Pade expansions. The simplest of them for the torque M_C and for the tangent force component $F_{\parallel}$ are the linear-fractional approximations. But, for the normal friction force component, corresponded Pade approximation, naturally, became of the second order:

$$
\begin{aligned}
M_C &= M_0\left(\frac{u}{u+mv} + 2\pi\left((2\mu_1 v^2 - \mu_2)uI_3 + \mu_1 u^3 I_5\right)\right),\ \frac{1}{m} = \frac{3\pi}{4} \\
F_{\parallel} &= F_0\left(\frac{v}{v+au} + 2\pi\left((\mu_1 v^3 - \mu_2 v)I_1 + 2\mu_1 v u^2 I_3\right)\right),\ \frac{1}{a} = \frac{4}{\pi} \\
F_{\perp} &= \frac{\mu F_0 uv}{(u+bv)(v+au)},\ \frac{1}{b} = \frac{4\pi^2}{3}
\end{aligned}
\tag{8}
$$

The model of the first order (8) preserves the values of the functions $F_{\parallel}(u,v)$, $F_{\perp}(u,v)$ and $M_C(u,v)$ defined by the formulas (7) at zero, as well as their behavior and the behavior of their first derivatives at infinity. But model of this type cannot completely preserve the values of all first partial derivatives of these functions at zero. To obtain a correct description of the behavior of the first derivatives at zero, it is required to use the second-order Pade´ approximations:

$$
\begin{aligned}
M_C &= M_0\left(\frac{u^2+muv}{v^2+muv+u^2} + 2\pi\left((2\mu_1 v^2 - \mu_2)uI_3 + \mu_1 u^3 I_5\right)\right),\ m = \frac{4}{3\pi} \\
F_{\parallel} &= F_0\left(\frac{v^2+auv}{v^2+auv+u^2} + 2\pi\left((\mu_1 v^3 - \mu_2 v)I_1 + 2\mu_1 v u^2 I_3\right)\right),\ a = \frac{\pi}{4} \\
F_{\perp} &= \frac{\mu F_0 uv}{(u+bu)(v+au)},\ \frac{1}{b} = \frac{4\pi^2}{3}
\end{aligned}
\tag{9}
$$

The second-order model (9) completely satisfies to the all integral model analytical properties.

The approximations (8) and (9) hold for positive values of u and v. They can be easily generalized to the case of arbitrary (in sign) velocities u and v by a formal change by absolute values in the denominators of the corresponding expressions.

2.3 Models based on Zhuravlev's approxomations

Another kind of the approximations which will be valid for all ranges of the velocities variation is Zhuravlev's approximations, firstly mentioned but not calculated in [7]:

$$
\begin{aligned}
&M_C = M_0\left(\frac{u}{\sqrt{u^2+mv^2}} + 2\pi\left((2\mu_1 v^2-\mu_2)uI_3+\mu_1 u^3 I_5\right)\right),\ \frac{1}{m}=\frac{16}{9\pi^2}\\
&F_{\parallel} = F_0\left(\frac{v}{\sqrt{v^2+au^2}} + 2\pi\left((\mu_1 v^3-\mu_2 v)I_1+2\mu_1 vu^2 I_3\right)\right),\ \frac{1}{a}=\frac{16}{\pi^2}\\
&F_{\perp} = \frac{\mu F_0 uv}{\sqrt{(u^2+bv^2)(v^2+au^2)}},\ \frac{1}{b}=\frac{16\pi^4}{9}
\end{aligned}
\tag{10}
$$

This model permits to escape using not smooth functions in the cases when velocities are changed their signs. Moreover, model (10) are defines by the same coefficients amounts as models based on Pade expansions.

3 DISK DYNAMICS

The equations of motion of a disk of radius R and mass m_d considered at the projections on the axes of fixed coordinate system $\{XY\}$ (Fig.2) hardly connected with vibration plane are:

$$
\begin{aligned}
&J\dot{\omega} = -M_C,\ m_d\ddot{\mathbf{r}} = \mathbf{F},\ \mathbf{r}=(x,y), \mathbf{F}=(F_x,F_y)\\
&N = m_d g - A_z m_d \nu_z^2 \sin \nu_z t
\end{aligned}
\tag{11}
$$

where A_z and ν_z are the amplitude and frequency of the plane oscillations, but $J = m_d R^2/2$ is the disk moment of inertia.

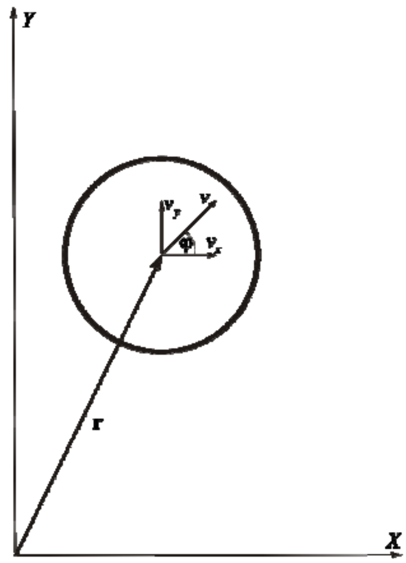

Figure 2: Disk kinematics.

Velocities of disk center mass are defined by formulas: $v_x = \dot{x}, v_y = \dot{y}$. In the moving polar coordinates system $\{v,\varphi\}$ with origin in projection of the mass center to the sliding plane (center of contact spot) they has form: $v_x = v\cos\varphi, v_y = v\sin\varphi, \dot{\varphi} = \omega$.

Connections of the net vector $\mathbf{F}$ components in fixed coordinate system with their components in moving coordinate systems with origin at the center of contact circle are given by the formulas:

$$\begin{aligned} F_x &= -F_{\parallel}\cos\varphi + F_{\perp}\sin(\varphi) \\ F_y &= -F_{\parallel}\sin\varphi - F_{\perp}\cos(\varphi) \end{aligned} \tag{12}$$

Here $F_{\parallel}$ and $F_{\perp}$ can be defined as by formulas (8)-(9) as formulas (10). Because the sliding velocity v can change the sign, it is convenient to use the model (10), in which:

$$\begin{aligned} F_0 &= fN = f(m_d g - A_z m_d \nu_z^2 \sin\nu_z t) \\ M_0 &= fNR = fR(m_d g - A_z m_d \nu_z^2 \sin\nu_z t) \end{aligned} \tag{13}$$

In result, the full equation system of the disk dynamics on the vibration rubbed plane has form:

$$J\dot{\omega} = -M_C,\ \ m_d\dot{v}_x = F_x,\ \ m_d\dot{v}_y = F_y \tag{14}$$

Transition from variables $(\omega, \dot{v}_x, \dot{v}_y)$ to variables (u, v, φ) gives:

$$J\dot{u} = -RM_C,\ \ m_d\dot{v} = -F_{\parallel},\ \ m_d v\dot{\varphi} = -F_{\perp} \tag{15}$$

From the last equation of this system (16) immediately follows conclusion about the shift of the disk trajectory from the straight line because $\dot{\varphi} \neq 0$ at $F_{\perp} \neq 0$. The first two equations are separated and, consequently, can be solve independently, because $M_C = M_C(u,v)$, $F_{\parallel} = F_{\parallel}(u,v)$. Moreover, presence the periodic functions caused the periodic disk motion.

4 CONCLUSIONS

- The combined dry friction effects have a basic influence on the disk dynamics.
- The periodic oscillations in vertical directions of the plane on which the disk is rest caused its periodic motion.

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